DISCRETE MAPS ON MANIFOLDS

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1. INTRODUCTION

Let M^n and N^n be second countable manifolds, and let $f: M^n \to N^n$ be a map (continuous function). The *branch set* $B_f \subset M^n$ is the set of points at which f fails to be a local homeomorphism; and f is *countable* (respectively, *discrete*) if $f^{-1}(y)$ is countable (respectively, consists of isolated points) for each $y \in N^n$.

1. THEOREM. *If* f *is countable, then* $\operatorname{int} B_f = \emptyset$, i.e. $\dim B_f \leq n-1$.

2. THEOREM. *If* f *is discrete, then* $\dim B_f = \dim f(B_f) \leq n-1$. *Specifically,* f *is open if and only if* $\dim B_f = \dim f(B_f) \leq n-2$.

In [13] Väisälä proved Theorem 1 for $n \leq 3$, and conjectured it for general n. The present proof for arbitrary n is shorter than Väisälä's proof, but builds on his earlier lemmas, and his clever ideas. The second sentence of Theorem 2 is already known (see (12)). Examples ((9) and (10)) show that the Theorems are sharp, and a mistake in a paper of Trohimčuk [11] is discussed. The author is grateful to the Institute for Advanced Study for its hospitality during the summer of 1977.

3. *Notation and terminology.* A map $f: M^n \to N^n$ is *light* if

$$\dim f^{-1}(y) \leq 0 \qquad \text{for every } y \in N^n.$$

Alexander-Spanier cohomology with integer coefficients and compact supports is used, and $\check{H}^m$ is augmented. The real numbers are denoted by $\mathbb{R}$, $[0, 1] \subset \mathbb{R}$ by I, the unit sphere in $\mathbb{R}^{n+1}$ by S^n, and the distance between x and y by $d(x, y)$. A subset $A \subset B$ is *residual* if $B - A$ is of the first category in $B \neq \emptyset$ [8].

2. THE PROOF OF THEOREM 1

4. LEMMA. *Let* $K \neq \emptyset$ *be compact and let* $B \subset I^m$ *be residual* $(m = 0, 1, \ldots)$. *If* $f: K \to I^m$ *is a light map with* $f|f^{-1}(B)$ *injective, then* $\check{H}^m(K) = 0$.

Proof. We use induction on m. For $m = 0$, I^0 is a single point, so $B = I^0$ and (since $f|f^{-1}(B)$ is injective), K is also a single point.

Suppose the lemma is true for $m - 1$, and consider $m \geq 1$. According to the Kuratowski-Ulam Theorem [8; Vol. I, p. 247, Corollary 1a] there is a residual set $C \subset I$ such that $B \cap (\{x\} \times I^{m-1})$ is residual in $\{x\} \times I^{m-1}$ for each $x \in C$.

Received August 5, 1977. Revision received October 6, 1977.

Michigan Math. J. 25 (1978).

Let $J \subset I$ be any interval with $H^m(f^{-1}(J \times I^{m-1})) \neq 0$, and let

$$0 \neq \alpha \in H^m(f^{-1}(J \times I^{m-1})).$$

There are subintervals J_1 and J_2 with $J_1 \cup J_2 = J$, $J_1 \cap J_2$ a single point $b \in C$, and length $\ell(J_r) < 2\ell(J)/3$ $(r = 1, 2)$. By inductive hypothesis,

$$\tilde{H}^{m-1}(f^{-1}(b \times I^{m-1})) = 0 \quad (\text{or } f^{-1}(b \times I^{m-1}) = \emptyset),$$

so in the Mayer-Vietoris Sequence inclusion induces a monomorphism

$$H^m(f^{-1}(J \times I^{m-1})) \to H^m(f^{-1}(J_1 \times I^{m-1})) \oplus H^m(f^{-1}(J_2 \times I^{m-1})).$$

Thus for either $r = 1$ or $r = 2$, $i_r^*(\alpha) \neq 0$.

