# CLASSIFICATION OF SO(3)-ACTIONS ON FIVE-MANIFOLDS WITH SINGULAR ORBITS

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## INTRODUCTION

We describe the smooth $SO(3)$-actions on simply-connected, connected, closed five-dimensional manifolds admitting at least one orbit whose dimension is strictly less than the dimension of the principal orbits. We will show that such an $SO(3)$-manifold must be diffeomorphic to $S^5$, $S^2 \times S^3$, or the connected sum $kX_{-1} \# \ell M_2$, $k, \ell \geq 0$, where the five-manifold $X_{-1}$ is diffeomorphic to the Wu-manifold $SU(3)/SO(3)$ and $M_2$ to the Brieskorn variety of the type (2,3,3,3,3).

Let $\mathscr{D}^i_{SO(3)}$ be the set of all smooth orientable $SO(3)$-manifolds of dimension five which admits no exceptional orbits (defined in I) and whose orbit spaces are diffeomorphic to the $i$-dimensional ball $D^i$, $i = 2$ or 3. Then using the techniques of Bredon [3], Hsiang and Hsiang [5] and Jänich [7], we can classify $\mathscr{D}^i_{SO(3)}$. Every manifold in $\mathscr{D}^2_{SO(3)}$ has two or three distinct orbit types; if exactly two distinct orbit types appear then the orbit structure is determined by the invariants $\{H, K; b\}$ where $SO(3)/H$ and $SO(3)/K$ are the orbit types and

$$b \in \Gamma = [S^1, N(H)/N(H) \cap N(K)]/\pi_o(N(H)/H);$$

$\Gamma$ is isomorphic to the trivial group, $Z_2$ or $Z_+$ depending on the subgroups $H$ and $K$. The pair $(H, K)$ is $(\{e\}, SO(2))$, $(Z_k, SO(2))$, or $(D_k, N(SO(2))$. If $M \in \mathscr{D}^2_{SO(3)}$ admits three orbit types, one of them is the fixed point type and this $M$ is determined by an equivalence class of a finite sequence of symbols $\{0,1,2\}$; the length of the sequence equals the number of the fixed points. For example, $S^5$ admits an $SO(3)$-action with two fixed points which is the one-point compactification of the irreducible linear action on $R^5$ [11]. We show that every manifold in $\mathscr{D}^2_{SO(3)}$ admitting two fixed points is equivalent to this action on $S^5$. A manifold in $\mathscr{D}^2_{SO(3)}$ with three fixed points is equivalent to the Wu-manifold $SU(3)/SO(3)$ with $SO(3)$ acting by the left coset-multiplication. For a manifold with four fixed points we have two equivalence classes. One of them is the equivariant connected sum

$$SU(3)/SO(3) \# SU(3)/SO(3),$$

and the other is $M_2$. From our classification theorem we know that there is exactly one $SO(3)$-action on $M_2$ and that this action has four fixed points, but this action is not natural in a sense that a manifold with four fixed points was constructed first and then it was identified as $M_2$ by Barden's classification (1). One might try to give a direct construction of this action. Every manifold in $\mathscr{D}^2_{SO(3)}$ with fixed points is simply-connected.

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The set $\mathscr{D}^3_{SO(3)}$ consists of one element, namely the triple suspension of the usual $SO(3)$-action on $S^2$. These results agree with the work of Richardson [11] who tried to classify all $SO(3)$-actions, not necessarily smooth, on $S^5$.

By the codimension of an action we mean the codimension of a principal orbit. The orbit space $M^*$ of a five dimensional simply-connected, connected closed codimension-two $SO(3)$-manifold $M$ is diffeomorphic to $D^2$ or $S^2$; for a codimension-three action the orbit space $M^*$ is diffeomorphic to a homotopy three-ball $\Delta^3$ or a homotopy three-sphere $\Sigma^3$. In any case, the boundary of the orbit space is precisely the image of the orbits whose dimensions are strictly less than the dimension of the principal orbits. Call these orbits singular. We notice that every manifold in $\mathscr{D}^2_{SO(3)}$ admits singular orbits and that $\mathscr{D}^2_{SO(3)}$ contains the class $\mathscr{D}_0$ of all simply-connected, connected, closed five-manifolds admitting singular orbits. On the other hand, the discussion of codimension-three actions involves the Poincaré conjecture. If the Poincaré conjecture is true, every codimension-three $SO(3)$-action on a simply-connected closed five-manifold with singular orbits is equivalent to the triple suspension of $S^2$ which is the single element of $\mathscr{D}^3_{SO(3)}$.

The organization of the paper is as follows: In I. we recall the minimum amount of standard theorems and definitions necessary and also recall some facts about the group $SO(3)$. We refer to Bredon [3] and Hsiang [6] for details of the fundamentals of the theory of differentiable actions. In II. we give examples of manifolds in $\mathscr{D}^2_{SO(3)}$ and indicate their fundamental groups, in III. we classify $\mathscr{D}^2_{SO(3)}$, and in IV. we give a topological classification for $\mathscr{D}_0$. In the appendix, we discuss simply-connected $SO(3)$-manifolds whose orbit spaces are three dimensional.

This is a part of the author's PhD thesis [4] with some extra observations about actions with fixed points and the appendix. The author expresses her gratitude to Wu-Yi Hsiang and Tomoyoshi Yoshida for their useful suggestions and help.

## I. PRELIMINARIES

Let $G$ be a compact connected Lie group and $M$ a smooth $G$-manifold (this means that the action of $G$ is smooth). One denotes by $G(x)$ the orbit of $G$ through $x$, and by $G_x$ the isotropy group at $x$. The orbit space is indicated by $M^*$ or $M/G$ or sometimes $M/\varphi$ when the action $\varphi$: $G \times M \to M$ is specifically mentioned. Let $\pi: M \to M^*$ be the orbit map. We give $M^*$ the quotient topology. If $H \subseteq G$ is a subgroup, write

$$(H) = \{K \subseteq G: K \text{ is conjugate to } H \text{ by an inner automorphism of } G\}.$$

Write $M_{(H)} = \{x \in M: G_x \in (H)\}$. If $X$ is a $G$-invariant subspace of $M$ we write $F(H,X) = \{x \in X: gx = x \text{ for all } g \in H\}$. We say that two $G$-manifolds $M_1$ and $M_2$ are equivalent if there is an equivariant diffeomorphism $F: M_1 \to M_2$.

1. We recall the much-cited notion of a slice representation for smooth actions. The references are abundant [3], [5], [7], [8], [10]. Here we adopt the one given in [5], [6].

*The slice representation.* Let $M$ be a smooth $G$-manifold, and assume $M$ is equipped with an invariant Riemannian metric. Let $x \in M$. Then the induced action of $G_x$ on the normal vectors of $G(x)$ at $x$ gives us a representation $\psi_x: G_x \to O(\ell)$ ($\ell = \dim M - \dim G(x)$), which is called the *slice representation* of $G_x$ at $x$. Then we have (8).

*Differential Slice Theorem.* Let $\eta$ be the normal vector bundle of the embedding $G/G_x \to G(x) \subset M$ and let $\xi$ be the canonical $G_x$-bundle $G_x \to G \to G/G_x$. Denote by $\alpha_\xi(\psi_x)$ the associated vector bundle $R^\ell \to G \underset{G_x}{\times} R \to G/G_x$. Then $\eta = \alpha_\xi(\psi_x)$. The left translations are naturally bundle maps of $\xi$ and they induce a natural $G$-action on $\eta = \alpha_\xi(\psi_x)$. With respect to this natural $G$-action on $\eta$ and a given invariant Riemannian metric on $M$, the usual exponential map gives an equivariant diffeomorphism between a sufficiently small disk bundle of $\eta$ and an equivariant tubular neighborhood of $G(x)$.

Because of this theorem, in dealing with smooth $SO(3)$-manifolds, we will not have to confront the question whether the double suspension of the Poincaré sphere is homeomorphic to $S^5$ (III.1A). The formulation of a topological $SO(3)$-action on the double suspension of the Poincaré sphere appeared in a paper of Richardson [11].

*Remark.* Since writing this paper, J. Cannon [16] has shown that the double suspension of the Poincaré sphere is homeomorphic to $S^5$ and so $S^5$ admits non-smoothable non-linear $SO(3)$-actions.

2. A partial ordering for $\{(G_x): x \in M\}$ exists. If $(H), (K) \in \{(G_x): x \in M\}$ then $(H) < (K)$ if $H$ is conjugate to a subgroup of $K$. Equivalently we may order the corresponding orbit types by $G/H > G/K$ if $(H) < (K)$.

2A. THEOREM [9]. *There exists a minimum $(H)$ with respect to the above ordering. $M_{(H)}$ is open dense in $M$. Equivalently there exists a maximum $G/H$ for $G$ on $M$.*

The maximum orbit type for orbits in $M$ guaranteed by this theorem is called the *principal orbit type* and orbits of this type are called *principal orbits.* The corresponding isotropy groups are called *principal isotropy groups.* If $P$ is a principal orbit and $Q$ is any orbit such that $\dim Q < \dim P$ then $Q$ is called a *singular orbit.* $Q$ has the type $G/K$ for some $K \supset H$ with $\dim(K/H) > 0$; such $G/K$ is called a *singular orbit type.* If $\dim P = \dim Q$ but $K$ is not conjugate to $H$, $Q$ is called an *exceptional orbit.* Suppose $Q = G(x)$ is an exceptional orbit. Let $S$ be a slice at $x$ ($G_x$ acts on $S$ by a representation $\psi_x: G_x \to O(\ell)$, where $\ell = \dim M - \dim G(x)$. We may think of $S$ as a sufficiently small closed disk in $R^\ell$ in the Differential Slice Theorem). *If $F(G_x,S)$ is of codimension 1 in $S$, $Q$* is called a *special exceptional orbit.*

2B. *The bundle structures of a G-manifold.*

*Theorem* [2]. *Let $G$ be a compact Lie group and $M$ be a manifold. Let $\phi: G \times M \to M$ be an action. Suppose $G/H$ appears as an orbit type. Then*

$$\xi: G/H \to M_{(H)} \to M_{(H)}/\phi$$

*is a fibre bundle with the associated principal bundle*

$$\hat{\xi}: N(H)/H \to F(H, M_{(H)}) \to M_{(H)}/\phi$$

*where* $N(H)$ *is the normalizer of* $H$. *Conversely let*

$$\xi: G/H \to E \to B$$

*be a fibre bundle with the structure group* $N(H)/H$. *Then we may give a G-action on* $E$ *such that each fibre is an orbit and*

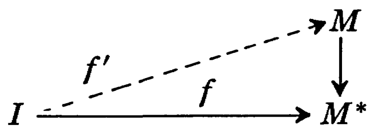
$$\hat{\xi}: N(H)/H \to F(H,E) \to B$$

*is the associated principal bundle. Furthermore, if* $\xi$ *is a differentiable fibre bundle, the action may be made into a differentiable action.*

3. *Topology of the G-manifold M and its orbit space* $M^*$. The reference to this subsection is Chapter IV of Bredon's book [3].

Let $G$ be a compact connected Lie group. Let $M$ be a smooth G-manifold, which is closed and connected.

3A. THEOREM (Equivariant Path Lifting). *Let* $f: I \to M^*$ *be any path. Then there is a lifting* $f': I \to M$ *such that* $f' = f$.



COROLLARY. *Suppose M is simply connected. Then* $M^*$ *is also simply connected.*

Write $B$, $E$, and $SE$ for the sets consisting of singular orbits, exceptional orbits, and special exceptional orbits respectively.

