ON REPRESENTATIONS OF ARTIN'S BRAID GROUP

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In [5], it is shown that the projective symplectic group $P\,Sp((n-2)/2, \mathbb{Z}_3)$ is an epimorphic image of B_n, Artin's Braid group on n strings. The method arises from machinery established by Hurwitz [10] for determining the action of B_n on branched coverings of the two-sphere. Redefining this action in terms of Fuchsian groups, a more direct proof of this result is obtained and the general method is shown to be allied to the methods of [8] of obtaining finite representations of the mapping class groups of related Fuchsian groups. These latter finite representations are discussed in Section 3. The link is provided in Section 2 by a general method of obtaining (infinite) symplectic representations of B_n, which is, in essence, a reformulation of results in [4].

1. PRELIMINARIES

A *Fuchsian group* is a discrete subgroup of $\mathscr{L} = PSL(2, \mathbb{R})$, the group of all conformal self-homeomorphisms of the upper half-plane U. A finitely-generated Fuchsian group of the first kind has a presentation of the form:

(1)
$$\text{Generators: } e_1, e_2, \ldots, e_r, p_1, \ldots, p_s, a_1, b_1, \ldots, a_g, b_g$$
$$\text{Relations: } e_i^{m_i} = 1\ (i = 1, 2, \ldots, r); \qquad \prod_{i=1}^{r} e_i \prod_{j=1}^{s} p_j \prod_{k=1}^{g} [a_j, b_j] = 1$$

A Fuchsian group with presentation (1) has *signature* $(g; m_1, \ldots, m_r; s)$. The e_i are elliptic elements, the p_i parabolic and the a_i, b_i hyperbolic. The quotient space U/Γ takes the structure of a Riemann surface obtained from a compact surface of genus g by deleting s points. The covering $U \to U/\Gamma$ is branched over r points corresponding to the fixed points of $e_1, e_2, \ldots, e_r$ and the *periods* m_i give the order of branching at these points.

Γ has a fundamental region in U whose hyperbolic area $\mu(\Gamma)$ is given by

$$\mu(\Gamma) = 2\pi\,[2(g-1) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + s]. \tag{2}$$

If Γ_1 is a subgroup of Γ of finite index n, then $\mu(\Gamma_1) = n\mu(\Gamma)$, which combined with (2) gives the Riemann-Hurwitz relation.

With Γ as at (1), an automorphism of Γ is called *type-preserving* if it maps parabolic elements into parabolic elements. Let F be a free group on $2g + r + s$

Received September 20, 1976. Revision received April 25, 1977.

Michigan Math. J. 25 (1978).

generators and N the normal closure of the relators given in (1) so that Γ is isomorphic to F/N. Every type-preserving automorphism ϕ of Γ is induced by an automorphism Φ of F. Furthermore, if we denote the generators of F by capital letters of the corresponding generators of Γ then

$$\Phi(E_i) = \lambda_i E_{\phi(i)}^{\varepsilon(\phi)} \lambda_i^{-1}, \qquad i = 1, 2, \ldots, r,$$

$$\Phi(P_j) = \mu_j P_{\phi(j)}^{\varepsilon(\phi)} \mu_j^{-1}, \qquad j = 1, 2, \ldots, s$$

and $\Phi(R) = \lambda R \lambda^{-1}$ where $R = \Pi E_i \Pi P_j \Pi [A_k, B_k]$, where $\varepsilon(\phi) = \pm 1$ and $i \mapsto \phi(i)$, $j \mapsto \phi(j)$ are permutations on r,s elements respectively (see [17]). Let $\mathfrak{A}(\Gamma)$ denote the group of type-preserving automorphisms ϕ of Γ which are also orientation-preserving; *i.e.,* $\varepsilon(\phi) = +1$. Notice that ϕ then maps each e_i into a conjugate of some e_j, with, necessarily, $m_i = m_j$, and each p_i into a conjugate of some p_j. If $\mathfrak{I}(\Gamma)$ denotes the group of inner automorphisms of Γ, then $\mathrm{Mod}\,\Gamma = \mathfrak{A}(\Gamma)/\mathfrak{I}(\Gamma)$ is the (Teichmüller) *modular group of* Γ.