Now suppose that $H^m(K) \neq 0$, i.e. there is an $\alpha \neq 0$ in

$$H^m(K) = H^m(f^{-1}(I \times I^{m-1})).$$

From the preceding argument there is a sequence of intervals I_s $(s = 0, 1, \ldots)$ such that $I_0 = I$, $\ell(I_s) \leq (2/3)^s$, and inclusion induces homomorphisms

$$i_s^* : H^m(K) = H^m(f^{-1}(I \times I^{m-1})) \to H^m(f^{-1}(I_s \times I^{m-1}))$$

with $i_s^*(\alpha) \neq 0$. If $\{c\} = \cap_s I_s$, $H^m(f^{-1}(c \times I^{m-1})) \neq 0$ by the Continuity Theorem [10; p. 318, Theorem 6]. Since each $f^{-1}(y)$ is at most 0-dimensional,

$$\dim (f^{-1}(c \times I^{m-1})) \leq m - 1$$

[9; pp. 91–92], and a contradiction results [9; p. 152]. Thus $H^m(K) = 0$.

5. LEMMA. *Let $G \subset \mathbb{R}^n$ be open, let $f : G \to f(G) \subset \mathbb{R}^n$ be a light map such that $f(G)$ is open in $\mathbb{R}^n$, and let $B \subset f(G)$ be a residual set such that f is open at each point of $A = f^{-1}(B)$ and $f|A$ is injective. Then f is a homeomorphism.*

Proof. For each $W \neq \emptyset$ open in G, $\dim f(W) = n$ [9; pp. 91–92], so that B meets int $f(W)$; thus A meets W, and since W is arbitrary, A is dense in G. Since $f|A$ is injective, if f is open, then f is a homeomorphism [13; p. 542, (2.5)].

Thus we may suppose that f fails to be open at some x in G, *i.e.*, there is an open neighborhood W of x in G such that $f(x) \notin \text{int} f(W)$. For any $r > 0$ sufficiently small and

$$V = \{y \in f(G) : d(f(x), y) < r\},$$

$V \subset f(G)$, the component U of $f^{-1}(V)$ containing x has $\bar{U}$ compact, $\bar{U} \subset W$ [13; p. 543, (3.3)], and $V - f(\bar{U}) \neq \emptyset$, so we may suppose that $\partial V - f(\bar{U}) \neq \emptyset$. By the Kuratowski-Ulam Theorem [8; Vol. I, p. 247, Corollary 1a] we may also choose r so that B meets ∂V in a residual set.

Now $f(\text{bdy } U) \subset \partial V \approx S^{n-1}$, so there is an $(n-1)$-cell $I^{n-1} \subset \partial V$ with

$$f(\text{bdy } U) \subset I^{n-1}.$$

From (4) applied to $f|\text{bdy } U : \text{bdy } U \to I^{n-1}$ with residual set $(B \cap I^{n-1})$, $\tilde{H}^{n-1}(\text{bdy } U) = 0$. A contradiction results from the exact sequence

$$\begin{array}{ccccc} \tilde{H}^{n-1}(\text{bdy } U) & \to & H^n_c(U) & \to & H^n(\bar{U}) . \\ & & \wr & & \| \\ & & \mathbb{Z} & & 0 \end{array}$$

6. LEMMA (Väisälä [13; p. 543, (3.4)]). *Suppose* $G \subset \mathbb{R}^n$ *is open and* $f : G \to \mathbb{R}^n$ *is a countable map. Then there is a set* B *residual in* $\text{int } f(G)$ *such that* $f^{-1}(B) = A$ *is residual in* G *and* f *is open at each point of* A. *Moreover, the set* $A_0 \subset A$ *at which* $f|A$ *is locally injective is dense in* A *and open in* A.

7. *Proof of Theorem* 1. Suppose $\text{int } B_f \neq \emptyset$ for some f. Since the question is local, we may suppose that $f : G \to \mathbb{R}^n$, where G is open in $\mathbb{R}^n$, and, by further restriction, that $B_f = G$. For the set A given by (6) there is a set D open in G such that $D \cap A \neq \emptyset$, and $f|D \cap A$ is injective. Since f is open at each point of A,

$$D \cap f^{-1}(B \cap \text{int } f(D)) = D \cap A .$$

For $G' = D \cap f^{-1}(\text{int } f(D))$ and $h : G' \to h(G')$ the restriction of f, $B_h = G'$, $h(G')$ is open, and h and $B' = B \cap h(G')$ satisfy the hypotheses of (5). Then h is a homeomorphism, and a contradiction results.