3B. THEOREM. *If* $\ell = n - 2$, *then the orbit space* $M^*$ *is a two-manifold and* $\partial M^*$ *consists exactly of* $B^* \cup SE^*$.

3C. THEOREM. *If* $H_1(M; Z_2) = 0$ *and if a principal orbit is connected then there are no special exceptional orbits.*

3D. THEOREM. *Suppose* $H_1(M;Z) = 0$ *and* $\ell = n - 2$. *Suppose singular orbits exist. Then* $E^* = \phi$ *and* $M^*$ *is a 2-disk with boundary* $B^*$.

Let us make the following conventions: write $(M, \phi)$ for a smooth closed connected orientable manifold of dimension $n$ together with a smooth action $\phi: G \times M \to M$ of codimension 2. Write $(M, \phi, D^2)$ for $(M, \phi)$ with $M^* = D^2$, and similarly $(M, \phi, S^2)$ if $M^*$ is $S^2$. Let $\mathscr{S}_G = \{(M,\phi): \pi_1(M) = \{e\}\}$, and

$$\mathscr{D}^2_G = \{(M, \phi, D^2): M \text{ has no exceptional orbits}\}$$

$\mathscr{F}^2_G = \{(M, \phi, S^2)\}$. Then $\mathscr{S}_G \subset \mathscr{D}^2_G \cup \mathscr{F}^2_G$. If we take $G = SO(3)$, $n = 5$, then $\mathscr{S}_{SO(3)} \cap \mathscr{D}^2_{SO(3)} = \mathscr{D}_0$ (defined in the introduction).

4A. The group $SO(3)$ of all proper rotations of $SO(3)$ can be thought of as the group of $3 \times 3$ real orthogonal matrices of determinant $+1$. The references for this group are abundant, but Wolf [15] has an especially good account of finite subgroups of $SO(3)$.

4B. Every subgroup of $SO(3)$ is conjugate to one of the following: $SO(2)$ (the group of proper rotations in $R^2$ regarded as a subgroup of $SO(3)$), $O'(2)$ (the normalizer of $SO(2)$ in $SO(3)$), cyclic group $Z_k$ of order $k$, the dihedral group $D_m$ of order $2m$, the tetrahedral group $T$, the octahedral group $O$, and the icosahedral group $I$ (the groups of symmetries of the regular $n$-gon, the regular tetrahedron, the regular octahedron, and the regular icosahedron, respectively). The symmetry groups are given by (15)

$$D_m:\quad A^m = B^2 = 1,\quad BAB^{-1} = A^{-1}$$

$$T:\quad A^3 = P^2 = Q^2 = 1,\quad PQ = QP,\quad APA^{-1} = Q,\quad AQA^{-1} = PQ$$

$$O:\quad \begin{cases} A^3 = P^2 = Q^2 = R^2 = 1, & PQ = QP,\quad RAR^{-1} = A^{-1} \\ APA^{-1} = Q,\quad AQA^{-1} = PQ, & RPR^{-1} = QP,\quad RQR^{-1} = Q^{-1} \end{cases}$$

$$I:\quad A^3 = B^2 = C^5 = ABC = 1$$

4C. [11]. The coset spaces $SO(3)/SO(2)$ and $SO(3)/O'(2)$ are diffeomorphic to $S^2$ and $RP^2$ respectively. For a finite subgroup $H$, $SO(3)/H$ is a closed orientable 3-manifold with $H_2(SO(3)/H; Z) = 0$. Using the double covering $\rho: S^3 \to SO(3)$, we get

$$\pi_1(SO(3)/Z_k) = Z_{2k},\quad \pi_1(SO(3)/D_m) = \tilde{D}_m,$$
$$\pi_1(SO(3)/T) = \tilde{T},\quad \pi_1(SO(3)/O) = \tilde{O},\quad \pi_1(SO(3)/I) = \tilde{I},\quad \text{and}$$
$$H_1(SO(3)/Z_k) = Z_{2k},\quad H_1(SO(3)/D_{2m}) = Z_2 \oplus Z_2$$
$$H_1(SO(3)/D_{2m+1}) = Z_4,\quad H_1(SO(3)/T) = Z_3,$$
$$H_1(SO(3)/O = Z_2,\quad H_1(SO(3)/I) = 0.$$

4D. $N(H)/H$ for subgroups of $SO(3)$.

| $H$ | $SO(2)$ | $O'(2)$ | $Z_k$ | $D_2$ | $D_m$ | $T$ | $I$ | $\{1\}$ | $O$ |
|---|---|---|---|---|---|---|---|---|---|
| $N(H)$ | $O'(2)$ | $O'(2)$ | $N(T)$ | $O$ | $D_{2m}$ | $O$ | $I$ | $SO(3)$ | $O$ |
| $N(H)/H$ | $Z_2$ | 1 | $N(T)/Z_k$ | $D_3$ | $Z_2$ | $Z_2$ | 1 | $SO(3)$ | 1 |

$T$ is the maximal torus of $SO(3)$ containing $Z_k$.

*Table 1*

4E. *The irreducible representations of subgroups $H$ of $SO(3)$ in all dimensions less than or equal to $5 - 3 + \dim H$.* The irreducible representations of $H = SO(3)$ are (a) the trivial representation $SO(3) \to \{1\}$ (b) the usual matrix representation $\rho: SO(3) \to SO(3)$ (c) $\wedge: SO(3) \to SO(5)$ defined as follows: let $X$ be the space of all $3 \times 3$ real symmetric matrices of trace 0, and let $SO(3)$ act on it by $(g, a) \to gag^{-1}$, $g \in SO(3)$ and $a$ a symmetric matrix with trace 0. Bredon [3, p. 42–44] gives a detailed discussion of this representation. $\wedge$ is called the weight 2 representation.

$H = SO(2)$ is one dimensional, and so we want its irreducible representations in dimensions 1, 2, and 3. It has no irreducible representations in dimension 3. It has (a) the trivial representation $1: SO(2) \to \{1\}$ (b) for each positive integer $k$, $\rho_k: SO(2) \to SO(2)$ defined by

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$$

$O'(2)$ has (a) $1: O'(2) \to \{1\}$ (b) det: $O'(2) \to O(1)$, where

$$\det\left(\begin{array}{c|c} * & \\ \hline & -1 \end{array}\right) = -1 \quad \text{and} \quad \det\left(\begin{array}{c|c} * & \\ \hline & 1 \end{array}\right) = 1$$

(c) for each $k \geq 1$, $\bar{\rho}_k: O'(2) \to O(2)$ defined by

$$\bar{\rho}_k \,|\, SO(2) = \rho_k, \quad \bar{\rho}_k \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

For finite subgroups $H$, we will only need $H \to SO(2)$ in dimension 1 and 2. They are the following: $Z_\ell$ has (a) the trivial $Z_\ell \to 1$. (b) for each $1 \leq k \leq [\ell - 1/2]$, $\rho_k: Z_\ell \to SO(2)$ is given by

$$\rho_k(g) = \begin{pmatrix} \cos 2\pi k/\ell & -\sin 2\pi k/\ell \\ \sin 2\pi k/\ell & \cos 2\pi k/\ell \end{pmatrix}$$

$D_m$, using the expression in (4B), has two one-dimensional representations: (a) the trivial one, $A \to 1$, $B \to 1$ (b) $A \to 1$, $B \to -1$, *for all m*. If $m$ is even we have in addition to (a) and (b), (c) $A \to -1$, $B \to 1$, (d) $A \to -1$, $B \to -1$.

$T$ has (a) the trivial representation, (b) $Z_2 + Z_2 \to T \to Z_3$,

$$P, Q \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A \to \begin{pmatrix} \cos 2i\pi/3 & -\sin 2i\pi/3 \\ \sin 2i\pi/3 & \cos 2i\pi/3 \end{pmatrix}, \qquad i = 1 \text{ or } 2$$

$O$ has (a) the trivial representation, (b) $T \to O \to Z_2$, $R \to 1$ and $A, P, Q, \to -1$.

The icosahedral group has the trivial representation in the dimension one and nothing in $SO(2)$.

5. *Classification of simply connected, connected five-manifolds* [1], [13]. Let $M$ be a simply connected, connected, five-manifold. The second Stiefel-Whitney class of $M$ may be regarded as a homomorphism $w_2: H_2(M:Z) \to Z_2$. $w_2$ determines

(by suitable rearrangement of basis for $H_2(M;Z)$) a diffeomorphism invariant $i(M)$ of $M$. The theorem of Barden states that $H_2(M;Z)$ and $i(M)$ form a complete set of invariants for the diffeomorphism classification. For manifolds $i(M) = 0$, $H_2(M;Z)$ is the invariant. This is the Theorem A of Smale [13].

We will need the following identification found in Theorem 2.3 (1).

| $H_2(M;Z)$ | $w_2(M)$ | $i(M)$ | $M$ |
|---|---|---|---|
| $Z$ | $0$ | $0$ | $S^2 \times S^3$ |
| $Z_2$ | $\neq 0$ | $1$ | $X_{-1}$ |
| $Z_2 + Z_2$ | $\neq 0$ | $1$ | $X_1 = X_{-1} \# X_{-1}$ |
| $Z_2 + Z_2$ | $0$ | $0$ | $M_2$ |
| $Z_2 + \underbrace{(Z_2 + \dots + Z_2)}_{2n}$ | $\neq 0$ | $1$ | $X_{-1} \# \underbrace{M_2 \# \dots \# M_2}_{n}$ |
| $\underbrace{Z_2 + \dots + Z_2}_{2n}$ | $\neq 0$ | $1$ | $X_1 \# \underbrace{M_2 \# \dots \# M_2}_{n-1}$ |
| $\underbrace{Z_2 + \dots Z_2}_{2n}$ | $0$ | $0$ | $\underbrace{M_2 \# \dots \# M_2}_{n}$ |

## II. EXAMPLES OF $\mathscr{D}^2_{SO(3)}$

1. Let $SO(3)$ act on $R^6 = R^3 \times R^3$ by $\rho \oplus \rho$, where $\rho$ is the usual matrix representation of $SO(3)$. Then the restriction to $S^5 \subset R^6$ defines an $SO(3)$-action $\phi_0$ on $S^5$.

2. The irreducible representation $\Lambda: SO(3) \to SO(5)$ defines an action on $S^4$ and hence on $S^5$ in an obvious way [11], [12]. We write these $SO(3)$-manifolds as $(S^4, \Lambda)$ and $(S^5, \Lambda)$ respectively.

Richardson [11] shows that these are the only smooth $SO(3)$-actions on $S^5$ admitting three-dimensional orbits.

3. Consider $H \subset K \subset SO(3)$, where $H = Z_k$ and $K = SO(2)$ so that $K/H \approx S^1$. We have the sphere bundle $K/H \to SO(3)/H \xrightarrow{p} SO(3)/K$, where $p$ is the projection; so its mapping cylinder $M_p = SO(3)/H \times [0,1] \bigcup_p SO(3)/K$ is a (smooth) manifold. $SO(3)$ acts on $SO(3)/H \times [0,1]$ (respectively on $SO(3)/K$) by $\alpha(\beta H, t) = \alpha\beta H, t)$ (respectively $\alpha(\beta K) = \alpha\beta K)$), $\alpha, \beta \in SO(3)$. Hence it also acts on $M_p$, and so on $M_P \times S^1$ (identity on $S^1$) and similarly on $SO(3)/H \times D^2$. The boundary of $M_P \times S^1$ is $SO(3)/H \times S^1$. Let $M = SO(3)/H \times D^2 \bigcup_f M_P \times S^1$

where $f\colon SO(3)/H \times S^1 \to SO(3)/H \times S^1$ is an equivariant diffeomorphism. $M$ is a smooth $SO(3)$-manifold of dimension 5 whose principal orbit type is $SO(3)/H$ and the singular orbit type is $SO(3)/K$. $M^* = D^2$ and $M_{(K)}{}^* = \partial M^* (= S^1)$.

