NOTES ON ESSENTIALLY POWERS FILTRATIONS

L. J. Ratliff, Jr.

Dedicated to Louis Yager

1. INTRODUCTION

All rings in this paper are assumed to be commutative with an identity, and the undefined terminology is, in general, the same as that in [7].

In this paper a study is made of a special type of filtration on a ring A, an *essentially powers filtration* (*e.p.f.*—see 2.3 for the definition). Such filtrations have some interesting properties, and they are a useful and important concept since, as briefly noted after 2.3, each filtration on A can be closely approximated by them so knowledge about *e.p.f.*'s can be used to derive knowledge about more general filtrations on A.

In Section 2 three characterizations of an *e.p.f.* f on a Noetherian ring A in terms of the Rees ring $\mathscr{R}(A,f)$ are proved, and then some lattice theoretic properties of such filtrations are given in Section 3. In Section 4 it is shown that with each filtration g on an analytically unramified semilocal ring R there exist infinitely many filtrations on R that are associated with g and all of these are *e.p.f.*'s if and only if one of them is an *e.p.f.* Section 5 contains a number of characterizations of an analytically unramified semilocal ring in terms of *e.p.f.*'s, and in Section 6 it is shown that some of the results obtained in this paper are applicable to the Chain Conjecture in altitude three.

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2. ESSENTIALLY POWERS FILTRATIONS AND REES RINGS

In this section, after giving the basic definitions and known results that are needed in the remainder of this paper, three characterizations of an *e.p.f.* f on a Noetherian ring A are given in terms of the Rees ring $\mathscr{R}(A,f)$. We begin by recalling the definition of a filtration.

Definition 2.1. A *filtration* $f = \{A_n\}$ on a ring A is a descending sequence of ideals A_n of A such that $A_o = A$ and $A_n A_m \subseteq A_{n+m}$, for all n and m.

The next definition gives a partial order on the set of all filtrations on a ring.

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Definition 2.2. Let $f = \{A_n\}$ and $g = \{B_n\}$ be filtrations on a ring A. Then $f \leq g$ in case $A_n \subseteq B_n$, for all n.

We are mainly interested in this paper in a type of filtration that has a number of the nice properties of filtrations of the form $\{I^n\}$, where I is an ideal. This specific type of filtration is defined in 2.3.

Definition 2.3. (*cf.* [1, Definition 2.14].) A filtration $f = \{A_n\}$ on a ring A is said to be an *essentially powers of an ideal filtration (e.p.f.)* in case there exists $k > 0$ such that $A_n = \sum_{1}^{k} A_{n-i}A_i$, for all $n \geq 1$, where $A_j = A$, if $j \leq 0$.

Therefore, if I is an ideal in A, then $f = \{I^n\}$ is an *e.p.f.* (Let $k = 1$ in 2.3.)

Since there are many useful and important filtrations on a ring A that are not powers of an ideal (such as the symbolic powers of a primary ideal or the integral closures of the powers of an ideal (see 4.6), and since every filtration on A is a "limit" of a sequence of *e.p.f.*'s (that is, if $f = \{A_n\}$, then the smallest filtration f_n on A whose first $n + 1$ terms are $A, A_1, A_2, \ldots, A_n$ is an *e.p.f.*, by (2.4.1), $f_1 \leq f_2 \leq \ldots \leq f_n \leq \ldots \leq f$, and "$\lim f_n = f$"), the study of such filtrations is important.

A number of properties of an *e.p.f.* are given in [1] and [2]. Those properties that are needed in this paper are summarized in the following remark.

Remark 2.4. Let $f = \{A_n\}$ be a filtration on a ring A. Then the following statements hold:

(2.4.1) [2, Proposition 2.8]. *f is an e.p.f. if and only if there exists $n > 0$ such that f is the least filtration on A whose first $n + 1$ terms are $A, A_1, \ldots, A_n$.*

(2.4.2) [1, (2.17), p. 26]. *f is an e.p.f. if and only if* $A_n = \sum \left(\prod_{1}^{k} A_i^{e_i} \right)$, *where k is as in* 2.3 *and the sum is over all nonnegative integers e_i such that*

$$e_1 + 2e_2 + \ldots + ke_k = n.$$

(2.4.3) *If A is Noetherian, then f is an e.p.f. if and only if there exists $m > 0$ such that, for all $j \geq m$, $A_{m+j} = A_m A_j$.*

(2.4.4) [1, Corollary 2.21 and the proof of Theorem 2.17]. *If A is Noetherian and f is an e.p.f., then, for all $n \geq 0$ and for all large h, $A_{hn} = (A_h)^n$.*

Proof. (2.4.1), (2.4.2), and (2.4.4) are proved in the cited references.

Proof of (2.4.3). If f is an *e.p.f.*, then the existence of such m is given by [1, Proposition 2.18 and Theorem 2.20]. The converse follows from (2.3) with $k = 2m$.

Many of the results in this paper concerning an *e.p.f.* will either be concerned with certain Rees rings of a Noetherian ring or will be derived using such rings in an auxiliary role, so we next define such rings.

Definition 2.5. Let $f = \{A_n\}$ be a filtration on a ring A. Then the *Rees ring* *of A with respect to f* is the ring $\mathscr{R} = \mathscr{R}(A,f) = A[tf,u] = A[u,tA_1,t^2A_2,\ldots]$, where t is an indeterminate and $u = 1/t$.

