NORMAL DIRECT SUMMANDS OF HYPOREDUCTIVE OPERATORS

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In [2], C. K. Fong showed that if S is a hyporeductive operator, N is normal, and if S is quasi-similar to N, then S is normal. In this paper we obtain an extension of Fong's result; in particular, we show that if there are any non-zero operators X and Y such that SX = XN and YS = NY, then S has a normal direct summand.

In what follows $\mathscr{H}$ will be a separable complex Hilbert space, N will be a fixed normal operator in $\mathscr{B}(\mathscr{H})$, and S will be a fixed *hyporeductive* operator in $\mathscr{B}(\mathscr{H})$; that is, S has the property that every hyperinvariant subspace reduces S.

If A and B are any two operators we will use the following notation:

$$\mathscr{L}(A,B) = \{Y: YA = BY\}$$
$$\mathscr{R}(A,B) = \{X: AX = XB\}.$$

(The letters $\mathscr{L}$ and $\mathscr{R}$ are chosen to reflect the position of Y or X with respect to A; in the defining equation Y appears on the left, X on the right of A.) For convenience we will refer to $\mathscr{L}(S,N)$ and $\mathscr{R}(S,N)$ as simply $\mathscr{L}$ and $\mathscr{R}$. $\mathscr{L}$ and $\mathscr{R}$ are not empty since the zero operator is in each. In addition, let $K_{\mathscr{L}}$ be the projection whose range is $\left[\bigcap \{\ker Y: Y \in \mathscr{L}\}\right]^{\perp}$ and let $R_{\mathscr{R}}$ be the projection whose range is $\bigvee \{\operatorname{ran} X: X \in \mathscr{R}\}$. Evidently, $\ker Y \supseteq \ker K_{\mathscr{L}}$ and $\operatorname{ran} X \subseteq \operatorname{ran} R_{\mathscr{R}}$ for each Y in $\mathscr{L}$ and X in $\mathscr{R}$.

THEOREM 1. *With the above notation, if* $K_{\mathscr{L}}$ *and* $R_{\mathscr{R}}$ *are both equal to the identity, then* S *is normal.*

Notice that Fong's result is a special case of Theorem 1, since if there exist quasiaffinities Y and X in $\mathscr{L}$ and $\mathscr{R}$ respectively, then $K_{\mathscr{L}} = R_{\mathscr{R}} = 1$ trivially. The proof below is based on the proof in [2].

Proof. First observe that if Y and X are in $\mathscr{L}$ and $\mathscr{R}$ respectively, and if C commutes with S and D commutes with N, then DY and YC are in $\mathscr{L}$ and XD and CX are in $\mathscr{R}$.

Suppose that $\mathscr{M}$ is a hyperinvariant subspace of the normal operator N. Denote

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by $\mathscr{R}\mathscr{M}$ the subspace $\bigvee \{X\mathscr{M} : X \in \mathscr{R}\}$. We assert that $\mathscr{R}\mathscr{M}$ is hyperinvariant for S, for if C commutes with S and X is in $\mathscr{R}$ then $CX \in \mathscr{R}$ and hence

$$C(X\mathscr{M}) = (CX)\mathscr{M} \subseteq \mathscr{R}\mathscr{M}$$

and it follows that $C(\mathscr{R}\mathscr{M}) \subseteq \mathscr{R}(\mathscr{M})$.

If $\mathscr{M}$ is hyperinvariant for N so is $\mathscr{M}^{\perp}$ and thus $\mathscr{R}(\mathscr{M}^{\perp})$ is hyperinvariant for S. Let P and Q be the projections with ranges $\mathscr{R}\mathscr{M}$ and $\mathscr{R}(\mathscr{M}^{\perp})$ respectively. We want to show that $Q = 1 - P$. Since S is hyporeductive, P and Q commute with S and since the range of Q is hyperinvariant, $QPQ = PQ$ and thus $PQ = QP$. Now if $Y \in \mathscr{L}$ and $X \in \mathscr{R}$, we have $NYX = YSX = YXN$ so YX commutes with N and thus $\mathscr{M}$ is invariant for YX. It follows that $Y(\mathscr{R}\mathscr{M}) \subseteq \mathscr{M}$ for all Y in $\mathscr{L}$. Likewise $Y(\mathscr{R}(\mathscr{M}^{\perp})) \subseteq \mathscr{M}^{\perp}$ and thus

$$Y(\mathscr{R}\mathscr{M}) \cap Y(\mathscr{R}(\mathscr{M}^{\perp})) = \{0\} \text{ for all } Y \text{ in } \mathscr{L}.$$