The Nielsen isomorphism maps $\mathrm{Mod}\,\Gamma$ onto the mapping class group (of homotopy classes of self-homeomorphisms) of the surface U/Γ [11].

The methods of proof in the later sections depend on the following known facts (see e.g. [11]). Suppose there exists a finite group G and a Fuchsian group Γ_0 such that the sequence

$$(3) \qquad 1 \to \Gamma \xrightarrow{i} \Gamma_0 \xrightarrow{j} G \to 1$$

is exact, with i the inclusion map. One can embed G in $\mathrm{Mod}\,\Gamma$ via $\hat{j}$ where $\hat{j}(g) = \bar{\phi}_g$ with $\phi_g(\gamma) = \gamma_0 \gamma \gamma_0^{-1}$ for every $\gamma \in \Gamma$ and $\gamma_0 \in \Gamma_0$ is such that $j(\gamma_0) = g$. $\hat{j}$ is a monomorphism since Γ_0 has trivial centre.

Define $\mathfrak{A}(\Gamma_0, \Gamma) = \{\phi \in \mathfrak{A}(\Gamma_0) : \phi(\Gamma) = \Gamma\}$. For such a ϕ, $\phi\phi_g\phi^{-1} = \phi_{g'}$ where $g' = j\phi(\gamma_0)$. Thus regarding $\phi \in \mathfrak{A}(\Gamma)$, $\bar{\phi} \in \mathscr{N}(\hat{j}(G))$, the normaliser of $\hat{j}(G)$ in Mod Γ. On the other hand, suppose $\bar{\phi} \in \mathscr{N}(\hat{j}(G))$. Now Mod Γ acts as a group of homeomorphisms of $T(\Gamma)$, the Teichmüller space of Γ. $\bar{\phi}$ will map $T(\Gamma_0)$, the fixed point set of $\hat{j}(G)$ in $T(\Gamma)$ onto itself. Choose $[\tau] \in T(\Gamma_0)$ such that $\tau(\Gamma_0)$ is a maximal Fuchsian group. With the exception of a finite number of signatures for Γ_0, which will not arise in later arguments, this is always possible [7], [16]. Now $\bar{\phi}[\tau] = [\sigma]$ for some $[\sigma] \in T(\Gamma_0)$. Thus $\tau(\Gamma)$ is normal in both $\tau(\Gamma_0)$ and $\sigma(\Gamma_0)$ and $\tau(\Gamma_0) \subsetneq N$ where N is the normaliser of $\tau(\Gamma)$ in $\mathscr{L}$, unless $\tau(\Gamma_0) = \sigma(\Gamma_0)$. Thus $\tau^{-1}\sigma \in \mathfrak{A}(\Gamma_0)$ and $\tau^{-1}\sigma|_\Gamma = \phi$. Thus

$$(4) \qquad 1 \to \mathfrak{I}(\Gamma) \to \mathfrak{A}(\Gamma_0, \Gamma) \xrightarrow{\mu_1} \mathscr{N}(\hat{j}(G)) \to 1$$

is exact.

Also, if $\mathrm{Mod}(\Gamma_0, \Gamma) = \mathfrak{A}(\Gamma_0, \Gamma)/\mathfrak{I}(\Gamma_0)$ then the sequence

$$(5) \qquad 1 \to \hat{j}(G) \to \mathscr{N}(\hat{j}(G)) \to \mathrm{Mod}(\Gamma_0, \Gamma) \to 1$$

is exact (see [11]).