Many facts about Rees rings are known. We will need only the three readily seen facts listed in 2.6.

Remark 2.6. Let $f = \{A_n\}$ be a filtration on a ring A and let $\mathscr{R} = \mathscr{R}(A,f)$. Then the following statements hold:

(2.6.1) *$\mathscr{R}$ is a graded subring of $A[t,u]$ and each element in $\mathscr{R}$ can be written in the form $\sum_{-m}^{n} c_i t^i$ with $c_i \in A_i$.*

(2.6.2) *u is a regular element in $\mathscr{R}$ and $u^i\mathscr{R} \cap A = A_i$, for all $i \geq 0$.*

(2.6.3) *If g is another filtration on A and $f \leq g$ (2.2), then $\mathscr{R}(A,f) \subseteq \mathscr{R}(A,g)$.*

The following theorem is the main result in this section. Special cases of parts of the theorem are given in [4] (where A is a Noetherian Krull domain), and in [20] (where A is a local ring). Also, closely related results are given in [5, Proposition 2.1] and in [19, Lemma 2] (which result is incorrect). Finally, (2.7.1) $\Leftrightarrow$ (2.7.4) is given in [2, Theorem 2.9].

THEOREM 2.7. (*cf.* [4, Theorem 2.2] and [20, Lemma 5.6].) *Let $f = \{A_n\}$ be a filtration on a Noetherian ring A, and let $\mathscr{R} = \mathscr{R}(A,f)$ (2.5). Then the following statements are equivalent:*

(2.7.1) *$\mathscr{R}$ is Noetherian.*

(2.7.2) *$\mathscr{N} = (tA_1,t^2A_2,\ldots)\mathscr{R}$ is a finitely generated ideal.*

(2.7.3) *$\mathscr{R}$ is finitely generated over A.*

(2.7.4) *f is an e.p.f.*

Proof. It is clear that (2.7.1) $\Rightarrow$ (2.7.2) and that (2.7.3) $\Rightarrow$ (2.7.1). Also, (2.7.4) $\Rightarrow$ (2.7.3), since (2.7.4) implies that $\mathscr{R} = A[u,tA_1,\ldots,t^kA_k]$ (by (2.4.1)), where k is as in 2.3.

Finally, assume that (2.7.2) holds and let $f_1,\ldots,f_m$ be a basis of $\mathscr{N}$. Since $\mathscr{N}$ is homogeneous and $f_i \in \mathscr{N}$, it may be assumed that $f_i = a_it^{e_i}$ is homogeneous and $e_i > 0$. Let $k = \max\{e_i; i = 1,\ldots,m\}$, so $\mathscr{N} = (tA_1,\ldots,t^kA_k)\mathscr{R}$. Let $n > k$ and $a \in A_n$, so $x = at^n \in \mathscr{N}$, hence $x = \sum g_i f_i$, for some $g_i \in \mathscr{R}$, and it may be assumed that $g_i = b_it^{n-e_i}$ is homogeneous. Therefore

$$a = \sum a_ib_i \in \sum_{i=1}^{n} A_{e_i}A_{n-e_i} \subseteq \sum_{j=1}^{k} A_jA_{n-j}.$$

Thus $A_n = \sum_{j=1}^{k} A_jA_{n-j}$, for $n > k$, hence it follows from (2.3) that f is an *e.p.f.*

If $\mathscr{N}$ in (2.7.2) is finitely generated, then $(\mathscr{N},u)\mathscr{R}$ is finitely generated and $\mathscr{R}$ is Noetherian, by 2.7. On the other hand, it follows from [20, Lemma 5.5] that if $(\mathscr{N},u)\mathscr{R}$ is finitely generated, then $\mathscr{R}$ need not be Noetherian.

We continue to study *e.p.f.*'s in the next two sections, and 2.7 will be useful in a number of results in these sections.

3. LATTICE PROPERTIES OF E.P.F'S

It is known [9] that the set $\mathscr{F}$ of all filtrations on a ring A is a lattice with a multiplication (operations defined in 3.1), the three operations $(\vee, \wedge, \cdot)$ are compatible with the partial order $f \leq g$ (2.2), and they are commutative and associative. It is shown in 3.1 that the subset of all *e.p.f.*'s on A is closed under joins and products, but does not form a sublattice of $\mathscr{F}$.

THEOREM 3.1. *Let $f = \{A_n\}$ and $g = \{B_n\}$ be e.p.f.'s on a Noetherian ring A. Then the following statements hold:*

(3.1.1) $f \vee g = \left\{ \sum_{i+j=n} A_i B_j \right\}$ *is an e.p.f.*

(3.1.2) $f \cdot g = \{A_n B_n\}$ *is an e.p.f.*

(3.1.3) $f \wedge g = \{A_n \cap B_n\}$ *is not necessarily an e.p.f.*

Proof of (3.1.1). It is easily verified that $f \vee g$ is a filtration on A. Also, since f and g are *e.p.f.*'s, $A[tf, u]$ and $A[tg, u]$ are finitely generated over A, by (2.7). Further, it follows from the definition that $\mathscr{R}(A, f \vee g) = A[tf, tg, u]$. Therefore $\mathscr{R}(A, f \vee g)$ is finitely generated over A, so $f \vee g$ is an *e.p.f.*, by 2.7.