By assumption $\bigcap \{\ker Y : Y \in \mathscr{L}\} = \{0\}$ and thus $(\mathscr{R}\mathscr{M}) \cap (\mathscr{R}(\mathscr{M}^{\perp})) = \{0\}$, that is, $PQ = 0$. Moreover,

$$(\mathscr{R}\mathscr{M}) \vee (\mathscr{R}(\mathscr{M}^{\perp})) \supseteq [\mathscr{R}(\mathscr{M} \vee \mathscr{M}^{\perp})] = \mathscr{R}\mathscr{H} = \mathscr{H}$$

since $R_{\mathscr{R}} = 1$. Hence $PQ = QP = 0$, $P + Q = 1$, and $Q = 1 - P$, that is, $\mathscr{R}(\mathscr{M}^{\perp}) = (\mathscr{R}\mathscr{M})^{\perp}$.

If σ is a Borel subset of the complex plane and if E is the spectral measure of N, denote by $F(\sigma)$ the projection with range $\mathscr{R}(\operatorname{ran} E(\sigma))$. The above paragraph shows that if $\sigma \cap \tau = \emptyset$, then $F(\sigma)F(\tau) = 0$. Hence if $\sigma \cap \tau = \emptyset$, then

$$F(\sigma \cup \tau) = F(\sigma) + F(\tau).$$

From the latter fact it follows readily that $F(\sigma \cap \sigma') = F(\sigma)F(\sigma')$ for any two Borel sets σ and σ', and that if $\{\sigma_i\}$ is a disjoint family of Borel sets then $F\left(\bigcup \sigma_i\right) = \sum F(\sigma_i)$. Hence F is a spectral measure, and we can define a normal operator M by setting $M = \int \lambda \, dF(\lambda)$.

We assert that for all Y in $\mathscr{L}$, $E(\sigma)Y = YF(\sigma)$. Recall that $Y(\mathscr{R}\mathscr{M}) \subseteq \mathscr{M}$ for any subspace $\mathscr{M}$ hyperinvariant for N, or in particular $Y(\operatorname{ran} F(\sigma)) \subseteq \operatorname{ran} E(\sigma)$, so that $E(\sigma)YF(\sigma) = YF(\sigma)$ for any σ. Thus it is also true that $E(\tilde{\sigma})YF(\tilde{\sigma}) = YF(\tilde{\sigma})$ and thus $E(\sigma)YF(\tilde{\sigma}) = E(\sigma)E(\tilde{\sigma})YF(\tilde{\sigma}) = 0$, where $\tilde{\sigma}$ is the complement of σ. We now have

$$YF(\sigma) = E(\sigma)YF(\sigma) = E(\sigma)YF(\sigma) + E(\sigma)YF(\tilde{\sigma}) = E(\sigma)Y,$$

and the assertion is proved. It now follows that the spectrum of N contains that of M (*i.e.*, $E(\sigma) = 0$ implies $F(\sigma) = 0$), since $\bigcap \{\ker Y : Y \in \mathscr{L}\} = \{0\}$. Thus if ϕ is a step function on $\sigma(N)$ we have $\phi(N)Y = Y\phi(M)$, and by approximating we conclude that $NY = YM$ for all $Y \in \mathscr{L}$. Since also $YS = NY$ we know that $Y(M - S) = 0$ for all Y in $\mathscr{L}$, and again using the fact that $K_{\mathscr{L}} = 1$ we have shown that $S = M$ and S is normal.

THEOREM 2. *Let* S *be hyporeductive and* N *normal, and let* $K_{\mathscr{L}}$ *and* $R_{\mathscr{R}}$ *be defined as above. Then the ranges of* $K_{\mathscr{L}}$ *and* $R_{\mathscr{R}}$ *are reducing subspaces of* S, $K_{\mathscr{L}}$ *commutes with* $R_{\mathscr{R}}$, *and the restriction of* S *to* $\operatorname{ran} K_{\mathscr{L}} \cap \operatorname{ran} R_{\mathscr{R}}$ *is normal.*