3. THE PROOF OF THEOREM 2

8. LEMMA. *If* $f : M^n \to N^n$ *is discrete, then* $\text{int } f(B_f) = \emptyset$.

Proof. Suppose there exists a discrete f with $\text{int } f(B_f) \neq \emptyset$. Since M^n is second countable, there is an $x \in M^n$ such that $\text{int } f(X \cap B_f) \neq \emptyset$ for every open neighborhood X of x in M^n. By restriction to a sufficiently small X, we obtain a map $\alpha : X \to \mathbb{R}^n$, where X is open in $\mathbb{R}^n$, and by further restriction we may suppose [13, p.543, (3.3)] that α is proper. There is an open n-cell $D \subset \text{int } \alpha(B_\alpha)$, and the restriction of α to $\alpha^{-1}(D)$ will again be denoted by f. Then f maps G into D^n, where G is open in $\mathbb{R}^n$, D^n is the open unit n-ball, and $f(G) = f(B_f) = D^n$. Let $B \subset D^n$ be the residual set given by (6), and let $B_i \subset B$ be the set of y such that $f^{-1}(y)$ has at most i points. Since f is discrete and proper, $B = \bigcup_{i \geq 1} B_i$.

For $y \in B - B_i$, let x_j $(j = 1, 2, \ldots, i + 1)$ be points of $f^{-1}(y)$, and let V_j be mutually disjoint open neighborhoods of x_j. Since f is open at each point of $f^{-1}(y)$ (by definition of B), $y \in \bigcap_j \text{int } f(V_j) = U$, and hence $U \subset D^n - B_i$. Thus $B - B_i$ is open in B, so that each B_i is closed in B. Since B is residual in D^n, it follows from the Baire Category Theorem that some B_i is somewhere dense in D^n, i.e.

for some open $V \subset D^n$, $\bar{B}_i \supset V$. Since B_i is closed in B, $V \cap B = V \cap B_i$.

Let $g: f^{-1}(V) \to V$ be the restriction of f (so that $V = g(B_g)$), and choose $y \in V \cap B$ such that $f^{-1}(y)$ is maximal, say $\{x_1, \ldots, x_m\}$ where $m \leq i$. Let V_j be mutually disjoint open neighborhoods of x_j, let $W = \bigcap_j \operatorname{int} g(V_j)$, let $W_j = V_j \cap g^{-1}(W)$, and let $h_j: W_j \to W$ be restrictions of g. Now $h_j(W_j) = W$ and $h_j | (W_j \cap g^{-1}(B))$ is injective, so by (5) each h_j is a homeomorphism. This contradicts $W \subset V = g(B_g)$.

Lemma 8 (and Theorem 2) cannot be extended to countable maps:

9. *Example. There is a countable, proper map* $f: \mathbb{R} \to \mathbb{R}$ *with* $f(B_f) = \mathbb{R}$.

Proof. We actually define $f: I \to I$, where $I = [0, 1]$, $f^{-1}(0) = \{0\}$, and $f^{-1}(1) = \{1\}$. Let C be the Cantor set, viewed as the set of sequences of 0's and 1's, and define a continuous surjection $g: C \to I$ by $g(\{a_n\}) = \sum_{n \geq 1} a_n \cdot 2^{-n}$. For each component (interval) J of $I - C$, $g(\partial J)$ is a single point. Extend g to $\bar{J}$ by folding $\bar{J}$ in half, i.e. each of the two half intervals of $\bar{J}$ is mapped isometrically onto its image interval. This defines a map f of I onto I.

Each $g^{-1}(y)$ has at most two points, each $f^{-1}(y) \cap J$ has at most two points, and there are a countable number of intervals J; thus f is countable (and some $f^{-1}(y)$ are not discrete). Each midpoint of each J is in B_f, and C is the set of limit points of these midpoints; thus $C \subset B_f$, so that $I = f(B_f)$, as desired.