This construction yields the following $SO(3)$-manifolds.

3A. Denote by $M(1)$ the manifold obtained by taking $H = \{e\}$, $K = SO(2)$ and $f =$ the identity map. Then

$$M(1) = SO(3) \times D^2 \bigcup_{id} M_P \times S^1.$$

One can think of $M(1)$ as $SO(3) \times D^2/{\sim}$, where $\sim$ corresponds to collapsing each orbit of $SO(3)$ through $(\alpha,x)$, $x \in \partial \in D^2$ by the restricted action of $SO(2)$.

Similarly, let $M(k)$ be the $SO(3)$-manifold with $H = Z_k$, $K = SO(2)$, and $f =$ the identity, for $k = 2, 3, \ldots$. Then $M(k) = SO(3)/Z_k \times D^2 \bigcup_{id} M_{P_k} \times S^1$ is of the principal orbit type $SO(3)/Z_k$ and the singular orbit type $SO(3)/SO(2)$. $M_{p_k}$ is the mapping cylinder of $p_k\colon SO(3)/Z_k \to SO(3)/SO(2)$.

Let $H = D_k$ $(k \geq 2)$, $K = O'(2)$; $f =$ the identity. Let

$$p'_k\colon SO(3)/D_k \to SO(3)/O'(2)$$

denote the projection. Let $Mp'_k = SO(3)/D_k \times D^2 \bigcup_{id} Mp'_k \times S^1$. $M'(k)$ is an $SO(3)$-manifold whose principal and singular orbit types are $SO(3)/Z_k$ and $SO(3)/O'(2)$ respectively.

3B. Let $H = O'(1)$, $K = O'(2)$, $p\colon SO(3)/O'(1) \to SO(3)/O'(2)$. Let

$$\hat{f}_j\colon N(H)/H \times S^1 \to N(H)/H \times S^1$$

be defined by $\hat{f}_j(\alpha,t) = (\alpha\sigma^j(t), t)$, where $\alpha \in N(H)/H$ and $\sigma =$ preferred generator of $\pi_1(N(H)/H) = Z$. This defines for $j = \pm 0, 1, 2, 3, \ldots$, an $SO(3)$-equivariant map

$$f_j\colon SO(3)/O'(1) \times (N(H)/H \times S^1) \circlearrowleft N(H)/H \approx SO(3)/O'(1) \times S^1.$$

Thus we obtain $M'(1,j) = SO(3)/O'(1) \times D^2 \bigcup_{f_j} M_p \times S^1$.

3C. *Remark.* $\pi_1(M(k)) = \{e\}$, $k = 1, 2, \ldots$, $\pi_1(M'(k)) \neq \{e\}$, $k \geq 2$, and $\pi_1(M'(1,j)) \neq \{e\}$, $j = \pm 0, 1, 2, \ldots$.

4A. One of the most interesting and important $SO(3)$-manifolds whose orbit spaces are diffeomorphic to the disk $D^2$ is the homogeneous space

$$X_{-1} = SU(3)/SO(3),$$

which is known as the Wu-manifold. $SO(3)$ acts on $X_{-1}$ by the left coset-multiplication. $F(SO(3),X_{-1}) = N(SO(3))/SO(3) = \{SO(3), \bar{\omega}SO(3), \bar{\omega}^2 SO(3)\}$ where $\bar{\omega}$ is the complex matrix

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

with $\omega$ = the cube root of 1. Hence it admits three fixed points. One can show that the principal orbit type is $SO(3)/(Z_2 + Z_2)$, and the singular orbit type other than the fixed points is $SO(3)/O'(2)$ either by a direct computation or as a corollary to our classification theorem (III.2B). $SO(3)$ can act on $X_{-1}$ in no other way. One finds another description of $X_{-1}$ in (1). Dennis Sullivan pointed out to us that $X_{-1}$ is indeed $SU(3)/SO(3)$. By taking connected sums

$$\underbrace{X_{-1} \# X_{-1} \# \ldots \# X_{-1}}_{n \text{ times}}$$

one obtains $SO(3)$-manifolds with $n + 2$ fixed points. Depending on how one connects $X_{-1}$'s, one obtains various distinct actions with same number of fixed points. All these manifolds are simply connected and have the nontrivial second Stiefel-Whitney class, $w_2(M)$ (IV).

4B. For $n = 2, 3, \ldots$, consider ordered $n$-tuples $(j_1, j_2, j_3, \ldots, j_n)$ with values in $\{0,1,2\}$ such that $j_i \neq j_{i+1}$, $1 \leq i \leq n$, and $j_1 \neq j_n$. We construct an $SO(3)$-manifold $M^{j_1 \cdots j_n}$ admitting three orbit types $SO(3)/(Z_2 + Z_2)$, $SO(3)/O'(2)$, $SO(3)/SO(3)$ with $F(SO(3),M)$ consisting of $n$ distinct points.

Let $X$ be the space with $\partial X = I_1 \cup \ldots \cup I_n \cup J_1 \cup \ldots \cup J_n$ as indicated in Figure 1 on page 294.

Fix $Z_2 + Z_2 \subset SO(3)$ as the group of matrices $\{P, Q, PQ, 1\}$ where

$$P = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

$Z_2 + Z_2$ is a subgroup of $N_0$, $N_1$, $N_2$ where $N_0$ is the subgroup consisting of matrices of the form

$$\left(\begin{array}{c|c} * & \\ \hline & \pm 1 \end{array}\right);$$

*i.e.*, $N_0 = O'(2)$, and

$$N_1 = \left\{\left(\begin{array}{c|c} \pm 1 & \\ \hline & * \end{array}\right)\right\}, \quad N_2 = \left\{\left(\begin{array}{c|c|c} * & & * \\ \hline & \pm 1 & \\ \hline * & & * \end{array}\right)\right\}.$$

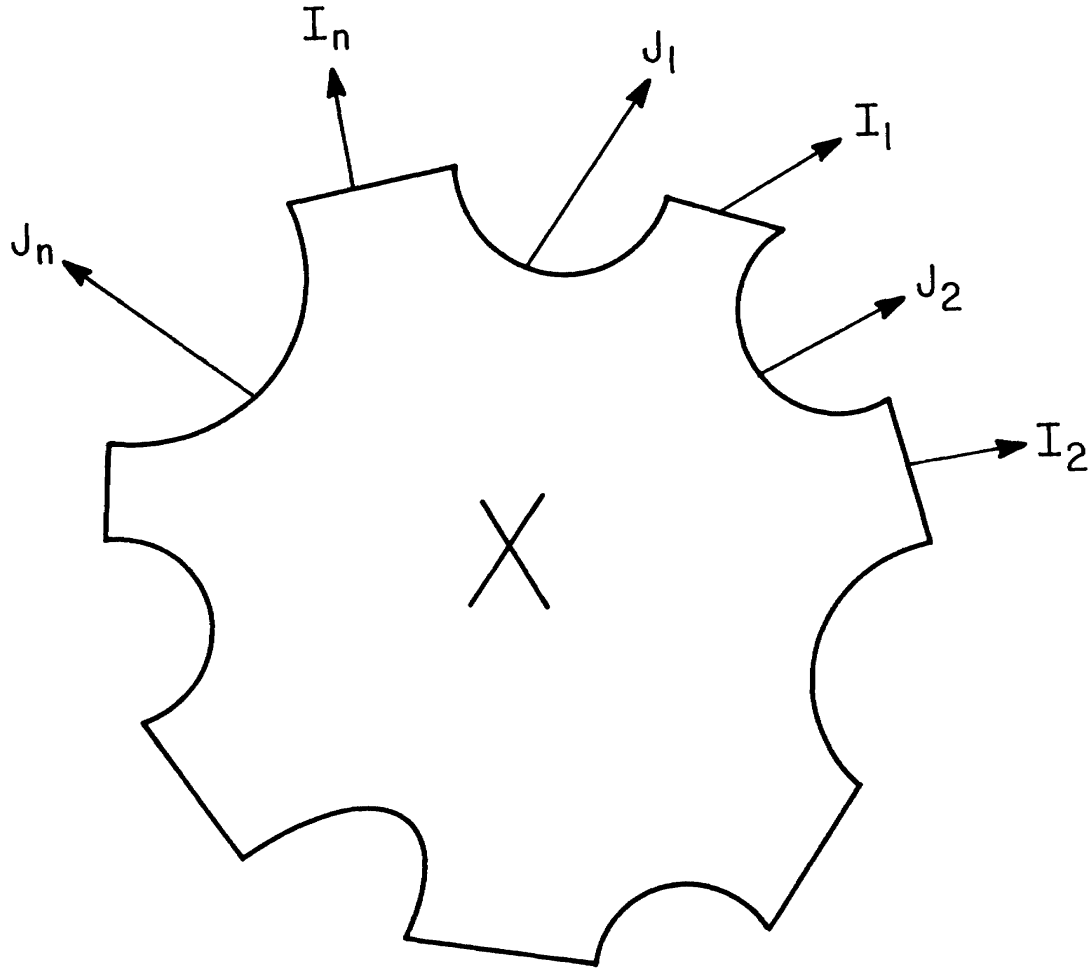



Figure 1.

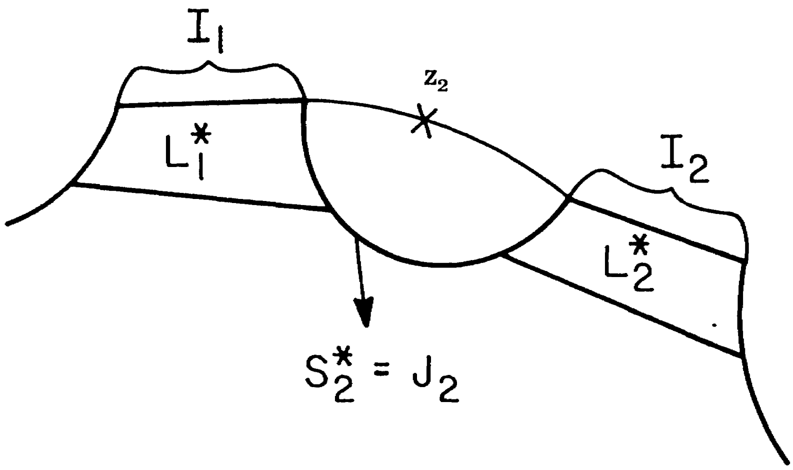



Figure 2.

Let $p_i: SO(3)/Z_2 + Z_2 \to SO(3)/N_i$, $i = 0, 1, 2$ be the standard projection and let $M_{p_i}$ be the mapping cylinder of $P_i$. $M_{p_i}$ is a smooth $SO(3)$-manifold because $N_i/(Z_2 + Z_2) \to SO(3)/Z_2 + Z_2) \to SO(3)/N_i$ is an $S^1$-bundle. Now let

$$M' = (SO(3)/(Z_2 + Z_2)) \times X.$$

$SO(3)$ acts on $M'$ by $\alpha([\beta],x) = ([\alpha\beta],x)$ for $\alpha \in SO(3)$, $[\beta] \in SO(3)/(Z_2 + Z_2)$ and $x \in X$. We have $(M')^* = X$. Each $(SO(3)/(Z_2 + Z_2)) \times I_i$ is an invariant submanifold of $M'$. Let $L_i = M_{pj(i)} \times I_i$. The "bottom" part of $L_i$ is $(SO(3)/(Z_2 + Z_2)) \times I_i$. Obtain $M''$ by attaching $L_i$ to $M'$ along

$$(SO(3)/(Z_2 + Z_2) \times I_i$$

by the identity map for $i = 1,2,\ldots,n$.

