OPERATORS OF CLASS C_{00} OVER MULTIPLY-CONNECTED DOMAINS

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INTRODUCTION

Let R be a domain in the complex plane bounded by n + 1 nonintersecting analytic Jordan curves, let $C(\partial R)$ be the space of continuous functions on the boundary of R, and let $\text{Rat}(\bar{R})$ be the uniform closure in $C(\partial R)$ of the space of rational functions with poles off of $\bar{R}$. Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the algebra of bounded linear operators on $\mathscr{H}$. M. B. Abrahamse and R. G. Douglas [4] have recently initiated the study of contractive unital $\mathscr{L}(\mathscr{H})$-valued representations of $\text{Rat}(\bar{R})$; that is, algebra homomorphisms

$$\sigma: \text{Rat}(\bar{R}) \to \mathscr{L}(\mathscr{H})$$

such that $\|\sigma(f)\| \leq \|f\|$ and $\sigma(1) = I_{\mathscr{H}}$. The Sz.-Nagy-Foiaş model theory for contraction operators [11] can be viewed as statements about representations of the disc algebra $\text{Rat}(\bar{D})$ (D the unit disk). Thus the theory begun by Abrahamse and Douglas can be viewed as a generalization of the Sz.-Nagy-Foiaş theory to multiply-connected domains.

In this paper we shall deal with some of the specific questions concerning such representations raised by Abrahamse and Douglas in their paper. A representation σ is said to be of class C_{00} if σ is continuous from the topology of bounded pointwise convergence on R in $\text{Rat}(\bar{R})$ to the double strong operator topology in $\mathscr{L}(\mathscr{H})$. A representation is said to be of class C_0 if its unique extension to $H^\infty(R)$ has a nontrivial kernel. It can be shown that these definitions are consistent with those of Sz.-Nagy and Foiaş for the case that R = D. In Section 2 of this paper we show that if $\sigma: \text{Rat}(\bar{R}) \to \mathscr{L}(\mathscr{H})$ is a representation of class C_{00} such that $\sigma(z) = N + K$, where N is normal with spectrum contained in the boundary of R and K is trace class, then σ is of class C_0. This answers Question 6 of [4].

Associated with any completely contractive unital representation of $\text{Rat}(\bar{R})$ (see the definition in Section 3) is a functional model analogous to the Sz.-Nagy-Foiaş functional model for a representation of the disc algebra. As in the disc case, the simplest form of the model occurs when the representation is C_{00}. The model is determined by a characteristic function, which in the disc case is uniquely determined by the representation. In the general case, as was pointed out by

Received December 2, 1976. Revision received May 16, 1977.

Portions of this paper were written while the author was supported by the N.S.F. as a participant in the Operator Theory Institute at the University of New Hampshire during the summer of 1976.

Michigan Math. J. 25 (1978).

Abrahamse and Douglas, there is a high degree of nonuniqueness; specifically, essentially different models can give rise to unitarily equivalent representations. In Section 3 of this paper we analyze this nonuniqueness for the case where $\mathscr{H}$ is finite-dimensional and $\sigma(z)$ has distinct eigenvalues. This enables us to show that not even the rank of the model is a unitary invariant of the representation, and thus to give negative answers to Questions 1 and 2 of [4].

In Section 4 we obtain an estimate on the possible rank of a model inducing a given representation. In particular, it will follow that a model inducing a representation unitarily equivalent to the representation arising from a rank 1 model can have rank at most $n + 1$ (the number of boundary components of R). It will also follow that the rank is a unitary invariant if it is infinite.

Section 5 is an attempt to shed some light on Question 4 of Abrahamse and Douglas. We obtain an implicit formula for the characteristic function of a model of a special type. In the case $R = D$, the formula specializes to a well-known formula which can be used to obtain the characteristic function completely in terms of the representation; in the general case, the formula of necessity (in view of the above-mentioned nonuniqueness) involves some quantities which make sense only in terms of a specific model. For those familiar with the disc model theory, this should provide insight into the complications arising from the multiple-connectivity of the underlying region.