Proof. First we show that $\ker K_{\mathscr{L}}$ is hyperinvariant for S. Let $CS = SC$, and let Y be in $\mathscr{L}$. Then YC is in $\mathscr{L}$; thus if $Yf = 0$ for all Y in $\mathscr{L}$, it must be that $YCf = 0$ for all Y in $\mathscr{L}$ and thus that $\bigcap \{\ker Y : Y \in \mathscr{L}\}$ is invariant under C. Since $\ker K_{\mathscr{L}} = \bigcap \{\ker Y : Y \in \mathscr{L}\}$, $\ker K_{\mathscr{L}}$ is hyperinvariant for S, and therefore $\ker K_{\mathscr{L}}$ and $\operatorname{ran} K_{\mathscr{L}}$ reduce S. It is equally easy to show that $\operatorname{ran} R_{\mathscr{R}}$ is hyperinvariant for S. Since $1 - K_{\mathscr{L}}$ and $R_{\mathscr{R}}$ have ranges that are hyperinvariant for S, it follows as in the proof of Theorem 1 that $R_{\mathscr{R}}$ commutes with $1 - K_{\mathscr{L}}$ and hence with $K_{\mathscr{L}}$.

Because of the above facts, $K_{\mathscr{L}}R_{\mathscr{R}}S$ is the restriction of S to $\operatorname{ran} K_{\mathscr{L}} \cap \operatorname{ran} R_{\mathscr{R}}$. Notice that if Y is in $\mathscr{L}$, then $YK_{\mathscr{L}} = Y$: for if $K_{\mathscr{L}}f = 0$, then

$$f \in \ker K_{\mathscr{L}} = \bigcap \{\ker Y : Y \in \mathscr{L}\},$$

so $Yf = 0$ also; on the other hand, if $K_{\mathscr{L}}f = f$, then clearly $YK_{\mathscr{L}}f = Yf$. Likewise if X lies in $\mathscr{R}$, then for any f in $\mathscr{H}$, Xf lies in $\bigvee \{\operatorname{ran} X : X \in \mathscr{R}\}$ and thus $R_{\mathscr{R}}Xf = Xf$, so that $R_{\mathscr{R}}X = X$.

We now show that if $Y \in \mathscr{L}(S,N)$, then $YR_{\mathscr{R}} \in \mathscr{L}(K_{\mathscr{L}}R_{\mathscr{R}}S, N)$. In fact, if $YS = NY$ then of course $YSR_{\mathscr{R}} = NYR_{\mathscr{R}}$, and since $YK_{\mathscr{L}} = Y$ we have

$$YK_{\mathscr{L}}SR_{\mathscr{R}} = NYR_{\mathscr{R}}.$$

Finally since $R_{\mathscr{R}}^2 = R_{\mathscr{R}}$ and $R_{\mathscr{R}}$ commutes with S and $K_{\mathscr{L}}$, we have

$$(YR_{\mathscr{R}})(K_{\mathscr{L}}R_{\mathscr{R}}S) = N(YR_{\mathscr{R}}),$$

that is, $YR_{\mathscr{R}} \in \mathscr{L}(K_{\mathscr{L}}R_{\mathscr{R}}S, N)$. Similarly we can show that if $X \in \mathscr{R}(S,N)$, then $K_{\mathscr{L}}X \in \mathscr{R}(k_{\mathscr{L}}S, N)$.

In order to apply Theorem 1 we consider the operators $\hat{S} = (K_{\mathscr{S}}R_{\mathscr{R}}S) \oplus 0$ and $\hat{N} = N \oplus 0$ acting on the space $\mathscr{H} \oplus \mathscr{H}$. $\hat{N}$ is normal, and in consequence of Lemma 5 of [1], $\hat{S}$ is hyporeductive. The above paragraph shows that $\mathscr{L}(\hat{S},\hat{N})$ contains all operators of the form $YR_{\mathscr{R}} \oplus 1$ with Y in $\mathscr{L}(S,N)$; in addition, a simple matrix multiplication shows that the operator

$$\hat{Y}_o = \begin{pmatrix} 0 & 0 \\ 1 - K_{\mathscr{S}}R_{\mathscr{R}} & 0 \end{pmatrix}$$

is also in $\mathscr{L}(\hat{S},\hat{N})$. (Recall that $K_{\mathscr{S}}$ and $R_{\mathscr{R}}$ commute.) Thus we know that

$$\begin{aligned} \bigcap \{\ker \hat{Y} \colon \hat{Y} \in \mathscr{L}(\hat{S},\hat{N})\} &\subseteq \ker \hat{Y}_o \cap \bigcap \{\ker (YR_{\mathscr{R}} \oplus 1) \colon Y \in \mathscr{L}(S,N)\} \\ &= [\ker (1 - K_{\mathscr{S}}R_{\mathscr{R}}) \oplus \mathscr{H}] \\ &\quad \cap \left[\bigcap \{\ker YR_{\mathscr{R}} : Y \in \mathscr{L}\} \oplus 0 \right] \\ &= \left[\ker (1 - K_{\mathscr{S}}R_{\mathscr{R}}) \cap \bigcap \{\ker YR_{\mathscr{R}} : Y \in \mathscr{L}\} \right] \oplus 0 \\ &= \left[(\operatorname{ran} K_{\mathscr{S}}R_{\mathscr{R}}) \cap \bigcap \{\ker YR_{\mathscr{R}} : Y \in \mathscr{L}\} \right] \oplus 0. \end{aligned}$$