The induced action of $SO(3)$ on $M''$ has two orbit types, $SO(3)/(Z_2 + Z_2)$ and $SO(3)/O'(2)$.

$(M'')^*$ is shown below.

$M''$ has $n$ boundary components $S_1, S_2, \ldots, S_n$, such that

$$S_i^* = J_i \cup M^*_{pj(i-1)} \cup M^*_{pj(i)}.$$

Recall we have taken $j_i \neq j_{i+1}$. Richardson (12) shows each $S_i$ is equivalent to $S^4$ with $SO(3)$ acting on it by $(S^4,\Lambda)$, where $\Lambda: SO(3) \to SO(5)$ is the irreducible

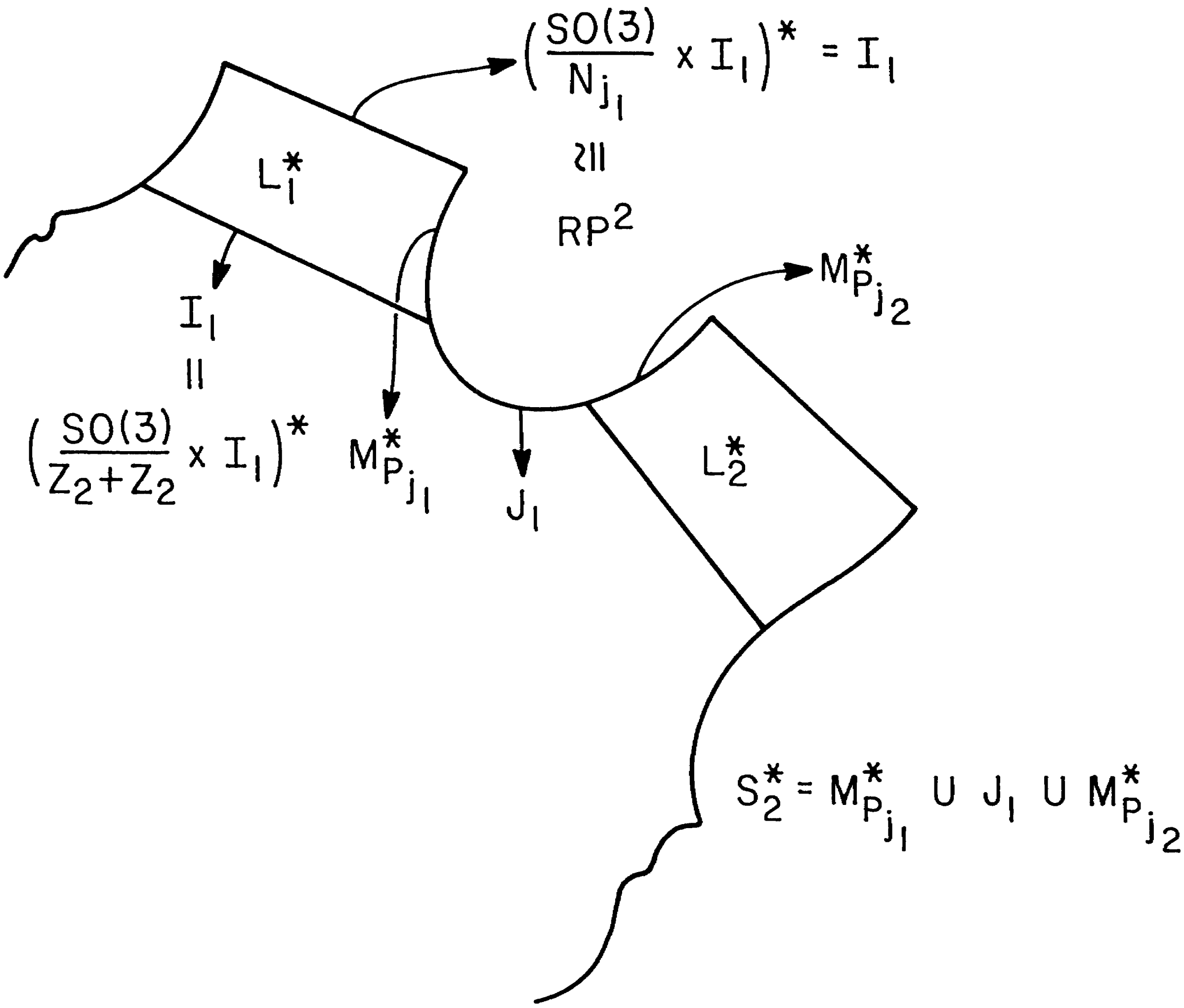



Figure 3.

representation of $SO(3)$ in dimension *5* (I.4E). Hence we can equivariantly fill in each $S_i$, in other words, we attach n 5-balls $D_i$ along $S_i$'s. Denote the center of $D_i$ by $z_i$. Then the resulting $SO(3)$-manifold has $n$ fixed points $z_1, \ldots, z_n$. We call this manifold $M^{j_1 \cdots j_n}$. Locally the orbit space, $(M^{j_1 \cdots j_n})^*$ looks like the figure 2.

4C. PROPOSITION. $\pi_1(M^{j_1 \cdots j_n}) - \{e\}$ *for* $n \geq 2$.

*Proof.* From the construction we have that

$$M^{j_1 \cdots j_n} = M'' \cup (D_1 \cup D_2 \cup \ldots \cup D_n),$$

where $D_i$ is diffeomorphic to a five-ball and

$$\begin{aligned} M'' &= M' \cup L_1 \cup \ldots \cup L_n; \\ M' &= (SO(3)/(Z_2 + Z_2)) \times X \ (X \text{ as in Figure 1}); \\ L_i &= M_{p_{j(i)}} \times I_i, \end{aligned}$$

with $M_{p_{j(i)}}$ the mapping cylinder of $p_{j_i}: SO(3)/(Z_2 + Z_2) \to SO(3) \times N_{j_i}$, and $M' \cap L_i = (SO(3)/(Z_2 + Z_2)) \times I_i$. Applying the Van Kampen Theorem repeatedly, we see that $\pi_1(M'') = \{e\}$; hence $\pi_1(M^{j_i \cdots j_n}) = \{e\}$.

## III. EQUIVARIANT CLASSIFICATIONS

In this section we classify $\mathscr{D}^2_{SO(3)}$ (See I.3, for definition). The main tool for this is the classification theorem due to Bredon, [1], Hsiang and Hsiang [6], and Jänich [8]. This will include the classification of simply connected five-manifolds whose orbit spaces are diffeomorphic to $D^2$.

1A. LEMMA. $(M, \phi) \in \mathscr{D}^2_{SO(3)}$ *has one of the following set of orbit types:* (a) $\{SO(3), SO(3)/SO(2)\}$, (b) $\{SO(3)/Z_k, SO(3)/SO(2)\}$, (c) $\{SO(3)/D_k, SO(3)/O'(2)\}$, (d) $\{SO(3)/Z_2, SO(3)/O'(2)\}$, (e) $\{SO(3)/Z_2 + Z_2), SO(3)/O'(2), SO(3)/SO(3)\}$.

*Proof.* This follows as the result of a case-by-case enumeration of representations of subgroups of $SO(3)$. We start with the possibilities for the principal orbit type. Since principal orbits are three dimensional, their isotropy groups must be conjugate to a discrete group. So if $(H)$ is the principal orbit type, $H$ is conjugate to $\{e\}$, $Z_k$, $D_k$, $T$, $O$ or $I$. Let $x \in M$ be on a singular orbit (we know by (I.3) singular orbits exist). $G_x$ must be conjugate to $SO(2)$, $O'(2)$, or $SO(3)$ (note $\dim(G/G_x) < \dim(G/H) = 3$, so $\dim(G/G_x) = 0$, 1, or 2; $\dim G_x = 3, 2$, or 1). Say $(G_x) = (K)$, where $(K) = (SO(2))$, $(O'(2))$ or $SO(3)$. By the slice representation theorem (I.1), there is a small normal disk (slice) $S_x$ to $G(x)$ at $x$. Now $S_x$ is $D^{\ell}$, $\ell = 5\text{-}\dim(SO(3)/K)$ ($\ell = 3$ or 5); and $\phi \mid G_x$ on $S_x$ is equivalent to a representation $\psi_x: G_x \to O(\ell)$, and a principal, exceptional or singular orbit $G_x(y)$, $y \in S_x$, for $\psi_x$ corresponds to a principal, exceptional or singular orbit $G(y)$ for $\phi$. Thus since $\phi$ has no exceptional orbits, neither does $\psi_x$. Suppose $H = \{e\}$. Let $x \in M$ be on a singular orbit. We must seek $K \subset SO(3)$ so that $G_x \in (K)$ and $\psi_x: G_x \to O(\ell)$, $\ell = 3$ or 5, has $\{e\}$ as the principal isotropy group. From (I.4E) we see

$$\rho_1 + 1 : SO(2) \to SO(3)$$

is the only possibility. Therefore $G_x \in (SO(2))$ and we have the case (a). Suppose $H = Z_k$, $k \geq 3$. For the same reasoning as for $H = \{e\}$, we must have $G_x \in (SO(2))$ with $\rho_k + 1: SO(2) \to SO(3)$; so we have case (b). Similarly if $H = D_k$, $k \geq 3$, we are in case (c). Suppose $H = Z_2$. Then $(G_x, \psi_x)$ can be

$$\rho_2 + 1 : SO(2) \to SO(3)$$

or $\bar{\rho}_1 + 1$: $O'(2) \to O(3)$ $(Z_1 = O'(1))$. But since $\partial M^* = B^*$ (the image of the set of singular orbits $B$) is connected we cannot have both $(SO(2))$ and $(O'(2))$ as the singular orbit types. Hence we must either have the case (b) with $k = 2$ or the case (d). Suppose $H = D_2 = Z_2 + Z_2$. Then $\psi_x$ can be either

$$\bar{\rho}_2 + 1 : O'(2) \to O(3),$$

or $\Lambda: SO(3) \to SO(5)$. If for all $x \in B$, $G_x \in (O'(2))$, we have the case (c) with $k = 2$. If for some $x \in B$, $(G_x, \psi_x)$ is $(SO(3), \Lambda)$, then there exists $y \in S_x$ so that $(G_y, \psi_y)$ is $(O'(2), \bar{\rho}_2 + 1)$. Thus we are in the case (e). $H = T$, $O$, or $I$ cannot happen since $SO(2)$ or $O'(2)$ has no finite subgroups conjugate to $H$ and $SO(3)$ has no representation in dimension 5 with the principal isotropy group for such an $H$.

1B. *Remark.* In the above setting when $K = SO(2)$ or $O'(2)$, the representation of $K(\rho_k + 1$ etc.) on $R^3$ is via a representation into $O(2) \subset O(3)$ and is transitive on the unit sphere $S^1$ in the orthogonal complement $R^2 \times \{0\}$ to the fixed point set $F(K, R^3) = \{0\} \times R$. This point is crucial in our classification. It enables us to use the classification theorem V.6.2 [3, pp. 326–333] for the differentiable case. Also this fact makes the orthogonal extension in the proof of Theorem 4.7 [5, p. 762] possible, although we have $k = 1$.