Proof of (3.1.2). $f \cdot g$ is readily seen to be a filtration on A, so, by (2.4.3), let $m > 0$ such that $A_{m+j} = A_m A_j$, for all $j \geq m$, and let $p > 0$ such that $B_{p+i} = B_p B_i$, for all $i \geq p$. Then it follows from (2.4.3) that, for all $j \geq \max\{m,p\}$,

$$A_{mp+j} B_{mp+j} = (A_m)^p A_j (B_p)^m B_j \subseteq A_{mp} B_{mp} A_j B_j \subseteq A_{mp+j} B_{mp+j}.$$

Therefore, since $mp \geq \max\{m,p\}$, $A_{mp+j} B_{mp+j} = A_{mp} B_{mp} A_j B_j$, for all $j \geq mp$, so $f \cdot g$ is an *e.p.f.*, by (2.4.3).

Proof of (3.1.3). This follows from [11, Example 2.2] and [1, Theorem 2.51], where it is shown that, even if $f = \{I^n\}$ and $g = \{J^n\}$ with I and J ideals in A, $f \wedge g$ need not be an *e.p.f.*

4. ANALYTICALLY UNRAMIFIED SEMI-LOCAL RINGS AND E.P.F.'S

In this section, we show in 4.2 and 4.8 that if f is a filtration on an analytically unramified semi-local ring R, then there are infinitely many filtrations on R that are related to f and all of these are *e.p.f.*'s if and only if one of them is an *e.p.f.* Then a number of related results are proved. We begin with a definition.

Definition 4.1. If $f = \{A_n\}$ is a filtration on a ring A, then, for each $k \geq 1$, $f^{(k)}$ denotes the filtration $\{A_{kn}\}$.

From the definition of f^k in (3.1.2), it is clear that $f^k \leq f^{(k)}$.

With this definition, we now give two somewhat surprising characterizations of an *e.p.f.* in an analytically unramified semi-local ring.

THEOREM 4.2. *Let R be an analytically unramified semi-local ring, and let $f = \{A_n\}$ be a filtration on R. Then the following statements are equivalent:*

(4.2.1) *f is an e.p.f.*

(4.2.2) *For all $k \geq 1$, $f^{(k)}$ (4.1) is an e.p.f.*

(4.2.3) *There exists $k \geq 1$ such that $f^{(k)}$ is an e.p.f.*

Proof. Let $\mathscr{R} = \mathscr{R}(R,f)$ and, for $k > 0$, let $\mathscr{R}_k = R[u^k, t^k A_k, t^{2k} A_{2k}, \ldots]$. Then $\mathscr{R}_k \cong \mathscr{R}(R, f^{(k)})$. Also, $\mathscr{R}$ is integral over $\mathscr{R}_k$; for, u is a root of $X^k - u^k$ and if $a \in A_n$, then $a^k \in (A_n)^k \subseteq A_{nk}$ and $t^n a$ is a root of $X^k - t^{nk} a^k$, hence the generators of $\mathscr{R}$ are integral over $\mathscr{R}_k$, so $\mathscr{R}$ is integral over $\mathscr{R}_k$.

Assume that (4.2.1) holds, and let $k \geq 1$. Then $\mathscr{R}$ is finitely generated over R, by 2.7, so $\mathscr{R}$ is finitely generated over $\mathscr{R}_k$, hence $\mathscr{R}$ is a finite $\mathscr{R}_k$-algebra (by the first paragraph of this proof). Therefore, since $\mathscr{R}$ is Noetherian, $\mathscr{R}_k$ is Noetherian, by [3, Theorem 2]. Hence, since $\mathscr{R}(R, f^{(k)}) \simeq \mathscr{R}_k$, $f^{(k)}$ is an *e.p.f.*, by (2.7), so (4.2.1) $\Rightarrow$ (4.2.2).

It is clear that (4.2.2) $\Rightarrow$ (4.2.3).

Finally, assume that (4.2.3) holds, so $\mathscr{R}_k \simeq \mathscr{R}(R, f^{(k)})$ is Noetherian and is finitely generated over R, by 2.7. Now $\mathscr{R}_k \subseteq \mathscr{R}_k[u] \subseteq \mathscr{R}$, and $\mathscr{R}_k[u]$ and $\mathscr{R}$ have the same total quotient ring, so, by the first paragraph of this proof, $\mathscr{R}_k[u]$ and $\mathscr{R}$ have the same integral closure, say $\mathscr{S}$. Also $\mathscr{R}_k[u]$ is finitely generated over R and is separably generated over R (since u is an indeterminate), so, since R is analytically unramified, $\mathscr{S}$ is a finite $\mathscr{R}_k[u]$-algebra, by [12, Lemma 2.4]. Therefore $\mathscr{S}$ is Noetherian and is a finite $\mathscr{R}$-algebra, so $\mathscr{R}$ is Noetherian, by [3, Theorem 2]. Thus f is an *e.p.f.*, by 2.7, so (4.2.3) $\Rightarrow$ (4.2.1).

Remark 4.3. The proof of 4.2 shows that (4.2.1) $\Rightarrow$ (4.2.2) $\Rightarrow$ (4.2.3) in an arbitrary Noetherian ring.