2. SYMPLECTIC REPRESENTATIONS OF B_n

Let B_n denote the *Artin Braid group* on n-strings ($n \geq 3$). B_n has a faithful representation as a group of automorphisms of the free group F_n on n generators $X_1, X_2, \ldots, X_n$ and we will take this as our definition of B_n. Thus

$$B_n = \{\sigma \in \mathrm{Aut}(F_n) : \sigma(X_i) = T_i X_{\sigma(i)} T_i^{-1} \text{ where } T_i \in F_n,\ i \mapsto \sigma(i) \text{ is a permutation of } 1, 2, \ldots, n, \text{ and } \sigma(X_1 X_2 \ldots X_n) = X_1 X_2 \ldots X_n\}.$$

It is well-known that B_n is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ where

$$\sigma_j(X_j) = X_{j+1}, \quad \sigma_j(X_{j+1}) = X_{j+1}^{-1} X_j X_{j+1}, \quad \text{and } \sigma_j(X_k) = X_k \qquad \text{for } k \neq j, j+1.$$

Let $Sp(2g, \mathbb{Z})$ denote the symplectic group of $2g \times 2g$ matrices S with integral entries, i.e. all S such that $S^t J S = J$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

LEMMA 1. *Suppose* Γ, Γ_0 *are Fuchsian groups as at* (3) *and* Γ *has signature* $(\gamma; —; 0)$, $\gamma \geq 2$. *If there is a representation* $\mu_2 : B_n \to \mathfrak{A}(\Gamma_0, \Gamma)$ *then* B_n *has a representation in* $Sp(2\gamma, \mathbb{Z})$.

Proof. Following μ_2 by the homomorphism μ_1 at (4), we obtain a representation of B_n in Mod Γ. But Mod Γ maps onto $Sp(2\gamma, \mathbb{Z})$ under the mapping μ_3 induced by $\Gamma \to \Gamma / [\Gamma, \Gamma]$ [14; p. 356].

Let Γ_0 have signature $(0; m^{(n)}; 0)$ where $m|n$. (Here $m^{(n)}$ indicates that the period m is repeated n times.) Let $\mathbb{Z}_m$ denote the cyclic group of residues (mod m) and define $j: \Gamma_0 \to \mathbb{Z}_m$ by $j(e_i) = 1$ for $i = 1, 2, \ldots, n$. Note that, for j to be a homomorphism one must have $m|n$. The kernel of j, Γ, is torsion-free and so has signature $(\gamma; —; 0)$ where, by the Riemann-Hurwitz formula, γ is given by

$$2\gamma = (n-2)(m-1). \tag{6}$$

Recall that, in order that Γ_0 be Fuchsian, $\mu(\Gamma_0)$ defined at (2) must be positive. Thus $n \geq 4$, and if $n = 4$, then $m = 4$.

For every $\phi \in \mathfrak{A}(\Gamma_0)$, $\phi(e_i) = t_i e_{\phi(i)} t_i^{-1}$ and so $j\phi = j$. Thus $\phi(\Gamma) = \Gamma$ and so $\mathfrak{A}(\Gamma_0, \Gamma) = \mathfrak{A}(\Gamma_0)$.

Now let B_n be represented as a group of automorphisms of F_n as before. Let $\pi: F_n \to \Gamma_0$ be given by $\pi(X_i) = e_i$ so that K, the kernel of π, is the normal closure of the elements $\{X_i^m, i = 1, 2, \ldots, n, X_1 X_2 \ldots X_n\}$. K is invariant under the B_n-automorphisms and so π induces a homomorphism $\mu_2: B_n \to \mathfrak{A}(\Gamma_0)$. At this stage, we note the following result which will be required later.

LEMMA 2. μ_2 *is surjective.*

Proof. As noted in Section 2, any $\phi \in \mathfrak{A}(\Gamma_0)$ is induced by an automorphism Φ of F_n where $\Phi(X_i) = \lambda_i X_{\phi(i)} \lambda_i^{-1}$, $i = 1, 2, \ldots, n$ and

$$\Phi(X_1X_2\dots X_n) = \lambda(X_1X_2\dots X_n)\lambda^{-1}.$$

Thus $i_{\lambda^{-1}} \circ \Phi \in B_n$. Let $\pi(\lambda) = \ell \in \Gamma_0$. Now $\mathfrak{I}(\Gamma_0) \subseteq \mu_2(B_n)$ [12]. Let $\tau \in B_n$ be such that $\mu_2(\tau) = i_\ell$. Then $\mu_2(\tau \circ i_{\lambda^{-1}} \circ \Phi) = \phi$.