On the other hand if $K_{\mathscr{S}}R_{\mathscr{R}}f$ lies in $\bigcap \{\ker YR_{\mathscr{R}} : Y \in \mathscr{L}\}$, then for all $Y \in \mathscr{L}$ we have $0 = YR_{\mathscr{R}}(K_{\mathscr{S}}R_{\mathscr{R}}f) = Y(K_{\mathscr{S}}R_{\mathscr{R}}f) = YR_{\mathscr{R}}f$, since $K_{\mathscr{S}}R_{\mathscr{R}} = R_{\mathscr{R}}K_{\mathscr{S}}$ and $YK_{\mathscr{S}} = Y$. But this means that $R_{\mathscr{R}}f \in \bigcap \{\ker Y \colon Y \in \mathscr{L}\}$, that is, $K_{\mathscr{S}}R_{\mathscr{R}}f = 0$. Hence $(\operatorname{ran} K_{\mathscr{S}}R_{\mathscr{R}}) \cap \bigcap \{\ker YR_{\mathscr{R}} : Y \in \mathscr{L}\} = 0$ and so

$$\bigcap \{\ker \hat{Y} \colon \hat{Y} \in \mathscr{L}(\hat{S},\hat{N})\} = 0.$$

Now let

$$\hat{X}_o = \begin{pmatrix} 0 & 1 - K_{\mathscr{S}}R_{\mathscr{R}} \\ 0 & 0 \end{pmatrix}.$$

By a similar computation,

$$\bigvee \{\operatorname{ran} \hat{X} : \hat{X} \in \mathscr{R}(\hat{S}, \hat{N})\} \supseteq (\operatorname{ran} \hat{X}_o) \vee \left(\bigvee \{\operatorname{ran}(K_{\mathscr{S}} X \oplus 1) : X \in \mathscr{R}(S, N)\} \right)$$

$$= \left[\ker K_{\mathscr{S}} R_{\mathscr{R}} \vee \bigvee \{\operatorname{ran} K_{\mathscr{S}} X : X \in \mathscr{R}\} \right] \oplus \mathscr{H}.$$

It is easy to see that $\bigvee \{\operatorname{ran} K_{\mathscr{S}} X : X \in \mathscr{R}\} = \operatorname{ran} K_{\mathscr{S}} R_{\mathscr{R}}$; for instance,

$$\operatorname{ran} K_{\mathscr{S}} R_{\mathscr{R}} = K_{\mathscr{S}}(\operatorname{ran} R_{\mathscr{R}}) = K_{\mathscr{S}}\left(\bigvee \{\operatorname{ran} X : X \in \mathscr{L}\} \right)$$

$$= \bigvee \{K_{\mathscr{S}}(\operatorname{ran} X) : X \in \mathscr{L}\} = \bigvee \{\operatorname{ran} K_{\mathscr{S}} X : X \in \mathscr{L}\}.$$

Thus $\bigvee \{\operatorname{ran} \hat{X} : \hat{X} \in \mathscr{R}(\hat{S}, \hat{N})\}$ is all of $\mathscr{H} \oplus \mathscr{H}$.

We have shown that the hypotheses of Theorem 1 apply to $\hat{S}$ and $\hat{N}$. Hence $\hat{S}$ is normal and we are done.

To show that Theorem 1 is really an extension of Fong's original result, we conclude with an example where S and N are not quasi-similar but $K_{\mathscr{S}} = R_{\mathscr{R}} = 1$. Let $\mathscr{H}$ be a three-dimensional Hilbert space, and let

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Quasi-similarity is the same as similarity for finite-dimensional spaces; S and N are clearly not similar since the multiplicities are wrong. On the other hand, $\mathscr{L}(S, N)$ contains the operators

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

the intersection of whose kernels is 0. Hence $K_{\mathscr{S}} = 1$. Similarly, $\mathscr{R}(S,N)$ contains

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and $R_{\mathscr{R}} = 1$.

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