Now we consider the notion of simultaneous conjugacy classes of a pair of subgroups $H \subset K$ of a group $G$. We say $H \subset K$ and $H' \subset K'$ are simultaneously conjugate, $H \subset K \sim H' \subset K'$, if and only if there is an element $g \in G$ such that $gHg^{-1} = H'$ and $gKg^{-1} = K'$. We write $(H \subset K)$ for the conjugacy class of $H \subset K$.

1C. *Arithmetic of* $SO(3)$ Recall the definitions in terms of generators and relations (I.4B), $D_2 = Z_2 + Z_2 : P^2 = Q^2 = 1$, $PQ = QP$ $T: P^2 = Q^2 = 1$, $PQ = QP$, $APA^{-1} = Q$, $AQA^{-1} = PQ$, and $A^3 = 1$. We may pick

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$PQ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with $H_0 = \langle Q \rangle$, $H_1 = \langle P \rangle$, $H_2 = \langle PQ \rangle$ ($\langle P \rangle$ equals the cyclic group generated by

$P$, etc.). Let $T_0$, $T_1$, $T_2$ be the maximal tori of $SO(3)$ containing $H_0$, $H_1$, $H_2$, respectively, and let $N_i = N(T_i)$, $i = 0, 1, 2$. Notice that earlier in II4A we made the convention that $O'(1) = H_1$, $SO(2) = T_0$, and $O'(2) = N_0$. We will use the notations interchangeably. We observe

$$AT_iA^{-1} = T_{i+2(\text{mod } 3)}, \qquad AN_iA^{-1} = N_{i+2(\text{mod } 3)}$$
$$A^{-1}T_iA = T_{i+1(\text{mod } 3)}, \qquad A^{-1}N_iA = N_{i+1(\text{mod } 3)}$$

Now using these matrices, it is easy to check

$$F(H_1, SO(3)/N_0) = \{A^2N_0\} \cup N(H_1)/N_0.$$

These correspond to the simultaneous conjugacy classes for $(H) = (O'(1))$ and $(K) = (O'(2))$; *i.e.*, there are two simultaneous conjugacy classes, $(H_1 \subset N_0)$ and $(H_1 \subset N_1)$. Clearly if $H \subset K \in (H_1 \subset N_1)$, $K/H$ cannot be isomorphic to $S^1$.

1D. LEMMA. *Let $H \subset K$ be subgroups appearing in* (a)–(d) *in Lemma* 1A. *Let $H' \in (H)$, $K' \in (K)$, with $H' \subset K'$ and $K'/H' \approx S^1$. Then $H' \subset K' \sim H \subset K$.*

The proof is immediate.

1E. THEOREM. *Equivalence classes of $(M,\phi) \in \mathscr{D}^2_{SO(3)}$ with two orbit types are classified by the invariants $\{H, K; b\}$ where $(H,K)$ ranges over the sets $(\{e\}, SO(2))$, $(Z_k, SO(2))$, $(D_k, O'(2))$, $(O'(1), O'(2))$, and $b = 0$ if $(H,K) = (Z_k, SO(2))$ or $(D_k, O'(2))$, and $b \in Z_2$ if $(H,K) = (\{e\}, SO(2))$, and $b \in Z_+$ if $(H,K) = (O'(1), O'(2))$.*

This theorem says that for each $k$, $k \geq 2$ there is a unique $SO(3)$-manifold $M(\{Z_k, SO(2)\})$, whose principal orbit type is $SO(3)/Z_k$ (the singular orbit type is automatically $SO(2)$). Similarly, $(M,\phi) \in \mathscr{D}^2_{SO(3)}$ whose principal orbit type is $SO(3)/D_k$ is unique for each $k$, $k \geq 2$. There are exactly two non-equivalent $SO(3)$-manifolds which have $SO(3)$ and $SO(3)/SO(2)$ as their orbit types. We will call them $M(\{\{e\}, SO(2); 0\})$ and $M(\{\{e\}, SO(2); 1\})$.

*Proof.* Given $(M,\phi) \in \mathscr{D}^2_{SO(3)}$, let $SO(3)/H$ and $SO(3)/K$ be the principal and singular orbit types respectively. By Lemma 1A, $(SO(3)/H, SO(3)/K)$ must be one of (a)–(d). Let $\pi: M \to M^* = D^2$ be the orbit map. Denote by $B$ the boundary circle of $D^2$. Then $\pi^{-1}(B) = M_{(K)}$ and $\pi^{-1}(D^2 - B) = M_{(H)}$. Write

$$S = N(H) \cap N(K)/H.$$

The Tube Theorem [3, p. 242] and the remark (1B) imply that for some small $\varepsilon > 0$, $\pi^{-1}([0,\varepsilon] \times S^1)$ is an invariant tubular neighborhood of $M_{(K)}$ which is equivalently diffeomorphic to $M_p \times_S Q$, where $p: G/H \to G/K$ is some equivariant homomorphism, and $Q$ is a principal $S$-bundle, $S \to Q \to S^1$. By Lemma 1D, $p$ can be chosen to be the standard projection. By V.4. (3), we see that $(M,\phi)$ is a proper $SO(3)$-manifold satisfying the hypothesis of V.6.2 [3, p. 257]. By the remark (1B), of this section, the classification is valid for the smooth case. Hence equivalence classes of $(M,\phi) \in \mathscr{D}^2_{SO(3)}$ with orbit types $SO(3)/H$ and $SO(3)/K$, are the same as equivalence classes of *proper* $(M,\phi) \in \mathscr{D}^2_{SO(3)}$ with these orbit types. By V.6.2 (3) the latter set is in one-one correspondence with

$$\Gamma = \left[ S^1, \; N(H) \cap N(H) \backslash N(H) \right] \Big/ \pi_0 N(H)/H$$

where $N(H) \cap N(H) \backslash N(H)$ is the orbit space of the left $N(H) \cap N(K)$-action on $N(H)$. It is easy to check the following chart.

| $H$ | $K$ | $N(H) \cap N(K) \backslash N(H)$ | $\pi_0 N(H)/H$ | $\Gamma$ |
|---|---|---|---|---|
| $e$ | $SO(2)$ | $RP^2$ | 1 | $Z_2$ |
| $Z_k$ | $SO(2)$ | 1 | $Z_2$ | 1 |
| $k \geq 3, D_k$ | $O'(2)$ | 1 | $Z_2$ | 1 |
| $Z_2 + Z_2$ | $O'(2)$ | $Z_3$ | $D_3$ | 1 |
| $O'(1)$ | $O'(2)$ | $\dfrac{N_1}{N_1 \cap N_0} = S^1$ | $Z_2$ | $Z_+$ |

*Table 3.*

Hence if $(H, K) = (\{e\}, SO(2))$, then $(M, \phi)$ depends on $b \in Z_2$. If $(H, K) = (Z_k, SO(2))$ or $(D_k, O'(2))$ then $(M, \phi)$ is unique. And if $(H, K) = (O'(1), O'(2))$, then $(M, \phi)$ depends on $b \in Z_+$.

According to [3, p. 255], $b \in \Gamma$ corresponds to an $S$-reduction $Q$ of the principal $N(H)/H$-bundle over $S^1$ (which in our case $\approx N(H)/H \times S^1$). We have then

$$M_b = SO(3)/H \times D^2 \bigcup_{f_b} M_p \times_S Q,$$

where $f_b \colon SO(3)/H \times_S Q \to SO(3)/H \times_{N(H)/H} (N(H)/H \times S^1)$ is the induced equivariant map.

We also notice that with the notion of properness and the remark (1B), we may use Theorem 4.1 [6, p. 757] for our classification. The value $b = 0$ corresponds to

$$\begin{array}{ccccc} S & \longrightarrow & S \times S^1 & \longrightarrow & S^1 \\ \downarrow & & \downarrow b & & \| \\ N(H)/H & \longrightarrow & N(H)/H \times S^1 & \longrightarrow & S^1, \end{array}$$

where $b(x,t) = (x,t)$, $x \in S$, $t \in S^1$. That is, for each $H \subset K$, $b = 0$ corresponds to $M = SO(3)/H \times D^2 \bigcup_{f_0} M_p \times S^1$, where $f_0 \colon SO(3)/H \times S^1 \to SO(3)/H \times S^1$ is the identity. Recall the examples in II. We see that $M(1) = M(\{\{e\}, SO(2); 0\})$, $M(k) = M(\{Z_k, SO(2)\})$, and $M'(1,0) = M(\{O'(1), O'(2)\})$. Since in II we saw that

$(S^5, \phi_0)$ is not equivalent to $M(1)$, we conclude that it corresponds to

$$M(\{\{e\}, SO(2); 1\}).$$

Now look at $M_b = M(\{O'(1), O'(2); b\})$. We want to show that $M_b$ is equivalent to $M'(1,i)$ for some $i \in Z$. (II 3B). We shall use the notations $O'(1) = H_1$, $O'(2) = N_0$ as in (1C). Let

$$S = N(H_1) \cap N(N_0)/H_1 = N_1 \cap N_0/H_1 = Z_2.$$

Let $p: SO(3)/H_1 \to SO(3)/N_0$ be the projection. We show that $M_p \times_S Q \cong M_p \times S^1$.

First of all we have a trivial $SO(3)/N_0$-bundle over $S^1 = M^*_{N_0}$ since

$$N(N_0)/N_0 = 1.$$

Thus by (I.2B), $M_{(N_0)} = SO(3)/N_0 \times S^1$. We also have that $\partial(M_p \times_S Q)$ (being the boundary of the (trivial) $SO(3)/H$-bundle over $D^2$) is trivial; *i.e.,*

$$\partial(M_p \times_S Q) \cong SO(3)/H_1 \times S^1.$$

So the sphere bundle (1) $N_0/H_1 \to \partial(M_p \times_S Q) \to SO(3)/N_0 \times S^1$ with the structural group $S$, is the same as the product of the sphere bundle (2) $N_0/H_1 \to SO(3)/H_1 \to SO(3)/N_0$ with $S^1$. Since $M_p \times_Q S$ is the associated disk bundle of (1), obviously it must be the product of the associated disk bundle $M_p$ of (2) with $S^1$. So $M_p \times_S Q \simeq M_p \times S^1$. Thus $M_b$ is diffeomorphic to

$$SO(3)/H_1 \times D^2 \bigcup_{f_b} M_p \times S^1,$$

where

$$f_b: SO(3)/H_1 \times_S (S \times S^1) \to SO(3)/H_1 \times_{N(H)/H} (N(H)/H \times S^1)$$

is the reduction corresponding to $b \in Z_+$. Hence $M_b$ is equivalent to some $M'_{1,j}$ constructed in (II.3B) and thus $\pi_1(M_b) \neq e$.

Together with the remark (II 3C), we have proved

1F. THEOREM. *If $(M,\phi) \in \mathscr{D}^2_{SO(3)}$ has two orbit types and $\pi_1(M) = \{e\}$, then it must be equivalent to one of* $(S^5, \phi_0)$, or $M(k)$, $k = 1, 2, \ldots$, *given in* II. 3A.

2. *Classification of $\mathscr{D}^2_{SO(3)}$ with fixed points.*

2A. THEOREM. *If $(M,\phi) \in \mathscr{D}^2_{SO(3)}$ has three orbit types then they must be $SO(3)/(Z_2+Z_2)$, $SO(3)/O'(2)$, and $SO(3)/SO(3)$. It must admit at least two fixed points. The orbit invariant is the equivalence class $[j_1, j_2, \ldots, j_n]$ by cyclic permutation, reflexion, and the diagonal action of $\sigma_3$ (the symmetric group on three letters), of an n-tuple $(j_1, j_2, \ldots, j_n)$, where $j_i \in \{0,1,2\}$ subject to $j_i \neq j_{i+1}$, and $j_1 \neq j_n$. The $SO(3)$-manifold corresponding to $[j_1, \ldots, j_n]$ is equivalent to $M^{j_1 \cdots j_n}$ of the examples* II.4B.

