A CHARACTERIZATION OF THE CLASS OF STARLIKE UNIVALENT FUNCTIONS

M. S. Robertson

1. INTRODUCTION

Let S^* denote the class of functions $f(z)$ analytic in D $\{z: |z| < 1\}$, normalized so that $f(0) = f'(0) - 1 = 0$ and univalently starlike in D. Then for $z \in D$ $f(z)$ satisfies the inequality $\operatorname{Re}(zf'(z)/f(z)) > 0$ and has the representation

$$\log(f(z)/z) = -2\int_0^{2\pi} \log(1 + ze^{it})\, d\alpha(t) \tag{1.1}$$

for some increasing function $\alpha(t)$ with $\alpha(2\pi) - \alpha(0) = 1$.

The properties of the elements f of the class S^* have been investigated extensively for many years. One of the more important early discoveries for this class was that $f(z)$ satisfies the inequality

$$|\sqrt{z/f(z)} - 1| < 1, \qquad z \in D. \tag{1.2}$$

This fact may also be expressed in the form

$$\operatorname{Re}\sqrt{f(z)/z} \geq 1/(1 + |z|) > 1/2, \qquad z \in D. \tag{1.3}$$

Then $f(z)/z$ is subordinate to $(1 + z)^{-2}$ in D and there exists an analytic function $\omega(z)$, $|\omega(z)| \leq |z| < 1$, such that $f(z)/z = (1 + \omega(z))^{-2}$, $z \in D$.

Proofs of this attractive result are due to Marx [2] and to Strohhächer [4]. The author [3] gave an alternate proof.

That the condition that $f(z)/z$ be subordinate to $(1 + z)^{-2}$ is not a sufficient condition that f belong to the class S^* is easily seen from the example $f(z) = z(1 + z^2)^{-2}$.

The main purpose of this note is to extend the earlier result of Marx and Strohhächer that we have mentioned. In doing so we modify (1.2) to obtain a new condition on f that is both necessary and sufficient that $f \in S^*$. This new condition provides a characterization of the class S^*. More precisely we have

THEOREM 1. *Let* $f(z)$ *be analytic in* $D\{z: |z| < 1\}$ *and normalized so that* $f(0) = f'(0) - 1 = 0$. *A necessary and sufficient condition that* $f \in S^*$ *is that for each real number* k, $-1 < k < 1$, *the function* $F_k(z)$, *defined by the equations*

Received April 21, 1978.

Michigan Math. J. 26 (1979).

$$F_k(z) = [kf(z)/f(kz)]^{1/2}, \quad F_k(0) = 1, \quad F_o(z) = [f(z)/z]^{1/2},$$

be analytic and subordinate to $(1 + kz)/(1 + z)$, $z \in D$, *or equivalently that*

$$\operatorname{Re} F_k(z) > (1 + k)/2, \quad |(1 + k)/F_k(z) - 1| < 1, \qquad z \in D, -1 < k < 1.$$

There are two special cases of Theorem 1 that should be observed. When $k = 0$ the theorem of Marx and Strohhächer follows at once. Secondly, on letting $k \to -1$, one has the interesting result that if $f \in S^*$ then $\operatorname{Re}\, [f(z)/(-f(-z))]^{1/2} > 0$ $(z \in D)$, an inequality that was brought to the author's attention several years ago by Professor F. R. Keogh. An independent and easy proof of this second special case can be found from the representation (1.1) since

$$-\arg \sqrt{\frac{f(z)}{-f(-z)}} = \int_0^{2\pi} \arg \left(\frac{1 + ze^{it}}{1 - ze^{it}}\right) d\alpha\,(t).$$

2. PROOF OF THEOREM 1

Let $f \in S^*$ and k be a real number for which $-1 < k < 1$. We prove first that $f(z)$ satisfies the inequality

$$\left|(1 + k)\sqrt{\frac{f(kz)}{kf(z)}} - 1\right| < 1, \qquad z \in D. \tag{2.1}$$

Because of the representation (1.1) it will be sufficient to establish (2.1) for functions f for which $f(z)/z$ is the derivative of a function mapping D onto a domain whose boundary is a convex polygon. Then $f(z)$ may be assumed to be of the form

$$f_n(z) = z \prod_{s=1}^{n} (1 + \varepsilon_s z)^{-2\alpha_s}, \qquad |\varepsilon_s| = 1, \alpha_s > 0, \sum_{s=1}^{n} \alpha_s = 1, \quad n = 1, 2, \ldots .$$