For this μ_2, Lemma 1 yields

THEOREM 3. *There is a representation of* B_n *in* $Sp(2\gamma,\mathbb{Z})$ *where* γ *is given by* (6) *for all* $m|n$, $n \geq 4$ *and if* $n = 4$, $m = 4$.

Since $F_n \subseteq F_{n'}$ for $n \leq n'$, there is an embedding of B_n into $B_{n'}$ given by $\sigma \mapsto \sigma'$ where $\sigma'(X_i) = \sigma(X_i)$ for $i = 1, 2, \dots, n$ and

$$\sigma'(X_i) = X_i \qquad \text{for } i = n+1, \dots, n'.$$

Thus

THEOREM 4. *There is a representation of* B_n *in* $Sp(2\gamma,\mathbb{Z})$ *where*

$$2\gamma = (n'-2)(m-1),$$

where $n' \geq n$, $m|n'$, $n' \geq 4$ *and if* $n' = 4$, $m = 4$.

In the above $\hat{j}(\mathbb{Z}_m)$ is a cyclic subgroup of Mod Γ corresponding to the branched cyclic covering $U/\Gamma \to U/\Gamma_0$ of the sphere U/Γ_0, branched over n points. In [4], homeomorphisms of the n-punctured sphere are lifted to fiber-preserving homeomorphisms of the surface U/Γ, and a presentation of the resulting subgroup of the mapping class group of U/Γ is obtained. This subgroup is the normaliser of the cyclic subgroup of order m corresponding to the branched cyclic covering and under the Nielsen isomorphism is isomorphic to $\mathscr{N}(\hat{j}(\mathbb{Z}_m))$. [On p. 438 of [4] one should also have the restriction that $k|n$.] Theorem 3 is immediately deducible from the results in [4] and the representations, by the above argument, in the two cases, are equivalent.

Remark. Theorem 3 is also proved in [13] and again it can be shown that the representations obtained are equivalent.

3. FINITE REPRESENTATIONS OF MOD Γ WHERE Γ HAS SIGNATURE (g;—;0)

Permutation representations of Mod Γ can be obtained as follows (see [2], [8]). Let G be a fixed finite group. Let

$$\mathscr{M}(G) = \{K\colon K \text{ a normal subgroup of } \Gamma \text{ such that } \Gamma/K \cong G\}.$$

Let Π_G denote the homomorphism Mod $\Gamma \to S(\mathscr{M}(G))$ given by $\Pi_G(\bar{\phi})(K) = \phi(K)$, $\phi \in \mathfrak{A}(K)$. Let $\mathscr{G} =$ Image of Π_G.

If Γ is generated by $a_1, b_1, \dots, a_g, b_g$ where $\prod_{j=1}^{g} [a_j, b_j] = 1$, these elements map onto $A_1, B_1, \dots, A_g, B_g$ in the abelian group $\Gamma/[\Gamma,\Gamma]\Gamma^p \cong \mathbb{Z}_p^{2g}$. As a vector space over $\mathbb{Z}_p$, this can be equipped with a bilinear form defined with respect to the basis $A_1, A_2, \ldots, A_g$, $B_1, B_2, \ldots, B_g$ by the matrix J where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, making it into a 2g-dimensional symplectic space V over $\mathbb{Z}_p$. $[\Gamma,\Gamma]\,\Gamma^p$ being characteristic, $\phi \in \mathfrak{A}(\Gamma)$ induces an automorphism ϕ^* of $\mathbb{Z}_p^{2g}$ which is an isometry of V. Also the map $\mathrm{Mod}\,\Gamma \to \mathrm{Sp}(2g, \mathbb{Z}_p)$ is onto [14].