*Remark.* Notice if there are only two distinct numbers appearing in $(j_1, \dots, j_n)$, it is in $[0,1,0,1,\dots,0,1]$. This is because dividing out by the diagonal action of $\sigma_3$, we can start our sequence with 0, 1; *i.e.,* our sequence can be taken as $[0,1,j_3,\dots,j_n]$. Given $n$, there are $k = 2^{n-2} - 2^{n-3} + 2^{n-4} \dots \pm 1$ many $(0,1,j_3,\dots,j_n)$'s satisfying $j_i \neq j_{i+1}$ and $j_n \neq 0$. So there are at most $k$ non-equivalent $SO(3)$-actions with $n$ fixed points. The number becomes reduced by dividing out by cyclic permutations and reflexion. For example when $n = 2$ or 3, $k = 1$, and when $n = 5$, $k = 5$ but the number of distinct classes is 2. In general computing the number of distinct actions for a given $n$ is possible but difficult.

2B. COROLLARY. *The $SO(3)$-manifold with 2 (respectively 3) fixed points is unique. Any manifold corresponding to* $\underbrace{[j_1,j_2,j_1,j_2,\dots,j_1,j_2]}_{\text{length } n}$ *is equivalent to* $M^{\underbrace{0101..01}_{\text{length}}}$. *We know that $(S^5,\Lambda)$ has two fixed points and the Wu-manifold $X_{-1} = SU(3)/SO(3)$ has three fixed points. Hence, $M^{01}$ must be equivalent to $(S^5,\Lambda)$ and $M^{012}$ to $SU(3)/SO(3)$.*

*Proof of* 2A. When $(M,\phi) \in \mathscr{D}^2_{SO(3)}$ has three orbit types, by Lemma (III.1A) they must be $\{SO(3)/(Z_2+Z_2), SO(3)/O'(2), SO(3)/SO(3)\}$, and the representation of $SO(3)$ at a fixed point $z \in M$ is given by $\Lambda\colon SO(3) \to SO(5)$, as in (I.4E). Thus $F = F(SO(3),M)$ is discrete and the compactness of $M$ implies that $F$ has only finitely many elements, say $F = \{z_1,\dots,z_n\}$. Notice by (I.3B) that $z_i^* = \{z_i\} \in \partial M^* = B$. For each $i$, choose a small 5-ball $D_i$ about $z_i$ such that $\phi|D_i$ is equivalent to $\Lambda$. Each $\partial D_i = S_i$ is $SO(3)$-invariant and $\phi|S_i$ is equivalent to $(H^4,\Lambda)$. The orbit space $M^* = D^2$ has the structure shown in Figure 4.

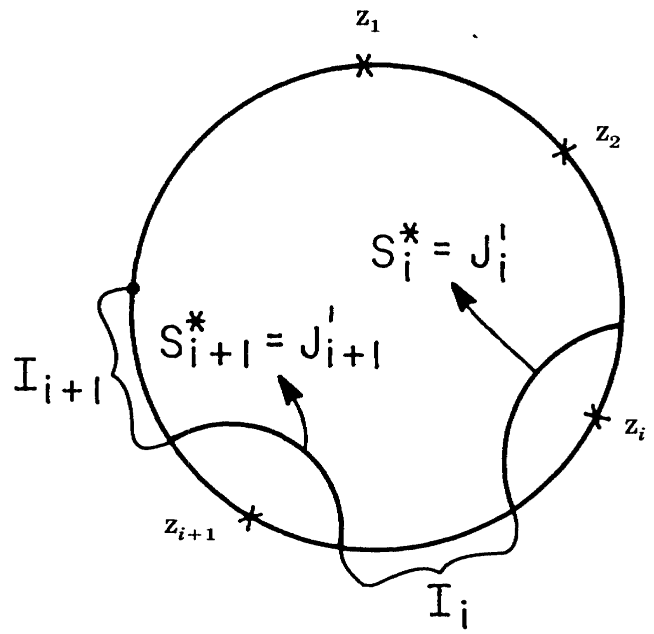



Figure 4.

In $M' = M - \overline{\bigcup_{i=1}^{n} D_i}$, the action has two orbit types $SO(3)/(Z_2 + Z_2)$ and $SO(3)/O'(2)$. $M'^* = \bigcup_{i=1}^{n} \overline{D^2 - D_i^*}$. $\partial M'^*$ consists of $(M'_{(O'(2))})^*$ and $\bigcup_{i=1}^{n} S_i^*$. Write $S_i^* = J_i'$ and $I_i$ = connected component of $(M'_{(O'(2))})^*$ as shown in Figure 5. We then have $G_x \in (Z_2 + Z_2)$ over $M'^* - \bigcup_{i=1}^{n} I_i$, and $G_x \in (O'(2))$ over $I_i$. The restriction of the action to $S_1 \cap M' = S_i$ is equivalent to $(S^4, \Lambda)$. More precisely $(S_i, \Lambda | S_i, J_i')$ is equivalent to $(S^4, \Lambda, I)$. Hence according to (12), $S_i$ is equivalent to $M_{pj(i-1)} \cup (SO(3)/(Z_2 + Z_2)) \times J_i \cup M_{pj_i}$, where $M_{pj_i}$ is the mapping cylinder $P_{j_i}: SO(3)/Z_2 + Z_2 \to SO(3)/N_{j_i}$ as defined in (II.1.4B). That is, we have the following diagram:

$$\begin{array}{ccc} S_i & \longrightarrow & S_i^* = J_i' \\ \approx\downarrow & & \approx\downarrow \\ M_{pj(i-1)} \cup (SO(3)/Z_2 + Z_2) \times J_i \cup M_{pj_i} & \to & [0,1] \cup J_i \cup [2,3] \end{array}$$

As pointed out in (II.4) we must have $j_i \neq j_{i+1}$. Extending $(S^4, \Lambda)$ to $D^5$, we recover $(M, \phi)$ uniquely from $(M', \phi | M', M'^*)$ [3, p. 282]. Therefore, the classification

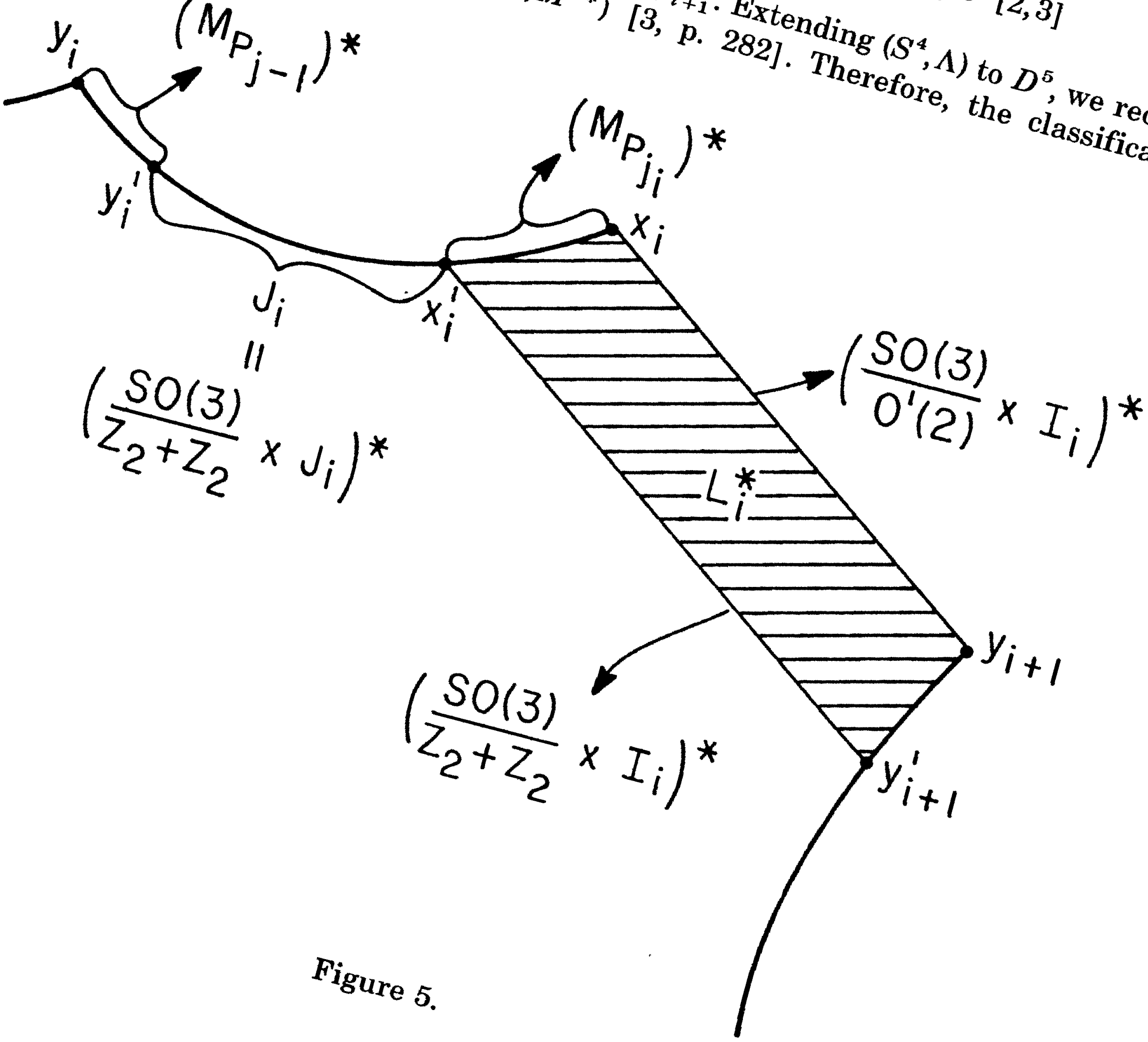



Figure 5.

of $(M,\phi) \in \mathscr{D}^2_{SO(3)}$ with three orbit types (and hence with $n$ fixed points) amounts to the classification of $(N,\psi,X)$ with $X$ as described in example (II.4), satisfying the following conditions:

(a) The principal orbit type is $SO(3)/(Z_2 + Z_2)$.

(b) The singular orbit type is $SO(3)/O'(2)$. The image of the singular orbits in $X$ is the union $B = I_1 \cup \ldots \cup I_n$.

(c) $\partial N - \bigcup_{i=1}^{n} \pi^{-1}(I_i) = \pi^{-1}(J'_1, \ldots, J'_n)$ is a disjoint union of $S^4$'s, $S_1, \ldots, S_n$, where each $(S_i, \psi \mid S_i, J'_i)$ is equivalent to $M_{pj_{(i-1)}} \cup ((SO(3)/Z_2 + Z_2) \times J_i) \cup M_{pj_i}$.

2B. LEMMA. *The triple* $(N,\psi,X)$ *satisfying* (a), (b) *and* (c) *corresponds to the equivalence class* $[j_1, \ldots, j_n]$.