Needed preliminaries concerning function theory on multiply-connected domains are given in Section 1. For simplicity, we choose to define the elements of the spaces needed in the sequel as functions analytic on R except for certain systematic jump discontinuities across cuts in R, as is done in [1], [4], rather than use the language of hermitean holomorphic vector bundles [3], [4]. The exposition proceeds under the assumption that the reader is familiar with the literature on H^p theory for multiply-connected domains for the scalar case [2], [12]. Many of the needed results are straightforward vector generalizations of results of [1], and detailed proofs will be omitted.

1. THE HILBERT SPACES $H^2_\alpha(R)$ AND OPERATOR-REPRODUCING KERNEL FUNCTIONS

Let R be a bounded domain in the complex plane bounded by $n + 1$ nonintersecting analytic Jordan curves, let $\mathscr{H}$ be a complex Hilbert space and let

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathscr{U}(\mathscr{H})^n$$

be an n-tuple of unitary operators on $\mathscr{H}$. We will define a Hilbert space $H^2_\alpha(R)$ which is a vector-valued generalization of the spaces $H^2_\alpha(R)$ defined in [1], [14]. To achieve this, let $C_1, \ldots, C_n$ be n pairwise disjoint cuts in the region R such that if C is the union of the C_k for $k = 1, \ldots, n$, then $R \setminus C$ is simply connected. For $k = 1, \ldots, n$, let U_k be an open set in R such that $\partial U_k \cap C = C_k$ and U_k lies on one side of the cut C_k. For $\alpha \in \mathscr{U}(\mathscr{H})^n$ as above, let $H_\alpha(R)$ be the set of $\mathscr{H}$-valued functions f on R such that $\|f\|$ is continuous on R, f is weakly analytic (*i.e.*, $\langle f(z), x\rangle_{\mathscr{H}}$ is analytic for each x in $\mathscr{H}$) on $R \setminus C$, and for w in C_k,

$$\lim_{\substack{z\to w\\ z \text{ in } U_k}} f(z) = \alpha_k f(w)\,,$$

where the limit is in the norm topology of $\mathscr{K}$. Thus $H_\alpha(R)$ is a space of $\mathscr{K}$-valued functions on R which are analytic except for certain systematic jump discontinuities across the cuts $C_1, \ldots, C_n$. If each component $\alpha_i\,(i = 1, \ldots, n)$ of α is the identity $I_{\mathscr{K}}$, so that $\alpha = e_{\mathscr{K}} = (I_{\mathscr{K}}, \ldots, I_{\mathscr{K}})$, then $H_{e_{\mathscr{K}}}(R) \equiv H_{\mathscr{K}}(R)$ is the space of $\mathscr{K}$-valued analytic functions on R. A $\mathscr{K}$-valued function f on R is said to be *norm automorphic* if $\|f\|$ is continuous on R. A function f is norm automorphic and analytic on $R\setminus C$ if f is in $H_\alpha(R)$ for some α in $\mathscr{U}(\mathscr{K})^n$. One then refers to α as the *index* of f.

Let $H^2_\alpha(R)$ be the space of functions f in $H_\alpha(R)$ such that there is a harmonic function u on R with $\|f\|^2 \le u$. Choose a point t in R, and for f in $H^2_\alpha(R)$ define $\|f\|$ as the infinum of the numbers $u(t)^{1/2}$ with u harmonic and $\|f\| \le u$. Then $H^2_\alpha(R)$ is a Hilbert space. Moreover, if m is harmonic measure for the point t, then every function f in $H^2_\alpha(R)$ defines via nontangential limits a boundary function $\hat{f}$ in $L^2_{\mathscr{K}}(m)$ (weakly measurable $\mathscr{K}$-valued functions on ∂R with norms square-integrable with respect to m) and the norm of f in $H^2_\alpha(R)$ is the same as the norm of $\hat{f}$ in $L^2_{\mathscr{K}}(m)$. Thus the space $H^2_\alpha(R)$ can be viewed as a closed subspace H^2_α of $L^2_{\mathscr{K}}(m)$.