A number of corollaries of 4.2 will now be given. The first (which is more precisely a corollary of 4.3) is concerned with monadic transformations of A with respect to $f^{(k)}$.

COROLLARY 4.4. *Let $f = \{A_n\}$ be an e.p.f. in a Noetherian ring A. Then for each $k \geq 1$ and for each regular element $a \in A_k$, $B = A[A_k/a, A_{2k}/a^2, \ldots]$ is finitely generated over A.*

Proof. $\mathscr{R}_k = A[u^k, t^k A_k, t^{2k} A_{2k}, \ldots] \subseteq A[t^k, u^k]$ and $K = (t^k a - 1)A[t^k, u^k]$ is the kernel of the natural homomorphism from $A[t^k, u^k]$ onto $A[1/a]$, so $B \simeq \mathscr{R}_k/(K \cap \mathscr{R}_k)$. Therefore, since f is an *e.p.f.*, $f^{(k)}$ is an *e.p.f.*, by 4.3, so $\mathscr{R}_k \simeq \mathscr{R}(A, f^{(k)})$ is finitely generated over A, by 2.7, hence B is finitely generated over A.

(2.4.4) showed that if f is an *e.p.f.*, then, for all large h, $f^{(h)}$ consists of powers of the ideal A_h. The next corollary shows that the converse holds for analytically unramified semi-local rings.

COROLLARY 4.5. *Let R and f be as in 4.2. Then the following statements are equivalent:*

(4.5.1) *f is an e.p.f.*

(4.5.2) *There exists* $k \geq 1$ *such that* $f^{(k)} = \{(A_k)^n\}$.

(4.5.3) *For all large* h, $f^{(h)} = \{(A_h)^n\}$.

Proof. It is clear that (4.5.3) $\Rightarrow$ (4.5.2), and if (4.5.2) holds, then $f^{(k)}$ is an *e.p.f.*, so f is an *e.p.f.*, by 4.2, hence (4.5.2) $\Rightarrow$ (4.5.1). Finally, (4.5.1) $\Rightarrow$ (4.5.3), by (2.4.4).

The following definition and remark will be used to give some further corollaries of 4.2.

Definition 4.6. Let I be an ideal in a ring A and let $f = \{A_n\}$ be a filtration on A. Then the *integral closure* I_a *of* I *in* A is defined to be the set of elements x in A which satisfy an equation of the form $x^n + a_1x^{n-1} + \ldots + a_n = 0$, where $a_i \in I^i$, and the *integral closure* f_a *of* f *in* A is $f_a = \{(A_n)_a\}$.

Remark 4.7. [10, Section 6]. With the notation of 4.6, the following statements hold:

(4.7.1) I_a *is an ideal in* A.

(4.7.2) $I \to I_a$ *is a semi-prime operation on* A (*that is:* $I \subseteq I_a$; $I \subseteq J$ *implies* $I_a \subseteq J_a$; $I_{aa} = I_a$; *and*, $I_aJ_a \subseteq (IJ)_a$), *so* f_a *is a filtration on* A.

The next corollary gives another characterization of an *e.p.f.* in an analytically unramified semi-local ring.

COROLLARY 4.8. *Let* R *and* f *be as in* 4.2. *Then the following statements hold:*

(4.8.1) *If there exist* $k \geq 1$ *and a filtration* g *on* R *such that* $g_a = (f^{(k)})_a$ *and* g *is an e.p.f., then* f *is an e.p.f.*

(4.8.2) *If* f *is an e.p.f., then, for all* $k \geq 1$ *and for all filtrations* g *on* R *such that* $g_a = (f^{(k)})_a$, g *is an e.p.f.*

Proof of (4.8.1). Assume g is an *e.p.f.* such that $g_a = (f^{(k)})_a$. Then $\mathscr{R} = \mathscr{R}(R,g)$ is finitely generated over R, by 2.7, and is separable over R. Therefore the integral closure $\mathscr{R}'$ of $\mathscr{R}$ is a finite $\mathscr{R}$-algebra, by [12, Lemma 2.4], so $\mathscr{S} = \mathscr{R}(R,g_a)$ is finitely generated over R (since $R \subseteq \mathscr{R} \subseteq \mathscr{S} \subseteq \mathscr{R}'$). Therefore, since

$$R \subseteq \mathscr{A} = \mathscr{R}(R,f^{(k)}) \subseteq \mathscr{R}(R,(f^{(k)})_a) = \mathscr{S},$$

$\mathscr{S}$ is finitely generated over $\mathscr{A}$. Hence, since $\mathscr{S}$ is integral over $\mathscr{A}$, $\mathscr{A}$ is Noetherian, by [3, Theorem 2]. Thus $f^{(k)}$ is an *e.p.f.*, by (2.7), and so f is an *e.p.f.*, by 4.2.