*Proof.* It is easy to check that $(N,\psi,X)$ satisfies the hypothesis of (V.6.2(3)); hence it corresponds to an element $b$ in

$$\Gamma = \left(\left[B, \frac{N(H)}{\cap\, N(K)} \Big\backslash N(H)\right] \Big/ \pi_0(N(H)/H)\right) \Big/ \sim.$$

where $K = O'(2)$, $H = Z_2 + Z_2$, $B = I_1 \cup \ldots \cup I_n$, and $\sim$ is the equivalence given by the action on $B$ of cyclic permutations and the reflection of $\{I_1, \ldots, I_n\}$. Now using the presentation for $N(Z_2 + Z_2) = \mathrm{O} = \{P,Q,R,A\}$ as in (I.4B) we see that

$$\frac{N(H)}{\cap\, N(K)} \Big\backslash N(H) \cong Z_3$$

is generated by $A/O = \{P,Q,R\}$, and $\pi_0 N(H)/H = D_3$ is generated by $A/\{P,Q\}$ and $R/\{P,Q\}$. $b \in \Gamma$ corresponds to a choice of the projection

$$p_{j_i}\colon SO(3)/(Z_2 + Z_2) \to SO(3)/N_{j_i}, j_i \in \{0,1,2\}, i = 1, \ldots, n.$$

$N$ is equivalent to

$$(SO(3)/(Z_2 + Z_2) \times X) \cup \bigcup_{i=1}^{n} (M_{pj_i} \times I_i).$$

But the condition (c) forces us to choose $j_i$'s so that $j_i \neq j_{i+1}$, and $j_1 \neq j_n$. Hence we have the equivalence class $[j_1, \ldots, j_n]$ as claimed. This completes the proof of the lemma.

Lemma 2B and the paragraph before that prove the first part of the theorem. It is also clear that $[j_1, \ldots, j_n]$ corresponds to $M^{j_1 \cdots j_n}$ of II. 4B.

It remains to show that $(M,\phi)$ with a single fixed point does not exist. Suppose that it did. Let $z \in M$ be the fixed point. Remove a small disk $D^5$ centered at $z$ from $M$. $N = M - D^5$ has the boundary equivalent to $(S^4, \Lambda)$. The orbit space $N^*$ is a half disk as indicated in Figure 6.

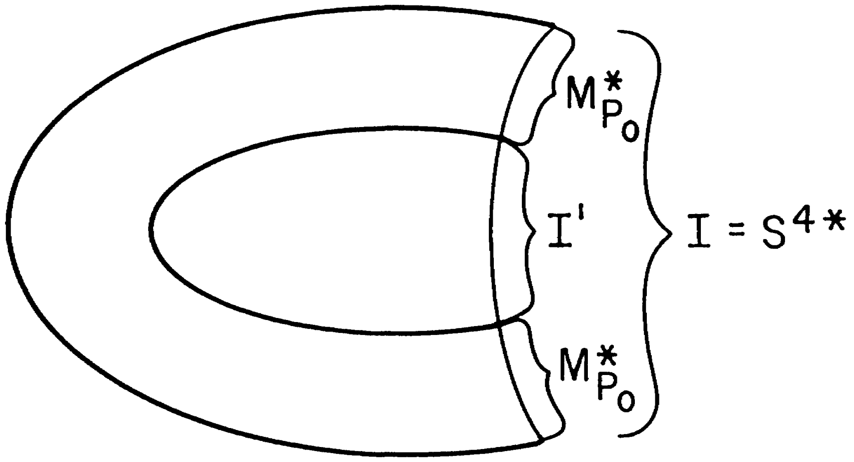



Figure 6.

By Lemma 2B, $N = \{SO(3)/(Z_2 + Z_2) \times X\} \cup \{M_{p_0} \times I\}$. Hence

$$\pi^{-1}(I) = M_{p_0} \cup \{SO(3)/(Z_2 + Z_2) \times I'\} \cup M_{p_0},$$

but this is not $S^4$. This finishes the proof of the theorem.

## IV. TOPOLOGICAL CLASSIFICATIONS

In this section we will identify the manifolds $M(k)$, $k = 1, 2, \ldots$, and $M^{j_1 \cdots j_n}$, $n \geq 2$. We already know that $M(\{\{e\}, SO(2); 1\})$ is equivalent to $(S^5, \phi_0)$, $M^{01}$ is diffeomorphic to $S^5$, and $M^{012}$ is $SU(3)/SO(3)$.

1A. THEOREM. *The manifolds $M(k)$ are diffeomorphic to $S^2 \times S^3$ for $k = 1, 2, \ldots$.*

*Proof.* From the construction,

$$M(1) = SO(3) \times D^2 \bigcup_{id} M_p \times S^1$$

where $p: SO(3) \to SO(3)/SO(2)$,

$$M_p = SO(3) \times [0,1] \bigcup_p SO(3)/SO(2).$$

Write $N_1 = SO(3) \times D^2$, $N_2 = M_p \times S^1$, $N_1 \cap N_2 = SO(3) \times S^1$. Then we have $H_*(N_1) \approx H_*(SO(3))$ and $H_i(N_2) = \bigoplus_{j=0} (H_j(M_p) \otimes H_{i-j}(S^1))$.

$$H_*(M_p) \approx H_*(SO(3)/SO(2)) \approx H_*(S^2).$$

From this we obtain Mayer-Vietoris sequence for $M = N_1 \cup N_2$, which gives $H_2(M; Z) = Z$.

To show $w_2(M) = 0$, we first notice that $w_2(N_1) = 0$ and $w_2(M_p \times S^1) = 0$. $w_2(N_1) = 0$ is obvious as $w_2(SO(3)) = 0$. $w_2(M_p \times S^1) = 0$ is also clear if we can show $w_2(M_p) = 0$. Notice first that we have $\nu: D^2 \to M_p \to S^2$ associated with the tangent sphere bundle $p\colon SO(3) \to SO(3)/SO(2)$ and its second Stiefel-Witney class is trivial ($S^2$ is almost parallelizable). Thus $w_2(M_p) = w_2(S^2) + w_2(\nu) = 0$. Now look at the Mayer-Vietoris cohomology sequence in $Z_2$.

$$\begin{array}{ccccccccc}
H^1(M) & \to & H^1(N_1) \oplus H^1(N_2) & \to & H^1(N_1 \cap N_2) & \overset{\beta}{\to} & H^2(M) & \overset{\alpha}{\to} & H^2(N_1) \oplus H^2(N_2) \\
\| & & \| & & \| & & & & \\
0 & \to & Z_2 + Z_2 & \to & Z_2 + Z_2 & \to & Z_2 & \to & Z_2 + Z_2 \\
 & & & & & & w_2 & & w_2^{(1)} \quad w_2^{(2)}
\end{array}$$

We see that $\beta$ is the zero map and so $\alpha$ is injective. If $w_2$, $w_2^{(1)}$, $w_2^{(2)}$ are $w_2(M)$, $w_2(N_1)$, $w_2(N_2)$ respectively we must have $\alpha(w_2) = (w_2^{(1)}, w_2^{(2)})$ ($w_2$ is natural with respect to cohomology maps) $= (0,0)$.

Therefore $w_2(M) = 0$. By Theorem A [13] we have that $M$ is diffeomorphic to $S^2 \times S^3$.

With exactly the same argument, we show that the manifold $M(k)$ is diffeomorphic to $S^2 \times S^3$, $k \geq 2$.

1B. THEOREM. $M^{j_1 \cdots j_{2k+1}} = X_{-1} \# \underbrace{M_2 \# \dots \# M_2}_{k-1}.$

$$M^{j_1 \cdots j_{2k}} = X_1 \# \underbrace{M_2 \# \dots \# M_2}_{k-2}$$

if all 0, 1, 2 appear among the $j_i$'s. $M^{\overbrace{ab \dots ab}^{2k}} = \underbrace{M_2 \# \dots \# M_2}_{k-1}$. Here $X_{-1}$, $X_1$, and $M_2$ are given in (1).

1C. LEMMA. $H_2(M^{j_1 \cdots j_n}; Z) = \underbrace{Z_2 + \dots + Z_2}_{n-2}.$

*Proof.* Writing $L = \bigcup_{i=1}^{n} L_i$, $M'_n \cap L = \bigcup_{i=1}^{n} (SO(3)/Z_2 + Z_2) \times I_i)$, we obtain the Mayer-Vietoris sequence

(1)

$$\begin{array}{ccccccccc}
H_2(M'_n) \oplus H_2(L) & \to & H_2(M''_n) & \underset{\varphi}{\to} & H_1(M'_n \cap L) & \underset{\psi}{\to} & H_1(M'_n) \oplus H_1(L) & \to & 0 \\
\| \qquad\quad \| & & & & \| & & \| \qquad\quad \| & & \\
0 \qquad\quad 0 & & & & n(Z_2 + Z_2) & & Z_2 + Z_2 \qquad nZ_2 & &
\end{array}$$

$$0 \to H_2(M''_n) \to \underbrace{Z_2 + \dots + Z_2}_{2n} \to \underbrace{Z_2 + \dots + Z_2}_{n+2} \to 0$$

Hence $H_2(M''_n) = \underbrace{Z_2 + \ldots + Z_2}_{n-2}$. See II 4B for the notations $M'$, $M''$. Hence

$$H_2(M_n) = \underbrace{Z_2 + \ldots + Z_2}_{n-2}$$

(Adding $D^5$'s to $M''_n$ does not change $H_2$). Here $n$ denotes the number of fixed points.

1D. PROPOSITION. $M^{j_1 \cdots j_{2K+1}} = X_{-1} \# \underbrace{M_2 \# \ldots \# M_2}_{k-1}$.

*Proof.* $H_2(M_{2k+1};Z) = \underbrace{Z_2 + \ldots + Z_2,}_{2k-1}$ hence by a theorem of Wall [14] $w_2(M_{2k+1}) \neq 0$. Hence by (I.5) the proposition follows.

1E. LEMMA. $w_2(M^{ab \ldots ab}) = 0$, *for any* $a \neq b \in \{0,1,2\}$. $w_2(M^{j_1 \cdots j_n}) \neq 0$ *if all* 0, 1, 2 *appear.*

To compute $w_2(M^{j_1 \cdots j_n})$ note that $w_2(M^{j_1 \cdots j_n}) = w_2((M^{j_1 \cdots j_n})'')$.

Now $w_2$ is a homomorphism: $H_2(M'';Z) \to Z_2$. Hence $w_2(M'') \neq 0$ if there is a basis element $\alpha \in H_2(M'',Z)$ with $w_2(\alpha) \neq 1$.

From the exact sequence (1) we see that a 2-cycle realizing a basis element of $H_2(M''_n)$ is in ker $\psi$. Write

$$H_1(M') = H_1(SO(3)/Z_2 + Z_2) = Z_2 + Z_2 = \{u_0, u_1, u_0 + u_1, 0\}$$

so that the following holds:

$$(p_i)_*(u_j) = \begin{cases} 0 & \text{if } i = j \qquad i = 0, 1, 2 \\ 1 & \text{if } i \neq j \qquad j = 0, 1 \end{cases}$$

where

$$\begin{array}{ccc} (p_i)_*: H_1(SO(3)/Z_2 + Z_2) & \to & H_1(SO(3)/N_i) \\ \| & & \| \\ Z_2 + Z_2 & & Z_2 \end{array}$$

Now choose a basis for

$$\oplus H_2((SO(3)/Z_2 + Z_2) \times I_i) = \bigoplus_{i=1}^{n} (Z_2 + Z_2)$$

as follows: for each $Z_2 + Z_2$ in the $i$th component, we choose a basis $(a_i, b_i)$ so $(P_{j_i *})(a_i) = 0$ and $(P_{j_i})_*(b_i) = 1$, *eg.*,

$j_i = 0$ then $a_i = u_0$, $\qquad b_i = u_i$ (or $u_0 + u_1$)

$j_i = 1$ then $a_i = u_1$, $\qquad$ etc.