We make first some necessary preliminary calculations. When x is any real number we shall mean by arc tan x the real number w, $-\pi/2 < w < \pi/2$, such that $\tan w = x$. For $-\pi/2 < t < \pi/2$ and $-1 < k < 1$ we need to establish that

$$\left|t - \arctan\left(\frac{k \sin 2t}{1 + k \cos 2t}\right)\right| < \pi/2. \tag{2.2}$$

The function $y = t - \arctan\left(\dfrac{k \sin 2t}{1 + k \cos 2t}\right) = \arg\left(\dfrac{1 + e^{2it}}{1 + ke^{2it}}\right)$ has a positive derivative $\dfrac{dy}{dt} = \dfrac{1 - k^2}{1 + 2k \cos 2t + k^2}$. It follows that y is an increasing function of t on $(-\pi/2, \pi/2)$ for fixed k, $|k| < 1$. This fact may be seen from the identity

(2.3) $$\tan y = \left(\frac{1-k}{1+k}\right)\tan t.$$

Since $\lim_{|t|\to\pi/2} |y| = \pi/2$ the inequality (2.2) follows at once.

The function $h(t) = \arc\tan\left(\frac{k \sin 2t}{1 + k\cos 2t}\right)$ has the sign of $(k \sin 2t)$. We conclude that

(2.4) $$\sin h(t) = +(k \sin 2t)[1 + 2k\cos 2t + k^2]^{-1/2},$$

(2.5) $$\cos h(t) = +(1 + k\cos 2t)[1 + 2k\cos 2t + k^2]^{-1/2} > 0.$$

From these preliminary observations we are now in a position to establish for $|t| < \pi/2$, $|k| < 1$, the identity

(2.6) $$\begin{aligned}\frac{1+k}{2}\left|\frac{1+e^{2it}}{1+ke^{2it}}\right| &= \frac{(1+k)\cos t}{[1 + 2k\cos 2t + k^2]^{1/2}}\\ &= \frac{\cos t\,(1 + k\cos 2t) + \sin t\,(k\sin 2t)}{[1 + 2k\cos 2t + k^2]^{1/2}} = \cos(t - h(t))\\ &= \cos y = \cos\left[\arg\left(\frac{1+e^{2it}}{1+ke^{2it}}\right)\right].\end{aligned}$$

If t_s, $s = 1,2,\ldots,n$, are real numbers such that $|t_s| < \pi/2$, then from the identity (2.6) we obtain

(2.7) $$(1+k)\prod_{s=1}^{n}\left|\frac{1+e^{2it_s}}{1+ke^{2it_s}}\right|^{\alpha_s} = 2\prod_{s=1}^{n}\left[\cos\arg\left(\frac{1+e^{2it_s}}{1+ke^{2it_s}}\right)\right]^{\alpha_s}$$

where $\alpha_s > 0$ and $\sum_{s=1}^{n}\alpha_s = 1$. Let

$$\mathcal{T}_s = t_s - \arc\tan\left(\frac{k\sin 2t_s}{1 + k\cos 2t_s}\right) = \arg\left(\frac{1+e^{2it_s}}{1+ke^{2it_s}}\right).$$

We have established in (2.2) the inequalities $|\mathcal{T}_s| < \pi/2$. Since $\phi(t) = \log\cos t$ is a convex function of t on $(-\pi/2, \pi/2)$ with $\phi''(t) = -\sec^2 t < 0$ it follows that

$$\sum_{s=1}^{n}\alpha_s\log\cos\mathcal{T}_s \le \log\cos\left(\sum_{s=1}^{n}\alpha_s\mathcal{T}_s\right), \qquad \alpha_s > 0, \sum_{s=1}^{n}\alpha_s = 1,$$

or

$$\log \prod_{s=1}^{n} (\cos \mathscr{T}_s)^{\alpha_s} \le \log \cos \left[\arg \prod_{s=1}^{n} \left(\frac{1+e^{2it_s}}{1+ke^{2it_s}} \right)^{\alpha_s} \right]. \tag{2.8}$$

From (2.7) and (2.8) we obtain

$$(1+k) \prod_{s=1}^{n} \left| \frac{1+e^{2it_s}}{1+ke^{2it_s}} \right|^{\alpha_s} \le 2 \cos \left[\arg \prod_{s=1}^{n} \left(\frac{1+e^{2it_s}}{1+ke^{2it_s}} \right)^{\alpha_s} \right], \tag{2.9}$$

or

$$\left| (1+k) \prod_{s=1}^{n} \left(\frac{1+e^{2it_s}}{1+ke^{2it_s}} \right)^{\alpha_s} - 1 \right| \le 1, \qquad |t_s| < \pi/2, \; |k| < 1,$$

and equality occurs if $|t_s| = \pi/2$ for any s.