Proof of (4.8.2). Assume that f is an *e.p.f.*, let $k \geq 1$, and let g be a filtration on R such that $g_a = (f^{(k)})_a$. Then $h = f^{(k)}$ is an *e.p.f.*, by 4.2, and $g = g^{(1)}$. Therefore $h_a = (g^{(1)})_a$ and h is an *e.p.f.*, so it follows from (4.8.1) that g is an *e.p.f.*

In particular, it follows from 4.8 that if I is an ideal in R and $f = \{A_n\}$ is a filtration on R such that $I^n \subseteq A_n \subseteq (I^n)_a$, for all n, then f is an *e.p.f.*

Remark 4.9. Let R and f be as in 4.2, and assume that f is an *e.p.f.* For each $n \geq 1$ let I_n be an ideal in R such that $A_n \subseteq I_n \subseteq (A_n)_a$. Then $\mathscr{I} = R[u, tI_1, t^2I_2, \ldots]$ is finitely generated over R (as in the proof of (4.8.1), since $\mathscr{R}(R,f) \subseteq \mathscr{I} \subseteq \mathscr{R}(R,f_a))$. Therefore, even if $\{I_n\}$ is not a filtration on R, it still holds that there exists $k > 0$ such that $I_n \subseteq \sum \left(\prod_1^k I_i^{e_i} \right)$.

Remark 4.10. If f and g are filtrations on an arbitrary Noetherian ring A such that $f \leq g \leq f_a$, and if g is an *e.p.f.*, then, for each filtration h on A such that $f \leq h \leq g$, h is an *e.p.f.*

Proof. $\mathscr{R} = \mathscr{R}(A,f) \subseteq \mathscr{R}(A,h) \subseteq \mathscr{R}(A,g) \subseteq \mathscr{R}(A,f_a) \subseteq \mathscr{R}'$, the integral closure of $\mathscr{R}$, and $\mathscr{R}(A,g)$ is finitely generated over A, by 2.7, so is finitely generated (and integral) over $\mathscr{R}(A,h)$. Therefore $\mathscr{R}(A,h)$ is Noetherian, by [3, Theorem 2], so h is an *e.p.f.*, by 2.7.

The next corollary results from combining 4.2, 4.5, and 4.8.

COROLLARY 4.11. *Let f and g be filtrations on an analytically unramified semi-local ring R such that $g_a = (f^{(h)})_a$, for some $h \geq 1$. Then the following statements are equivalent:*

(4.11.1) *f is an e.p.f.*

(4.11.2) *g is an e.p.f.*

(4.11.3) *There exists $k \geq 1$ such that $g^{(k)}$ is an e.p.f.*

(4.11.4) *There exists $k \geq 1$ such that $g^{(k)} = \{(B_k)^n\}$, where $g = \{B_n\}$.*

(4.11.5) *For all $k \geq 1$, $g^{(k)}$ is an e.p.f.*

Proof. This follows immediately from 4.2, 4.5, and 4.8.

For a final corollary, we recall the following definition.

Definition 4.12. Let I be an ideal in a ring A. I is said to be *normal* in case $(I^n)_a = I^n$, for all $n \geq 1$.

With this terminology, the following known result [18, Theorem 2] is an easy corollary of (4.8.2).

COROLLARY 4.13. *Let I be an ideal in an analytically unramified semi-local ring R. Then, for all large k, $(I^k)_a$ is normal* (4.12).

Proof. By (4.8.2) applied to $f = \{I^n\}$, $\{(I^n)_a\}$ is an *e.p.f.* Therefore, by (2.4.4), for all large k, $(I^{kn})_a = ((I^k)_a)^n$, for all $n \geq 0$, so $(I^k)_a$ is normal (see (4.7.2)).

5. CHARACTERIZATIONS OF ANALYTICALLY UNRAMIFIED SEMI-LOCAL RINGS AND E.P.F.'S

In this section, a number of necessary and sufficient conditions for a semi-local ring to be analytically unramified are given in terms of *e.p.f.*'s. We begin with a needed definition.

Definition 5.1. If f is a filtration on a ring A, then Rad f is the common radical of the ideals A_n, for $n > 0$.

Concerning 5.1, it follows easily from the definition of a filtration $\{A_n\}$ that Rad A_i = Rad A_j, for all nonzero i and j.

To prove the first two characterizations, use will be made of the fact [17, Lemma 1] that if there exists an open ideal Q in a semi-local ring R such that $(Q^n)_a \subseteq Q^{m(n)}$ and $m(n)$ tends to infinity with n, then R is analytically unramified. (Actually, this was only proved for a local ring in [17], but the proof given there continues to hold for the semi-local case.)

THEOREM 5.2. *The following statements are equivalent for a semi-local ring R:*

(5.2.1) *R is analytically unramified.*

(5.2.2) *For each filtration f on R which is an e.p.f., f_a is an e.p.f.*

(5.2.3) *There exists a filtration f on R such that* Rad f *is the Jacobson radical of R and f_a is an e.p.f.*

(5.2.4) *There exists an open ideal Q in R such that $\{(Q^n)_a\}$ is an e.p.f.*

Proof. (5.2.1) $\Rightarrow$ (5.2.2), by (4.8.2) (with $k = 1$ and $g = f_a$), and it is clear that (5.2.2) $\Rightarrow$ (5.2.4) $\Rightarrow$ (5.2.3).

Finally, assume that (5.2.3) holds and let $f = \{A_n\}$. Then, for all large h and for all $n \geq 0$, $(A_{hn})_a = (A_{ha})^n$, by (2.4.4) applied to f_a, hence $Q = A_{ha}$ is a normal ideal. Also, $\operatorname{Rad} A_{ha} = \operatorname{Rad} A_h$ is the Jacobson radical of R, by hypothesis and 5.1, so (5.2.1) holds, by [17, Lemma 1].