$j_i = 2$ then $a_i = u_0 + u_1$

Look at

$$
\bigoplus_{i=1}^{n} H_1\Big((SO(3)/Z_2+Z_2)\times I_i\Big)
\begin{array}{l}
\xrightarrow{\ \psi_1\ } H_1(SO(3)/Z_2+Z_2\times X) \\[2ex]
\xrightarrow{\ \psi_2\ } \displaystyle\bigoplus_{i=1}^{n} H_1(M_{p_{j(i)}}\times I_i)\oplus H_1(SO(3)/N_1)
\end{array}
$$

$\psi=(\psi_1,-\psi_2)$, $\psi_1=i_1+\dots+i_n$, where

$$i_j\colon H_1(SO(3)/Z_2+Z_2\times I_1)\to H_1(SO(3)/Z_2+Z_2\times X)$$

is the inclusion homomorphism, and $\psi_2=((p_{j_i})_*,\dots,(p_{j_n})_*)$. Notice $u_0+u_0=0$, $u_1+u_1=0$, $u_0+u_1+(u_0+u_1)=0$ in $H_1(SO(3)/Z_2+Z_2)$. Thus $a\in\ker\psi$ if and only if $a\in\ker\psi_1\cap\ker\psi_2$; so $\ker\psi$ consists of the following: Let $n=n_1+n_2+n_3$ where $n$ equals the number of $u_0$'s appearing in $(a_1,a_2,\dots,a_n)$. $n_2$ equals the number of $u_1$'s appearing in $(a_1,\dots,a_n)$, $n_3$ equals the number of $(u_0+u_1)$'s appearing in $(a_1\dots a_n)$. We have arranged the basis $(a_1,\dots,a_n,b_1,\dots,b_n)$ so that $b_i\notin\ker\psi$. Ker $\psi$ is generated by

$$
\left.\begin{array}{l}
(u_0,u_0,\underbrace{0,\dots,0}_{2n-2}) \\[3ex]
(u_0,0,u_0,\underbrace{0,\dots,0}_{2n-3}) \\[3ex]
\quad\vdots \\[3ex]
(u_0,\underbrace{0,\dots,0}_{n_1-2},u_0,\underbrace{0,\dots,0}_{2n-n_1})
\end{array}\right\}\ n_1-1\ \text{elements}
$$

$$
\left.\begin{array}{l}
(\underbrace{0,\dots,0}_{n_1},u_1,u_1,\underbrace{0,\dots,0}_{2n-n_1-2}) \\[3ex]
(\underbrace{0,\dots,0}_{n_1},u_1,0,u_1,\underbrace{0,\dots,0}_{2n-n_1-3}) \\[3ex]
\quad\vdots \\[3ex]
(\underbrace{0,\dots,0}_{n_1},u_1,\underbrace{0,\dots,0}_{n_2-2},u_1,\underbrace{0,\dots,0}_{2n-n_1-n_2})
\end{array}\right\}\ n_2-1\ \text{elements}
$$

$$\left.\begin{array}{l}
(\underbrace{0,\ldots,0}_{n_1+n_2},\, u_0+u_1,\, u_0+u_1,\, \underbrace{0,\ldots,0}_{2n-n_1-n_2-2}) \\[2ex]
(\underbrace{0,\ldots,0}_{n_1+n_2},\, u_0+u_1,\, 0,\, u_0+u_1,\, \underbrace{0,\ldots,0}_{2n-n_1-n_2-3}) \\
\quad\vdots \\[2ex]
(\underbrace{0,\ldots,0}_{n_1+n_2},\, u_0+u_1,\, \underbrace{0,\ldots,0}_{n_3-2},\, u_0+u_1,\, \underbrace{0,\ldots,0}_{2n-n_1-n_2-n_3})
\end{array}\right\} n_3-1 \text{ elements}$$

$$(u_0,\, \underbrace{0,\ldots,0}_{n_1-1},\, u_1,\, \underbrace{0,\ldots,0}_{n_2-1},\, u_0+u_1,\, \underbrace{0,\ldots,0}_{n_3-1}) \qquad 1 \text{ element}$$

Note $(n_1 - 1) + (n_2 - 1) + (n_3 - 1) + 1 = n - 2$. If one of $n_i$'s is zero, say $n = n_1 + n_2$, we don't have the last element $(u_0, \ldots, u_1, \ldots, u_0 + u_1)$ and $(n_1 - 1) + (n_2 - 1) = n - 2$. Write the basis of $H_2(M;Z) = Z_2 + \ldots + Z_2$ as $\{\beta_1, \ldots, \beta_{n-3}, \alpha\}$ where $\beta_1 = \varphi^{-1}(u_0, u_0, 0, \ldots, 0)$ etc. and

$$\alpha = \varphi^{-1}(u_0, 0, \ldots, 0, u_1, 0, \ldots, 0, u_0 + u_1, 0, \ldots, 0),$$

or $\{\beta_1, \ldots, \beta_{n-2}\}$, $\beta_1 = \varphi^{-1}(u_i, u_i, 0, \ldots, 0)$ etc. for appropriate $i$, if $M = M_n^{ab\ldots ab}$

*Claim* (1) $w_2(\beta_i) = 0$, (2) $w_2(\alpha) \neq 0$.

Once this claim is proven, we will have the result that $w_2(M_n^{ab\ldots ab}) = 0$, and $w_2(M^{j_1 \cdots j_n}) \neq 0$ if all 0, 1, 2 appear among the $j_i$'s.

*Proof of the claim.* (1) $\beta_i$ is realized by a two-sphere $C_i$ in $M^5$, where $C_i = C_i^1 \cup C_i^2$, $C_i^1$ and $C_i^2$ are mirror images to each other and both are contractible; hence we can set up 3-frames identical on $C_i^1$ and $C_i^2$. They agree on the boundary

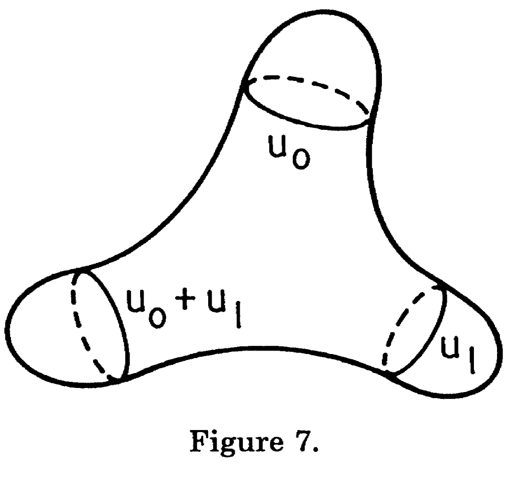



Figure 7.

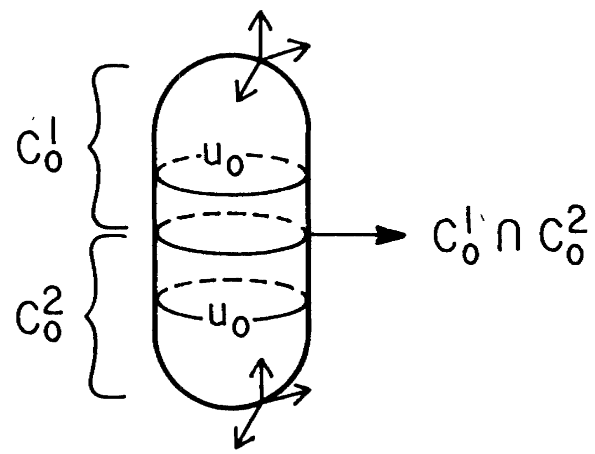



Figure 8.

$C_i^1 \cap C_i^2$ (modulo the orientations). Hence these frames agree on $C_i^1 \cap C_i^2$ (mod. 2). Hence we have established a 3-frame on $C_i$. So $w_2(\beta_i) = 0$.

(2) To show that $w_2(\alpha) \neq 0$, look at the Wu-manifold $M^{012}$. The second homology $H_2(M:Z)$ is isomorphic to $Z_2$ and the cycle $(u_0, u_1, u_0 + u_1, 0, 0, 0)$ is a generator of this group. Now by a theorem of Wall (14), we know that the second Stiefel-Whitney class $w_2(M)$ is nontrivial and hence $w_2(\alpha) \neq 0$. This means that we cannot set up a three-frame along any two-sphere realizing $\alpha$. Now from the construction it is clear that for any $n \geq 3$, $w_2$ does not vanish on

$$\alpha = \varphi^{-1}(u_0, 0, \ldots, u_1, 0, 0, \ldots, u_0 + u_1, 0, \ldots, 0, \ldots, 0);$$

*i.e.*, $w_2(\alpha) \neq 0$. This concludes the proof of Lemma 1E.

Now Theorem 1B follows from (I.5), 1D, and 1E.

## APPENDIX. THE CODIMENSION-THREE CASE

Let $M$ be a simply-connected, connected closed $SO(3)$-five-manifold of codimension three. Then by [3, p. 190] the orbit space $M^*$ is a simply-connected three dimensional manifold $\Delta^3$ (or $\Sigma^3$) with (or without) boundary. We will show that $M$ is diffeomorphic to $S^2 \times S^3$ or $S^5$.

The dimension of the principal orbit type $SO(3)/H$ being two, we see that $H$ is either $SO(2)$ or $N(SO(2))$. But the latter does not occur since

$$SO(3)/N(SO(2)) \approx RP^2$$

[3, p. 188]. Thus the principal orbit type is always $SO(3)/SO(2) \approx S^2$.

(1) Suppose all orbits are spheres. Then by (I.3) we know that the boundary $\partial M^*$ is empty. Hence $M^*$ is a homotopy sphere $\Sigma^3$. By (I.2B), we have a fibre bundle $SO(3)/SO(2) \to M \to \Sigma^3$, with the structural group $Z_2$. But such a bundle is trivial; hence we have that $M \approx S^2 \times \Sigma^3$. By Smale [13], $M$ is diffeomorphic to $S^2 \times S^3$. Thus if the Poincaré conjecture is true, $M$ is equivalent to $SO(3)/SO(2) \times S^3$ with the usual action.

(2) Next we notice that $SO(3)/N(SO(2))$ cannot occur as an exceptional orbit [3, p. 188]. So the only other possibility is that $M$ admits fixed points. Say $x$ is a fixed point. Choose a small five-ball about $x$ on which $SO(3)$ acts linearly with the principal orbit type $SO(3)/SO(2)$. There is exactly one such representation of $SO(3) \to SO(5)$, namely $\rho + 1$, where $\rho$ is the matrix representation. By (I.3) the boundary $\partial M^*$ is nonempty and corresponds to the fixed points. The manifold $M^*$ is therefore a homotopy ball $\Delta^3$. Thus the boundary $\partial M^*$ is diffeomorphic to $S^2$. Hence $M$ is uniquely determined by an element $b$ in

$$\Gamma = \left[ S^2, \frac{N(SO(2))}{N(SO(2))\,SO(3)} \right] \Big/ \pi_0 \left( \frac{N(SO(2))}{SO(2)} \right) = \{\text{singleton}\}.$$

Thus $M$ is equivalent to $(SO(3)/SO(2) \times \Delta^3)/\sim$ where $\sim$ collapses every $SO(3)/SO(2) \times \{t\}$, $t \in \partial\Delta^3$ to a point. The manifold $M$ is diffeomorphic to the standard five-sphere $S^5$. (It is easy to compute $H_2(M;Z) = 0$ and $w_2(M)$ is trivial). If $M^*$ is really $D^3$ then $M$ is equivalent to the triple-suspension of the $SO(3)$-manifold $S^2$ [11]. Hence if the Poincaré conjecture is true, this triple suspension is the only simply-connected $SO(3)$-manifold of dimension five which admits a sphere of fixed points.

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