10. *Remark.* Theorem 1 is false for light maps: by folding enough we may define a light map $f: \mathbb{R} \to \mathbb{R}$ with $B_f = \mathbb{R}$. In fact, as Väisälä noted in a letter to the author, any nowhere differentiable map serves as a counter example.

11. *Lemma. If* $f: M^n \to N^n$ *is light,* $\dim B_f \leq n - 2$, *and* $\dim f(B_f) \leq n - 1$, *then* f *is open.*

Proof. Suppose that f is not open, at $\bar{x} \in M^n$; then there is an open neighborhood W of $\bar{x}$ such that $f(\bar{x}) \notin \operatorname{int} f(W)$. Choose an open n-cell neighborhood V of $f(\bar{x})$ in N^n sufficiently small that the component U of $f^{-1}(V)$ containing $\bar{x}$ is contained in the interior of a closed n-cell of M^n. Then $g: U \to V$ defined by restriction of f is proper, and since $g(U) \neq V$, $\deg g = 0$. According to [6; pp. 32-33, (5.7) (and the following paragraph)], if there are points $x, y \in U$ with the local degrees $d_x = 1$ and $d_y = -1$, then $H_c^{n-1}(B_g; G) \neq 0$ for any module G ($\neq 0$) over a commutative ring. Since $\dim B_g \leq n - 2$, $d_x = 1$ (say) for every $x \in U - B_g$.

Since $\dim g(B_g) \leq n - 1$, $\dim g^{-1}(g(B_g)) \leq n - 1$ [9; pp. 91-92], so there exists $x \in U - g^{-1}(g(B_g))$. Let $x(i)$ $(i = 1, 2, \ldots, m)$ be the points of $g^{-1}(g(x))$. Since each $d_{x(i)} = 1$, $0 = \deg g = \sum_i d_{x(i)} = m$, and a contradiction results.

12. *Proof of Theorem* 2. According to [1], [2], and [12], if f is discrete open, then $\dim B_f = \dim f(B_f) \leq n - 2$. According to [5; p. 531, (2.4)], if f is light and $\dim f(B_f) \leq n - 2$, then f is open. The second conclusion results.

Thus we may suppose that f is not open, and $\dim f(B_f) \geq n - 1$. By Theorem

1 $\dim B_f \leq n-1$, and by (8) $\dim f(B_f) \leq n-1$, so that $\dim f(B_f) = n-1$. By (11) $\dim B_f = n-1$ also.

13. *Remarks.* In the smooth case more can be said ([4; p. 94, (2.3)] and [3; p. 500, (5.1)]): If $f: M^n \to N^n$ is C^3 light, then $B_f = \emptyset$,

$$\dim B_f = n-2 \quad \text{or} \quad \dim B_f = n-1;$$

the last case occurs if and only if f is not open.

The topological case differs: for $n \geq 5$ there are discrete open maps $f: S^n \to S^n$ with $B_f \approx f(B_f) \approx S^{n-4}$ [7; (5.6)].

4. TROHIMČUK'S PAPER

14. *Remark.* In [11] Trohimčuk claims to prove that there is no discrete open map $f: M^n \to N^n$ with $n \geq 3$ and $\dim B_f = 0$ (and thus $\dim f(B_f) = 0$). We now note that there is a gap in his proof (the author agrees), so the question is apparently still open.

The gap occurs in the proof of [11; Lemma 6], specifically in the middle of page 287 where he claims that $\bar{Q}$ is a closed 2-cell ("... $Q \cup L_0$ is homeomorphic to a closed disk, and the boundary of Q coincides with L_0 ..."). For an example, define $f: \mathbb{R}^3 \to \mathbb{R}^3$ by $f(u+iv, t) = ((u+iv)^2, t)$, so $B_f = f(B_f) = \{(0,0)\} \times \mathbb{R}$. Let Λ be the union of the line segment joining (0, 0, 1) to (0, 1, 0) and that joining (0, 0, −1) to (0, 1, 0); in a sufficiently fine triangulation of $\mathbb{R}^3$ the regular neighborhood $\bar{V}_r$ of Λ will be closed 3-ball meeting $\{(0,0)\} \times \mathbb{R}$ in two disjoint line segments, each contained in $V_r = \operatorname{int} \bar{V}_r$ except for its endpoints on $\operatorname{bdy}(V_r) = (V_r)_g = S_1$. Then $G_r = f^{-1}(V_r)$ is the interior of a solid torus $\bar{G}_r$ meeting $\{(0,0)\} \times \mathbb{R}$ in two disjoint line segments, each contained in G_r except for its endpoints on