For two Hilbert spaces $\mathscr{K}_1$ and $\mathscr{K}_2$, an element $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathscr{U}(\mathscr{K}_1)^n$ and an element $\beta = (\beta_1, \ldots, \beta_n)$ of $\mathscr{U}(\mathscr{K}_2)^n$, let $H_{\beta,\alpha}(R)$ be the set of functions F defined on R with values in $\mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ (bounded linear operators from $\mathscr{K}_1$ into $\mathscr{K}_2$) such that $\|F\|$ is continuous on R, F is weakly analytic ($\langle F(z)x, y\rangle_{\mathscr{K}_2}$ is analytic for each x in $\mathscr{K}_1$ and y in $\mathscr{K}_2$) on $R\setminus C$, and for w in C_k,

$$\lim_{\substack{z\to w\\ z \text{ in } U_k}} F(z) = \beta_k F(w)\,\alpha_k\,,$$

where the limit is in the strong topology of $\mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is in $\mathscr{U}(\mathscr{K}_1)^n$ and α^* is defined to be $\alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$, it is easily seen that H_{β,α^*} maps H_α into H_β under pointwise multiplication. If we let $H^\infty_{\beta,\alpha^*}(R)$ be the space of functions F in $H_{\beta,\alpha^*}(R)$ with norm bounded on R, then $H^\infty_{\beta,\alpha^*}(R)$ is a Banach space, each element of which maps $H^2_\alpha(R)$ into $H^2_\beta(R)$ via pointwise multiplication. An element F of $H^\infty_{\beta,\alpha^*}(R)$ determines via nontangential strong limits an element $\hat{F}$ of $L^\infty_{\mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)}$ (essentially bounded measurable $\mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$-valued functions on ∂R) such that the norm of F in $H^\infty_{\beta,\alpha^*}(R)$ is the same as that of $\hat{F}$ in $L^\infty_{\mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)}$. Thus $H^\infty_{\beta,\alpha^*}(R)$ can be identified as a closed subspace $H^\infty_{\beta,\alpha^*}$ of $L^\infty_{\mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)}$.

For $\alpha \in \mathscr{U}(\mathscr{K})^n$, the bundle shift operator S_α on $H^2_\alpha(R)$, defined as multiplication by z on $H^2_\alpha(R)$, has been extensively studied by Abrahamse and Douglas [3]. There it is shown that an operator F mapping $H^2_\alpha(R)$ into $H^2_\beta(R)$ intertwines S_α and S_β if and only if F is multiplication by an element $F = F(z)$ in $H^\infty_{\beta,\alpha^*}(R)$. In particular, the commutant of S_α can be identified as $H^\infty_{\alpha,\alpha^*}(R)$. It is also shown that for each $\alpha \in \mathscr{U}(\mathscr{K})^n$, there is an $E_\alpha \in H^\infty_{\alpha,e_{\mathscr{K}}}(R)$ with $E_\alpha^{-1} \in H^\infty_{e_{\mathscr{K}},\alpha^*}(R)$. It follows that

$$H^2_\alpha(R) = E_\alpha H^2_{\mathscr{K}}(R)\,, \tag{1.1}$$

and therefore many properties of the space $H^2_\alpha(R)$ follow from those of the more familiar $H^2_{\mathscr{K}}(R)$, as in [1].

LEMMA 1.1. *For* w *in* R, α *in* $\mathscr{U}(\mathscr{K})^n$, *the evaluation mapping* $e_\alpha(w)$: $f \to f(w)$ *is a bounded linear transformation of* $H^2_\alpha(R)$ *into* $\mathscr{K}$.

Proof. The result follows from equation (1.1) as for the scalar case done in [1].