The following corollary is interesting, since it shows that if there exists an open ideal Q in R such that the Rees ring of R with respect to Q has finite integral closure, then the integral closures of all the rings $\mathscr{R}(R,f)$ (with f an e.p.f.) are finite $\mathscr{R}(R,f)$-algebras.

COROLLARY 5.3. *The following statements are equivalent for a semi-local ring R:*

(5.3.1) *R is analytically unramified.*

(5.3.2) *For each e.p.f. f on R, the integral closure $\mathscr{R}'$ of $\mathscr{R} = \mathscr{R}(R,f)$ is a finite $\mathscr{R}$-algebra.*

(5.3.3) *There exists an e.p.f. f on R such that* Rad f *is the Jacobson radical of R and $\mathscr{R}(R,f)'$ is a finite $\mathscr{R}(R,f)$-algebra.*

(5.3.4) *There exists an open ideal Q in R such that $R[tQ,u]'$ is a finite $R[tQ,u]$-algebra.*

Proof. (5.3.1) $\Rightarrow$ (5.3.2), by [12, Lemma 2.4] (see the proof of (4.8.1)), and it is clear that (5.3.2) $\Rightarrow$ (5.3.4) $\Rightarrow$ (5.3.3).

Finally, if (5.3.3) holds, then $\mathscr{R}(R,f_a)$ is finitely generated over R, since $\mathscr{R}(R,f)$ is (by (2.7)) and $\mathscr{R}(R,f) \subseteq \mathscr{R}(R,f_a) \subseteq \mathscr{R}(R,f)'$, so f_a is an *e.p.f.*, by 2.7. Therefore (5.3.1) holds, by (5.2.3) $\Rightarrow$ (5.2.1).

6. THE CHAIN CONJECTURE IN ALTITUDE THREE AND E.P.F.'S

In this section, we show in (6.5.1) that the Chain Conjecture, 6.4, holds for altitude three local domains, if all filtrations of a special type in a Henselian local domain of altitude three are *e.p.f.*'s. Then in (6.5.2) (respectively, (6.5.3)) characterizations of analytically unramified (respectively, pseudo-geometric) Henselian local domains of altitude three that satisfy the Chain Conjecture are given in terms of *e.p.f.*'s.

Throughout this section, it will be necessary to use a number of facts concerning a certain over-ring I of a local domain R. These facts were proved in [15], but since this paper has not as yet appeared in print, we define I and summarize the needed facts in the following remark. (As in Section 5, *A' will be used to denote the integral closure of a ring A in its total quotient ring.*)

Remark 6.1. Let (R,M) be a local domain, let $a = \text{altitude } R \geq 1$, let $I = \cap \{R'_p : p \in \text{Spec } R' \text{ and depth } p = a - 1\}$, and let B be a ring such that $R \subseteq B \subseteq I$. Then the following statements hold:

(6.1.1) [15, (2.3.3) and (5.1)]. Altitude $B = a$.

(6.1.2) [15, (5.4)]. *$I = R'$ if and only if every height one prime ideal in R' has* depth $= a - 1$.

(6.1.3) [15, (9.7) and (4.4.1)]. *For each height one prime ideal q in I,* depth $q = a - 1$. *Moreover, for each $Q \in \text{Spec } I$ such that* height Q + depth $Q = a$, height $Q \cap B$ = height Q *and* depth $Q \cap B$ = depth Q.

(6.1.4) [15, (4.4.2)]. *If $P \in \text{Spec } B$ is such that* height P + depth $P = a$, *then there exists $Q \in \text{Spec } I$ such that $Q \cap B = P$,* height Q = height P, *and* depth Q = depth P.

(6.1.5) [15, (12.1)]. *If $a = 3$, then* height $MI = 3$.

Definition 6.3. If b is a nonzero nonunit in a local domain R, then $\mathscr{T}(bR) = R_b \cap R_S$, where $S = R - \cup \{p \in \text{Spec } R : p \text{ is a (minimal) prime divisior of } bR \text{ and depth } p = \text{altitude } R - 1\}$, and $b^{(n)} = b^n \mathscr{T}(bR) \cap R$, for each $n \geq 0$.

The facts concerning $\mathscr{T}(bR)$ and $b^{(n)}$ that are needed in this section are given in the following lemma.

LEMMA 6.3. *Let R, b, and S be as in* 6.2, *let* a = altitude R, *and let* $T = \mathscr{T}(bR)$. *Then the following statements hold:*

(6.3.1) $R \subseteq T \subseteq I$ (6.1).

(6.3.2) $T = \{c/b^n;\ n \geq 1$ *and c is in every* depth $a - 1$ *primary component of* $b^n R\}$.

(6.3.3) $b^n T = b^n R_S \cap T$, *so $b^n T$ and $b^{(n)}$ are finite intersections of* depth $a - 1$ *primary ideals.*

(6.3.4) $f = \{b^{(n)}\}$ *is a filtration on R.*

(6.3.5) $T = R[b^{(1)}/b, \ldots, b^{(n)}/b, \ldots]$.