If $z = e^{i\theta}$ and $\varepsilon_s = e^{i\nu_s}$ in the identity $f_n(z) = z \prod_{s=1}^{n} (1 + \varepsilon_s z)^{-2\alpha_s}$ we may set $\varepsilon_s z = e^{2it_s}$ and define t_s so that $-\pi/2 \le t_s \le \pi/2$. Thus for $|z| = 1$ and $-1 < k < 1$ we have shown by means of (2.9) that

$$\left| (1+k) \sqrt{\frac{f_n(kz)}{kf_n(z)}} - 1 \right| \le 1. \tag{2.10}$$

Because $k \ne \pm 1$ (2.1) follows for $f(z) = f_n(z)$, $z \in D$, by the maximum modulus theorem for an analytic function of z.

Turning next to the proof of the sufficiency part of Theorem 1 we let $f(z)$ be regular in D with the normalization $f(0) = f'(0) - 1 = 0$. Suppose now that the function

$$\psi(z) = \frac{1+k}{F_k(z)} - 1 = (1+k) \sqrt{\frac{f(kz)}{kf(z)}} - 1 \tag{2.11}$$

is analytic and bounded in D, $|\psi(z)| < 1$, for all k, $0 < k < 1$. We shall show that this situation implies that $\operatorname{Re} zf'(z)/f(z) > 0$, $z \in D$, and $f \in S^*$. Let $t = 1 - k$, $0 < t < 1$. Since

$$f(kz) = f(z - tz) = f(z) + \sum_{n=1}^{\infty} \frac{f^{(n)}(z)}{n!} (-tz)^n, \qquad |tz| < 1 - |z|,$$

we have

$$\psi(z) = (2-t) \left[\frac{f(z-tz)}{(1-t) f(z)} \right]^{1/2} - 1 = 1 - \left(\frac{zf'(z)}{f(z)} \right) t + O(t^2)$$

as $t \to 0$, $|z| < 1$. Since $|\psi(z)| < 1$ for $z \in D$ we obtain

$$1 + \mathrm{Re}\,(-zf'(z)/f(z))\,t + O(t^2) < 1, \quad \mathrm{Re}\,(-zf'(z)/f(z)) + O(t) < 0.$$

Letting $t \to 0$ we have $\mathrm{Re}\, zf'(z)/f(z) \geq 0, z \in D$. But at $z = 0$ the function $zf'(z)/f(z)$ does not vanish so we then have $\mathrm{Re}\,\dfrac{zf'(z)}{f(z)} > 0$, $z \in D$, and $f \in S^*$. This completes the proof of Theorem 1.

The following corollary arises from Theorem 1 by an easy computation.

COROLLARY. *Let* $f(z) \in S^*$. *Then for* $-1 \leq k \leq 1$, $z \in D$,

$$|f(kz)| \leq |k|\left(\frac{1+|z|}{1+k|z|}\right)^2 \cdot |f(z)|.$$

For $-(3 - 2\sqrt{2}) < k < 1$ *and* $0 < |z| < 1$, $|f(kz)| < |f(z)|$. *The function* $z(1+z)^{-2}$ *shows that the constant* $-(3 - 2\sqrt{2})$ *in the range for* k *is best possible.*

Since $[(1 + k)/F_k(z) - 1]$ is a bounded analytic function when $f \in S^*$, a number of applications will occur to the reader. For example, since

$$\sqrt{f(kz)/z} - \sqrt{f(z)/z} = [\sqrt{f(z)/z} - k\sqrt{f(kz)/(kz)}\,]\,\omega(z), \qquad |\omega(z)| \leq |z|,$$

an application of the Clunie method [1] leads to interesting inequalities for the coefficients of an arbitrary odd function of the class S^*. Indeed, if

$$g(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1} \in S^*$$

then for $-1 \leq k < 1$ and $n = 1,2,\ldots$ we obtain

$$(2.12) \qquad |b_{2n+1}|^2\left(\frac{1-k^n}{1-k}\right)^2 \leq 1 + \sum_{s=1}^{n-1} \frac{k^s(2 - k^s - k^{s+1})}{1-k}\,|b_{2s+1}|^2.$$

We omit the details of the proof.

REFERENCES

1. J. Clunie, *On meromorphic schlicht functions.* J. London Math. Soc. 34 (1959), 215–216.

2. A. Marx, *Untersuchungen über schlichte Abbildungen.* Math. Ann. 107 (1932), 40–67.

3. M. I. S. Robertson, *On the theory of univalent functions.* Ann. of Math. (2) 37 (1936), 374–408.

4. E. Strohhäcker, *Beitrag zur Theorie der schlichten Funktionen.* Math. Z. 37 (1933), 356–380.

1116 Lawson Avenue
West Vancouver, British Columbia
Canada V7T 2E5