If we set $k^\alpha_w = e_\alpha(w)^*$, then for each x in $\mathscr{K}$, $k^\alpha_w x = k^\alpha_w(z)x$ is an element of $H^2_\alpha(R)$, and has the reproducing property $\langle f, k^\alpha_w x\rangle_{H^2_\alpha(R)} = \langle f(w), x\rangle_{\mathscr{K}}$. We refer to $k^\alpha_w(z)$ as the (operator) kernel function for the space $H^2_\alpha(R)$.

THEOREM 1.2. *For any region* R *as above, there exists an* α *in the* n*-torus* T^n *such that for any* w *in* R, $k^\alpha_w(z)$ *has* n *zeros in* R.

Proof. The result is known (see [7, p. 118] or [8]) if $\alpha = (1, \ldots, 1)$ and arclength measure $d|z|$ is used to define the norm of the space $H^2(R)$ rather than harmonic measure m for the point t. The conclusion of the theorem now follows by the analysis of Section 7 of [1].

The following facts concerning kernel functions will be needed in Section 3.

LEMMA 1.3. *For any fixed* w *in* R, *the kernel of* $S^*_\alpha - w$ *is* $\{k^\alpha_w x : x \in \mathscr{K}\}$.

Proof. A simple computation,

$$\langle (S^*_\alpha - \bar{w})k^\alpha_w x, g\rangle = \langle k^\alpha_w x, (S_\alpha - w)g\rangle = \langle x, (w-w)g(w)\rangle_K = 0$$
$$\text{for all } g \text{ in } H^2_\alpha(R),$$

shows that $\{k^\alpha_w x: x \text{ in } K\} \subseteq \ker(S^*_\alpha - \bar{w})$. Conversely, if f is orthogonal to $\{k^\alpha_w x: x \text{ in } K\}$, then $f(w) = 0$, and hence $g(z) = (z-w)^{-1} f(z)$ is in $H^2_\alpha(R)$. Therefore, $f = (S - w)g \in \operatorname{Ran}(S_\alpha - w) \subseteq [\ker(S^*_\alpha - \bar{w})]^\perp$.

Another result of Abrahamse and Douglas basic for this paper is a generalization to multiply-connected regions of the Beurling-Lax theorem characterizing invariant subspaces of the unilateral shift. If Ω is an element of $H^\infty_{\alpha,\beta^*}(R)$, Ω is said to be *inner* if the boundary value function $\hat{\Omega}$ is isometric almost everywhere on ∂R. It is clear that for such a Ω, $\mathscr{M} = \Omega H^2_\beta(R)$ is a closed subspace of $H^2_\alpha(R)$ which is invariant under $\operatorname{Rat}(S_\alpha)$ (multiplication by $\operatorname{Rat}(\bar{R})$ functions on $H^2_\alpha(R)$). Theorem 12 of [3], translated to the language of this paper, is the converse assertion.

THEOREM 1.4. (Abrahamse and Douglas) *Let* α *be an element of* $\mathscr{U}(\mathscr{K})^n$.

(a) *A closed subspace* M *of* $H^2_\alpha(R)$ *is invariant for* $\operatorname{Rat}(S_\alpha)$ *if and only if there are a Hilbert space* $\mathscr{K}'$, *a* β *in* $\mathscr{U}(\mathscr{K}')^n$, *and an inner* $\Omega \in H^\infty_{\alpha,\beta^*}(R)$ *such that* $M = \Omega H^2_\beta(R)$.

(b) *Two such subspaces* $\Omega_1 H^2_{\beta_1}(R)$ *and* $\Omega_2 H^2_{\beta_2}(R)$ *are equal if and only if there exists a unitary operator* Ψ *from* $\mathscr{K}'_1$ *onto* $\mathscr{K}'_2$ *such that*

$$\beta_2 = \Psi\beta_1\Psi^* \quad \textit{and} \quad \Omega_1 = \Omega_2\Psi.$$

(For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathscr{U}(\mathscr{K})^n$ *and* Ψ *unitary,* $\Psi\alpha\Psi^*$ *means* $(\Psi\alpha_1\Psi^*, \ldots, \Psi\alpha_n\Psi^*)$*).*

In Sections 3 and 5 we will need the following result.