THEOREM 5. *If* $G = A$ *is an elementary abelian* p*-group of rank* $r < 2g$ *then* $\mathcal{G} \cong \mathrm{P\,Sp}(2g, \mathbb{Z}_p)$.

Proof. If $K \in \mathcal{M}(A)$, then $K \supseteq [\Gamma,\Gamma]\,\Gamma^p$ and the elements of $\mathcal{M}(A)$ are in one-to-one correspondence with subspace of dimension $2g - r$ in V. Thus $\mathcal{G}$ is isomorphic to the induced action of $\mathrm{Sp}(2g,\mathbb{Z}_p)$ on these subspaces. Via the orthogonal complement, these subspaces are in one-to-one correspondence with the subspaces of dimension r and we can assume that $r \leq g$. For every non-zero vector $\nu \in V$, there exist subspaces of dimension r such that their intersection is $\langle \nu \rangle$. Thus, if an isometry T of $\mathrm{Sp}(2g,\mathbb{Z}_p)$ fixes all r-dimensional subspaces, it fixes all one dimensional subspaces and so belongs to the centre of $\mathrm{Sp}(2g,\mathbb{Z}_p)$. Thus

$$\mathcal{G} \cong \mathrm{P\,Sp}(2g, \mathbb{Z}_p).$$

COROLLARY 6. *There are* N (r) *orbits in this permutation representation where* N (r) *is the number of isometry classes of subspaces of dimension* r.

If G is soluble, there will be a characteristic subgroup G_1 such that $A = G/G_1$ is an elementary abelian p-group. Let

$$\mathcal{M}(A, G) = \{N \subseteq \Gamma : N = p_K^{-1}(G_1) \text{ for } K \in \mathcal{M}(G)\}$$

where p_K is any epimorphism $\Gamma \to G$ with kernel K. N is uniquely determined by K since G_1 is characteristic. Thus we have $\Pi_{A,G} : \mathrm{Mod}\,\Gamma \to S(\mathcal{M}(A, G))$. Now $\mathcal{M}(A, G) \subseteq \mathcal{M}(A)$ and if $N \in \mathcal{M}(A,G)$ then every N' in the orbit of N under $\Pi_A(\mathrm{Mod}\ \Gamma)$ also belongs to $\mathcal{M}(A,G)$. Thus $\Pi_{A,G}$ (Mod Γ) is just $\mathrm{Sp}(2g,\mathbb{Z}_p)$ restricted to act on certain isometry classes of subspaces of the symplectic space V. If $A = \mathbb{Z}_p^{2g}$, $\mathcal{M}(A,G)$ consists of just one element $[\Gamma,\Gamma]\,\Gamma^p$. If rank of $A < 2g$, we can assume as before that the subspaces have dimension $r \leq g$. Again, for any non-zero vector ν of V and isometry class of subspaces of dimension r, there are two subspaces in that class whose intersection is $\langle \nu \rangle$. Thus if an isometry acts trivially on any isometry class of subspaces, it acts trivially on all isometry classes. Thus for $r < 2g$, $\Pi_{A,G}(\mathrm{Mod}\,\Gamma) \cong \mathrm{P\,Sp}(2g, \mathbb{Z}_p)$.

If the orbits of $\mathcal{M}(A,G)$ are $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ let $\mathcal{K}_i = \{K \in \mathcal{M}(G) : p_K^{-1}(G_1) \in \mathcal{M}_i\}$. Taking $\mathcal{G}$ acting on one $\mathcal{K}_i$ at a time, the situation is described in [8]. From that result one obtains that $\mathcal{G}$ is a subgroup of the generalised wreath product $(Q_1, Q_2, \ldots, Q_k) \wr \mathrm{P\,Sp}(2g, \mathbb{Z}_p)$ where Q_i is isomorphic to the action of $\mathrm{Mod}(\Gamma, N_i)$ for $N_i \in \mathcal{M}_i$ acting on $\mathcal{K}_{ii} = \{K \in \mathcal{M}(G) : p_K^{-1}(G_1) = N_i\}$.