Proof of (6.3.1). Since depth $a - 1$ prime ideals in R' lie over depth $a - 1$ prime ideals in R, it follows that

$T = R_b \cap R_S \subseteq \cap \{R_q : q \in \text{Spec } R,\ b \notin q, \text{ and height } q = 1\} \cap R_S \subseteq \cap \{R_q : q \in \text{Spec } R \text{ and depth } q = a - 1\} \subseteq \cap \{R'_p : p \in \text{Spec } R' \text{ and depth } p = a - 1\} = I$, and it is clear that $R \subseteq T$.

(6.3.2) follows readily from the definition of T.

Proof of (6.3.3). $b^n T = b^n(R_b \cap R_S) = R_b \cap b^n R_S \subseteq T$, so $b^n T = b^n R_S \cap T$. Also, $b^n R_S$ is a finite intersection of height one primary ideals, so $b^n T$ and $b^n T \cap R$ are. Finally, let p be a depth $a - 1$ prime divisor of bR and let $p' \in \operatorname{Spec} R'$ such that $p' \cap R = p$. Then $T \subseteq R_S \subseteq R_p \subseteq R'_{p'}$. Also, depth $p' = a - 1$, so $T \subseteq I \subseteq R'_{p'}$, by (6.3.1) and the definition of I, so

$$pR_p \cap T = (p'R'_{p'}, \cap I) \cap T.$$

Thus $\operatorname{depth} p'R'_{p'} \cap I = a - 1 = \operatorname{depth} pR_p \cap T$, by (6.1.3) and (6.3.1), hence it follows that each prime divisor of $b^n T$ and of $b^n T \cap R$ has depth $= a - 1$.

(6.3.4) is readily proved, and (6.3.5) follows from (6.3.2) and the definition of $b^{(n)}$ 6.2.

The following conjecture has been open since it was mentioned by M. Nagata in 1956 in [6, Problem 3″].

CHAIN CONJECTURE 6.4. *The integral closure of a local domain is catenary.*

For the history of this conjecture and a number of equivalences of it, see [16, Section 4].

It is known that this conjecture holds for local domains of altitude at most two, and it is not known if it holds for all local domains of altitude three. It is also known [16, (4.4)] that to prove the conjecture in altitude three, it is sufficient to prove that every Henselian local domain of altitude three is catenary. For these reasons, we restrict attention to Henselian local domains of altitude three in our last result, 6.5.

(6.5.1) shows that if all $\{b^{(n)}\}$ in every Henselian local domain of altitude three are *e.p.f.*'s, then the Chain Conjecture holds for altitude three. A case when the converse of (6.5.1) holds is given in (6.5.2), and a closely related result is given in (6.5.3). (Concerning (6.5.3), it is somewhat startling to realize that it is not known if the Chain Conjecture holds for pseudo-geometric Henselian local domains of altitude three.)

THEOREM 6.5. *Let (R,M) be a Henselian local domain of altitude* 3. *Let* (*) *denote the statement:* $\{b^{(n)}\}$ *is an e.p.f. Then the following statements hold:*

(6.5.1) *R is catenary if* (*) *holds for every nonzero b in M.*

(6.5.2) *If R is analytically unramified, then R is catenary if and only if* (*) *holds for every nonzero b in M.*

(6.5.3) *If R is pseudo-geometric, then R is catenary if and only if* (*) *holds for some nonzero b in M.*

Proof of (6.5.1). It will be proved that if b is a nonzero element in M such that $\{b^{(n)}\}$ is an *e.p.f.*, then every minimal prime divisor of bR had depth two. From this and the hypothesis, it follows that every height one prime ideal in R has depth two, so R is catenary (since altitude $R = 3$).

Therefore, fix a nonzero b in M and assume that $\{b^{(n)}\}$ is an *e.p.f.* Then $T = \mathscr{T}(bR)$

is finitely generated over R, by (6.3.5) and 4.4. Therefore assume it is known that there exists a minimal prime divisor N of MT such that height $N = 3$. Then, since altitude $T = 3$, by (6.3.1) and (6.1.1), N is isolated over M (that is, N is maximal and minimal in the set of prime ideals in T that lie over M). Therefore, since R' is quasi-local (since R is Henselian), it follows from C. Peskine's version of Zariski's Main Theorem [8, p. 119] that T is integral over R, and so T is a finite integral extension domain of R. Also, every prime divisor of bT has depth two, by (6.3.3), so it follows from integral dependence that every minimal prime divisor of bR has depth two, as desired. Therefore, it remains to show that MT has a minimal prime divisor of height three.

For this, let N' be a minimal prime divisor of MI, so height $N' = 3$, by (6.1.5), and so $N = N' \cap T$ is a height three maximal ideal in T, by (6.1.3). Therefore, since $MT \subseteq N$, there exists a minimal prime divisor P of MT such that $P \subseteq N$. If height $P = 1$, then depth $P = 2$, by (6.3.3), so height $MI < 3$, by (6.1.4), and this contradicts (6.1.5). If height $P = 2$, then depth $P = 1$, so again a contradiction to (6.1.5) follows from (6.1.4). Therefore, $P = N$ is a height three minimal prime divisor of MT.