THEOREM 1.5. ([5], [7], [8]) *If* R *is a domain as above with* n+1 *boundary components, there exists a complex-valued inner function* ψ *on* R, *such that* ψ *is analytic on a neighborhood of* $\bar{R}$, *has precisely* n+1 *zeros in* R, *and wraps each component of the boundary of* R *once around the unit disk.*

2. C_{00} AND C_0 REPRESENTATIONS OF RAT $(\bar{R})$

In this section we begin our study of contractive unital representations of Rat $(\bar{R})$ into $\mathscr{L}(\mathscr{H})$ discussed in the introduction. A result of W. Mlak [10] implies that, with certain absolute continuity conditions satisfied, any such representation σ has an extension (also denoted by σ) to the algebra $H^\infty(R)$ of bounded analytic functions on R. Representations arising this way are those continuous from the weak-* topology on $H^\infty(R)$ to the weak operator topology on $\mathscr{L}(\mathscr{H})$. For convenience, we assume that this extension has been carried out, so that σ is defined on $H^\infty(R)$. A contractive unital representation σ is said to be of class C_{00} if σ is continuous from the topology of bounded pointwise convergence on R in $H^\infty(R)$ to the double strong operator topology on $\mathscr{L}(\mathscr{H})$; that is, if whenever f_n tends to zero pointwise boundedly on R, then both $\sigma(f_n)$ and $\sigma(f_n)^*$ tend to zero strongly in $\mathscr{L}(\mathscr{H})$. The representation $\sigma: H^\infty(R) \to \mathscr{L}(\mathscr{H})$ is said to be of class C_0 if it has a nontrivial kernel. If σ is a completely contractive unital representation of $H^\infty(R)$ (to be discussed in the next section) then σ has a dilation to $L^\infty(\partial R)$, and an argument similar to that of [11, p. 122–123] shows that any such C_0 representation must also be of class C_{00} (i.e., $C_0 \subseteq C_{00}$). The converse direction is more delicate. For the case R = D, the following is a well-known result of Sz.-Nagy and Foiaş [11, Theorem VIII.11] stated in the language of representations.

THEOREM 2.1. (Sz.-Nagy-Foiaş) *If* ρ *is a contractive unital representation of* Rat $(\bar{D})$ *of class* C_{00} *such that*

(i) *the spectrum of* $\rho(z)$ *does not fill the unit disc* D, *and*

(ii) $I - \rho(z)\rho(z)^*$ *is trace class,*

then ρ *is of class* C_0.

Abrahamse and Douglas [4, Question 6] ask whether this theorem has an analogue for $\mathscr{L}(\mathscr{H})$-valued representations of Rat $(\bar{R})$. We now show that the answer is affirmative, even without assuming the presence of a dilation.

THEOREM 2.2. *If* σ *is a contractive unital representation of* Rat $(\bar{R})$ *belonging to class* C_{00} *such that* $\sigma(z) = N + K$, *where* N *is normal with spectrum contained in the boundary of* R *and* K *is trace class, then* σ *is of class* C_0.

Proof. Let ψ be an inner function as in Theorem 1.5. Since ψ is analytic in a neighborhood of $\bar{R}$ and the spectrum of $\sigma(z) = N + K$ is contained in $\bar{R}$, the operator $\sigma(\psi) = \psi(N + K)$ can be defined by the Riesz-Dunford functional calculus

$$\sigma(\psi) = \psi(N+K) = -\frac{1}{2\pi i}\int_\gamma (N+K-zI)^{-1}\psi(z)\,dz$$

for γ an appropriately chosen contour around $\bar{R}$. Since

$$(N + K - zI)^{-1} - (N - zI)^{-1} = (N + K - zI)^{-1} K (N - zI)^{-1},$$

$$\psi(N + K) = \psi(N) + \frac{1}{2\pi i} \int_\gamma (N + K - zI)^{-1} K (N - zI)^{-1} \psi(z)\, dz = \psi(N) + K_1 .$$