The subgroup $L = \bigcap_{K \in \mathcal{M}(G)} K$ is characteristic of finite index in Γ. Any such subgroup will lead to a finite representation of Mod Γ. In this connection we note the following result.

THEOREM 7. *Let* L *be a characteristic subgroup of* Γ. *Then if* H = Γ/L *either* H *is perfect or* H/[H,H] $\cong \mathbb{Z}^{2g}$ or $\mathbb{Z}_m^{2g}$ *for some* m.

Proof. Let A = H/[H,H] and let $\Pi: \Gamma \to A$. Π induces a homomorphism Π^*: Mod Γ → Aut (A). Let the images of the standard generators of Γ in A be $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ and the image of Π^* be C.

Various elements of $\mathfrak{A}(\Gamma)$ are known explicitly (see [9], [3]) and from these, we see that there are automorphisms of C which map each α_i onto each β_j or its inverse. Thus the order of all these generators in A must be the same, be it finite or infinite. Thus A is a factor group of $\mathbb{Z}_m^{2g}$ or $\mathbb{Z}^{2g}$. If it is a proper factor group, then there must be an additional relation holding in A which can be written in the form $\alpha_i^k = W = W(\alpha_1, \alpha_2, \ldots, \hat{\alpha}_i, \ldots \alpha_g, \beta_1, \ldots, \beta_g)$, where, if applicable, $k < m$, or a similar relation involving β_i^k. Now there are automorphisms of Γ which map a_i into $a_i b_i$ and leave all others fixed, and map b_i into $b_i a_i^{-1}$ and leave all others fixed. Thus in A, $(\alpha_i \beta_i)^k = W$ and so $\beta_i^k = 1$ and similarly for the other. This is a contradiction. Thus $A \cong \mathbb{Z}_m^{2g}$ or $\mathbb{Z}^{2g}$.

4. FINITE SYMPLECTIC REPRESENTATION OF B_n

In [5], Cohen utilises machinery set up by Hurwitz [10] for describing the action of the braid group on equivalence classes of Riemann surfaces with a fixed number of branch points and fixed structure over the 2-sphere, to obtain a representation of B_n on P Sp $((n-2)/2, \mathbb{Z}_3)$.

We briefly describe the general approach (see also [12]). Every compact Riemann surface is a branched-covering of the 2-sphere. Such a covering is determined topologically by the number of sheets m, the number of branch points n and a set of permutations T_i, $i = 1,2,\ldots,n$ on m objects (the sheets) which describe how the sheets hang together at the branch points. The T_i generate a transitive permutation group and are such that $T_1 T_2 \ldots T_m = 1$. A renumbering of the sheets will not affect the covering and so two coverings are defined to be topologically equivalent if and only if the sets of permutations are conjugate in S_m.

Cohen actually considers representations of the monodromy group defined by Hurwitz which is isomorphic to the braid group $B_n(S^2)$. This is a quotient group of Artin's braid group B_n.

Let C denote the set of equivalence classes of coverings of S^2 with fixed number m of sheets, n of branch points and such that each defining permutation has the same cycle structure. Pick a representative set of permutations $\{T_1, T_2, \ldots, T_n\}$ for an element of C and let θ be the mapping: $F_n \to \langle T_1, T_2, \ldots, T_n \rangle$ given by $\theta(X_i) = T_i$. A representation χ of B_n in the permutation group S (C) is then obtained by defining $\chi(\sigma)\{T_1, T_2, \ldots, T_n\} = \{\theta\sigma(X_1), \ldots, \theta\sigma(X_n)\}$.

In the case where $m = 3$ and the permutations are all transpositions,

$$\chi(B_n) \cong P\,Sp((n-2)/2, \mathbb{Z}_3)$$

[5].