$$\operatorname{bdy}(G_r) = (G_r)_g = S.$$

The set of (u, v, t) on S_1 defined by $v = 0$ and $u \geq 0$ consists of two arcs, as does the set defined by $u = 0$ and $v \geq 0$. Let $L_1 \subset S_1$ be the union of the two former arcs, together with (either) one of the latter two; then L_1 is an arc containing the four points of $S_1 \cap (\{(0,0)\} \times \mathbb{R})$, in their order on $\{(0,0)\} \times \mathbb{R}$, with two of them as endpoints. Each component Q of $S - f^{-1}(L_1)$ has $\bar{Q}$ a cylinder, *not* a disk as Trohimčuk claims.

Of course $\dim B_f \neq 0$, as is required in the hypothesis of [11; Lemma 6], so this example is not a counterexample. But observe that all that is *used* in the proof of [11; Lemma 6] is:

$$\dim(B_f \cap S) = \dim(f(B_f) \cap S_1) = 0,$$
$$f^{-1}(f(B_f)) = B_f, \quad f: B_f \approx f(B_f), \quad \text{and}$$
$$f: \mathbb{R}^3 - B_f \to \mathbb{R}^3 - f(B_f)$$

is a covering map. Thus the argument of [11; Lemma 6] should apply to this example; since it does not, the gap cited earlier appears to be significant.

5. ANY MAP IS QUASI-OPEN ALMOST EVERYWHERE

15. *Definitions.* For a map $f : X \to Y$ and $x \in X$, the component of $f^{-1}(f(x))$ containing x is denoted by Γ_x. The map f is called *quasi-open* [14] at x if, for every open neighborhood U of Γ_x, $f(x) \in \operatorname{int} f(U)$. A light quasi-open map is open. Lemma 6 (Väisälä [13; p. 543, (3.4)]) has a generalization from light maps to arbitrary maps (continuous functions):

16. PROPOSITION. *Let* X *and* Y *be locally connected, compact metric spaces, and let* $f : X \to Y$ *be any map. Then there is a set* B *residual in* Y *such that* f *is quasi-open at each point of* $f^{-1}(B)$.

Proof. The space Y has a countable base $\{V_i\}$ of open sets with $\bar{V}_i$ compact. Let $U_{i,j}$ be the components of $f^{-1}(V_i)$. Then each $U_{i,j}$ is open, and there are a countable number of them.

Now

$$\begin{aligned} \operatorname{bdy} f(U_{i,j}) &= f(\bar{U}_{i,j}) - \operatorname{int} f(U_{i,j}) \\ &\subset f(\operatorname{bdy} U_{i,j}) \cup (f(U_{i,j}) - \operatorname{int} f(U_{i,j})) \\ &\subset \operatorname{bdy} V_i \cup (f(U_{i,j}) - \operatorname{int} f(U_{i,j})), \end{aligned}$$

so that the closed subset $\operatorname{bdy} f(U_{i,j}) \subset Y$ has empty interior. Let

$$F = \bigcup_{i,j} \operatorname{bdy} f(U_{i,j});$$

then $B = Y - F$ is residual in Y.

Let $x \in f^{-1}(B)$, and let U be an open neighborhood of Γ_x. There is a $U_{i,j}$ with $\Gamma_x \subset U_{i,j} \subset U$. Since $f(x) \in B$, $f(x) \notin \operatorname{bdy} f(U_{i,j})$, so that

$$f(x) \in \operatorname{int} f(U_{i,j}) \subset \operatorname{int} f(U).$$

Hence f is quasi-open at each point of $f^{-1}(B)$.

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