Proof of (6.5.2). By (6.5.1) it suffices to prove that if R is analytically unramified and catenary, then (*) holds for every nonzero b in M. For this, since R is catenary every height one prime ideal in R has depth two, and so this holds for R', by [14, (2.6)]. Therefore, the ring I in 6.1 is the integral closure of R, by (6.1.2). Also, since R is analytically unramified, $I = R'$ is a finite R-algebra. Moreover, for each nonzero b in M, $\mathscr{T}(bR) \subseteq I$, by (6.3.1). Therefore,

$$b^n R \subseteq b^{(n)} = b^n T \cap R \subseteq b^n R' \cap R = (b^n R)_a,$$

for each $n \geq 1$, hence, since $\{b^n R\}$ is clearly an *e.p.f.*, it follows from (4.8.2) that $\{b^{(n)}\}$ is an *e.p.f.*, and so (*) holds for all nonzero b in M.

Proof of (6.5.3). Since a pseudo-geometric local domain is analytically unramified, by [7, (36.4)], it suffices, by (6.5.2), to show that if (*) holds for some nonzero b in M, then R is unmixed (for then R is catenary). For this, let $T = \mathscr{T}(bR)$, so T is a finite R-algebra and $T \subseteq R'$, by the middle paragraph of the proof of (6.5.1). Therefore, to prove that R is unmixed, it suffices to prove that T is unmixed. For this, let z be a prime divisor of zero in the completion T^* of T and let p^* be a minimal prime divisor of $(z, b)T^*$. Then p^* is a prime divisor of bT^*, by [21, Lemma 1, p. 394] applied to $T^*_{p^*}$, so $p = p^* \cap T$ is a prime divisor of bT, hence depth $p = 2$, by (6.3.3). Therefore T/p is a pseudo-geometric Henselian local domain of altitude two, so T/p is unmixed, by [13, Remark 2.23(iii)]. Hence, since p^* is a prime divisor of pT^*, by [7, (18.11)], it follows that depth $p^* = 2$. Therefore depth $z = 3$, hence T is unmixed.

In closing, it should be noted that by [13, Remark 2.23(i)] the conclusion of (6.5.1) is equivalent to: R is quasi-unmixed. Also, if R is analytically unramified (as in (6.5.2) and (6.5.3)), then R is quasi-unmixed if and only if R is unmixed, so each of these conditions is equivalent to the equivalent conditions in (6.5.2) and in (6.5.3).

REFERENCES

1. W. Bishop, *A theory of multiplicity for multiplicative filtrations.* Ph.D. Dissertation, Western Michigan Univ., Kalamazoo, Michigan, 1971.
2. ———, *A theory of multiplicity for multiplicative filtrations.* J. Reine Angew. Math. 277 (1975), 8–26.
3. P. M. Eakin, Jr., *The converse to a well known theorem on Noetherian rings.* Math. Ann. 177 (1968), 278–282.
4. P. M. Eakin, Jr. and W. Heinzer, *Non finiteness in finite dimensional Krull domains.* J. Algebra 14 (1970), 333–340.
5. H. T. Muhly and M. Sakuma, *Asymptotic factorization of ideals.* J. London Math. Soc. 38 (1963), 341–350.
6. M. Nagata, *On the chain problem of prime ideals.* Nagoya Math. J. 10 (1956), 51–64.
7. ———, *Local Rings.* Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers, New York, NY, 1962.
8. C. Peskine, *Une généralization du "Main Theorem" de Zariski.* Bull. Sci. Math. 90 (1966), 119–127.
9. J. W. Petro, *Some results in the theory of pseduo-valuations.* Ph.D. Dissertation, State Univ. of Iowa, Iowa City, Iowa, 1961.
10. ———, *Some results on the asymptotic completion of an ideal.* Proc. Amer. Math. Soc. 15 (1964), 519–524.
11. ———, *Concerning filtrations on commutative rings.* Manuscripta Math. 15 (1975), 261–274.
12. L. J. Ratliff, Jr., *A characterization of analytically unramified semi-local rings and applications.* Pacific J. Math. 27 (1968), 127–143.
13. ———, *On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals* (II). Amer. J. Math. 92 (1970), 99–144.
14. ———, *Four notes on saturated chains of prime ideals.* J. Algebra 39 (1976), No. 1, 75–93.
15. ———, *Valuation rings and the (catenary) chain conjectures.* J. Algebra, 52 (1958), 101–144.
16. ———, *Chain conjectures in ring theory.* Lecture notes in mathematics No. 647, Springer-Verlag, New York, N.Y. 1978.
17. D. Rees, *A note on analytically unramified local rings,* J. London Math. Soc. 36 (1961), 24–28.
18. M. Sakuma and H. Okuyama, *On a criterion for analytically unramification of a local ring.* J. Gakugei Tokushima Univ. 15 (1966), 36–38.
19. P. Samuel, *Sur les anneaux gradués.* An. Acad. Brasil. Ci. 30 (1958), 447–450.
20. P. G. Sawtelle, *Characterizations of unmixed and quasi-unmixed local domains.* Ph.D. Dissertation, Univ. of California, Riverside, California, 1971.
21. O. Zariski and P. Samuel, *Commutative Algebra.* Vol. II, D. Van Nostrand, Co., Inc., New York, NY, 1960.

Department of Mathematics
University of California
Riverside, California 92521