The above situation will now be described in terms of Fuchsian groups. The covering is of the form $U/\Gamma_1 \to U/\Gamma_0$ where Γ_0 has signature of the form $(0; \ell^{(n)}; 0)$ and is generated by $e_1, e_2, \ldots, e_n$. Γ_0 acts as a permutation group on the left Γ_1-cosets and the permutations T_i are the images of the generators e_i in this permutation group (see e.g. [15]). Let G denote the subgroup of S_m generated by these permutations. If G acts on $\{1,2,\ldots,m\}$, 1 corresponds to the coset Γ_1 and so Γ_1 is the inverse image of the stabiliser of 1 in G. Thus

$$C = \{K_0 : K_0 \text{ is a normal subgroup of } \Gamma_0, \Gamma_0/K_0 \cong G$$
$$\text{and each element } e_i K_0 \text{ has the same fixed cycle structure}\}.$$

The permutation representation of B_n on C is then just the natural representation of Mod Γ_0 on C, similar to that in Section 9.

With this description and our earlier results, an alternative proof of the result in [5] is obtained and the possibility of generalisation discussed.

THEOREM 8. *For* $n \geq 6$, $P\,Sp((n-2)/2, \mathbb{Z}_3)$ *is an epimorphic image of* B_n.

Proof. Let $m = 3$ and all permutations be transpositions so that $G \cong S_3$. Clearly n must be even, so that $n = 2n'$. Thus χ is equivalent to the mapping

$$B_n \to \mathfrak{A}(\Gamma_0) \to \text{Mod}\,\Gamma_0 \to S(C)$$

where Γ_0 has signature $(0; 2^{(n)}; 0)$. Now Γ_0 contains a torsion-free normal subgroup Γ of index 2 which is invariant under all the elements of $\mathfrak{A}(\Gamma_0)$. Thus from the exact sequence

$$1 \to \Gamma \to \Gamma_0 \xrightarrow{j} Z_2 \to 1$$

the mapping $B_n \to \text{Mod}\,\Gamma_0$ factors through $\mathcal{N}(\hat{j}(Z_2))$, the normaliser of $\hat{j}(\mathbb{Z}_2)$ in Mod Γ (see (4) and (5)). From the Riemann-Hurwitz relation, Γ has signature $(n'-1; —; 0)$. From the previous section we have a representation of Mod Γ in $S(\mathcal{M}(\mathbb{Z}_3))$. Now every element of C is an element of $\mathcal{M}(\mathbb{Z}_3)$ and there is an embedding $S(C) \to S(\mathcal{M}(Z_3))$. The following diagram then commutes

$$\begin{array}{ccc}
B_n & & \\
\downarrow & \searrow & \\
\text{Mod}\,\Gamma_0 & \leftarrow \mathcal{N}(\hat{j}(Z_2)) \subseteq & \text{Mod}\,\Gamma \\
\downarrow & & \downarrow \\
S(C) & \longrightarrow & S(\mathcal{M}(\mathbb{Z}_3))
\end{array}$$

Thus the image of B_n in S (C) is isomorphic to a subgroup of P Sp ((n − 2)/2, $\mathbb{Z}_3$) by Theorem 5.

To complete the proof and show that $\chi(B_n)$ is the whole of P Sp (n′ − 1, $\mathbb{Z}_3$), we use the results of Section 2 and some elementary symplectic geometry (see e.g. Chapter III of [1]). The mapping χ factors through the homomorphism μ_3: Mod $\Gamma \to$ Sp (2g, $\mathbb{Z}_3$) where 2g = n − 2, which is determined by the mapping which carries the standard generators of Γ onto the symplectic basis

$$A_1, B_1, \ldots, A_g, B_g$$

of the symplectic space V of dimension 2g over the finite field $\mathbb{Z}_3$. Let B denote the bilinear form on V and for X ∈ V, the transvection

$$\sigma_X(Y) = Y + B(X, Y)X.$$

The group of isometries of V, Sp (2g, $\mathbb{Z}_3$) is generated by the transvections. We will make frequent use of the following fact: for t ∈ Sp (2g, $\mathbb{Z}_3$), $t\sigma_X t^{-1} = \sigma_{t(X)}$.

Let D denote the image of B_n in Sp (2g, $\mathbb{Z}_3$). The result will follow once we have shown that all transvections lie in D. Recall from Lemma 2, that D is the image of the whole of $\mathcal{N}(\hat{j}(\mathbb{Z}_2))$ since $B_n \to \mathcal{N}(j(\mathbb{Z}_2))$ is surjective. But, by Theorem 3, the comments there and [3], the action of generators of $\mathcal{N}(\hat{j}(\mathbb{Z}_2))$ on the standard generators of Γ is determined. It follows that D contains the following isometries: σ_{A_i} for i = 1,2,...,g, σ_{B_1}, σ_{B_g} and $t_1, t_2, \ldots, t_{g-1}$ where

$$t_i(A_i) = A_i - B_i + B_{i+1} \quad \text{and} \quad t_i(A_{i+1}) = B_i + A_{i+1} - B_{i+1}$$

and t_i fixes the others.

We proceed by induction on i where $V_i = \langle A_1, B_1, \ldots, A_i, B_i \rangle$ showing that D contains all transvections corresponding to vectors in V_i. In V_1, we need only consider the four vectors A_1, B_1, $A_1 \pm B_1$. $\sigma_{A_1}, \sigma_{B_1}$ are already in D, $\sigma_{A_1}\sigma_{B_1}\sigma_{A_1}^{-1} = \sigma_{\sigma_{A_1}(B_1)} = \sigma_{A_1+B_1} \in D$. Likewise $\sigma_{B_1}(A_1) = A_1 - B_1$ so that $\sigma_{A_1-B_1} \in D$. Note that, using suitable combinations of $\sigma_{A_i}, \sigma_{B_i}$, any non-zero vector in $\langle A_i, B_i \rangle$ can be carried into A_i.

Now assume D contains all transvections corresponding to vectors in V_{i-1}, and consider V_i. First $\sigma_{A_i} \in D$. Now

$$t_{i-1}\sigma_{A_i}^{-1}t_{i-1}(A_{i-1}) = A_{i-1} - A_i \quad \text{and} \quad \sigma_{A_i}^{-1}t_{i-1}^{-1}\sigma_{A_{i-1}}\sigma_{B_{i-1}}(A_{i-1} - A_i) = -B_i$$

so that $\sigma_{B_i} \in D$. Now consider $X = Y + \alpha A_i + \beta B_i$ where $Y \in V_{i-1}$ and not both α, β are zero. As noted above we can map X into $Y + A_i$. If

$$Y = Z + \gamma A_{i-1} + \delta B_{i-1}, \qquad Z \in V_{i-2}$$

where not both γ, δ are zero, then X can be mapped into $Z - A_{i-1} + A_i$. If both γ, δ are zero, first apply t_{i-1} to $Y + A_i$ and then repeat the above steps. Using the inverse of the element described above, $Z - A_{i-1} + A_i$ is carried into $Z - A_{i-1} \in V_{i-1}$. This completes the inductive step and D = Sp (2g, $\mathbb{Z}_3$).

If we consider the more general situation of any number m of sheets, but all permutations still transpositions, G will be isomorphic to S_m and the representation will factor through Mod $\Gamma \to S(\mathcal{M}(A_m))$ where Γ has signature (g; —; 0) and A_m is the alternating group on m elements. Since for $m \geq 5$, A_m is simple this seems a difficult problem (c.f. [6]). For m = 4, A_m has a characteristic subgroup isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with quotient $\mathbb{Z}_3$ and the image of Mod Γ in $S(\mathcal{M}(A_4))$ is a subgroup of a wreath product with quotient group $P\,Sp(2g, \mathbb{Z}_3)$. Using the same methods as in the above theorem, $P\,Sp(2g, \mathbb{Z}_3)$ is a quotient of the image of B_n in this case.

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