If $\{x_i\}_{i=1}^{\infty}$ is any orthonormal basis for H and U is any unitary operator on H, then

$$\left| \sum_{i=1}^{N} \langle U K_1 x_i, x_i \rangle \right|$$
$$\leq \frac{1}{2\pi} \int_\gamma \left\{ \left| \sum_{i=1}^{N} \langle U (N + K - zI)^{-1} K (N - zI)^{-1} \psi(z) x_i, x_i \rangle \right| \right\} dz$$
$$\leq \frac{1}{2\pi} \int_\gamma \mathrm{Tr}\, \{U (N + K - zI)^{-1} K (N - zI)^{-1} \psi(z)\}\, dz \leq M\, \mathrm{Tr} K ,$$

where $M = \frac{1}{2\pi} \sup_{z \in \gamma} \{\|(N + K - zI)^{-1}\| \, \|(N - zI)^{-1}\|\} \ell(\gamma) < \infty$, and Tr represents the trace norm. By Lemma II.4.1 of [9], it follows that K_1 is also trace class. Since N is normal with spectrum contained in ∂R, it follows that $\psi(N)$ is unitary. Hence $I - \psi(N + K)\psi(N + K)^*$ is also trace class. Define a representation ρ of Rat$(\bar{D})$ by $\rho(f) = \sigma(f \circ \psi)$. Then $\rho(z) = \sigma(\psi) = \psi(N + K)$, and since σ is of class C_{00} as a representation of Rat$(\bar{R})$, it follows easily from the definitions that ρ is C_{00} as a representation of Rat $\bar{D}$. Since $\rho(z) = \psi(N) + K_1$ is a compact perturbation of a unitary operator, the spectrum of $\rho(z)$ cannot fill the unit disk. Theorem 2.1 implies ρ, and hence also σ, is of class C_0.

3. MODELS FOR COMPLETELY CONTRACTIVE UNITAL REPRESENTATIONS OF CLASS C_{00}

One way to construct a contractive unital representation of Rat$(\bar{R})$ of class C_{00} is as follows. Let $\mathscr{K}$ be a complex Hilbert space, let α and β be two elements of $\mathscr{U}(\mathscr{K})^n$, let $\Omega \in H^\infty_{\alpha,\beta^*}(R)$ be inner (see Section 1 for definitions), and let

$$\mathscr{H} = H^2_\alpha(R) \ominus \Omega H^2_\beta(R).$$

Define a representation σ: Rat$(\bar{R}) \to \mathscr{L}(\mathscr{H})$ by $\sigma(f) = T_f = P_{\mathscr{H}} M_f | \mathscr{H}$, where M_f is the operator of multiplication by f and $P_{\mathscr{H}}$ is the orthogonal projection onto $\mathscr{H}$. We note that $\mathscr{H}$ is a semiinvariant subspace for Rat(S_α) in the sense of Sarason [13], and hence the above formula does define a representation of Rat$(\bar{R})$. Identifying $\mathscr{H}$ as a subspace of $L^2_{\mathscr{K}}(m)$ via nontangential boundary values, we note that any such representation σ has the form $\sigma(f) = P_{\mathscr{H}} \tau(f) | \mathscr{H}$, where τ is the *-representation $\tau : f \to M_f$ on $L^2_{\mathscr{K}}(\partial R)$ of $C(\partial R)$, and $\mathscr{H}$ is semiinvariant for $\tau(\mathrm{Rat}(\bar{R}))$. When σ arises from a *-representation τ of $C(\partial R)$ in this way, τ is said to be a ∂R-*dilation* of σ. It is unknown whether every contractive unital representation of Rat $(\bar{R})$ has a ∂R-dilation; however, Arveson [6] has shown that every completely contractive unital (c.c.u.) representation does, and the two classes of representations in fact coincide. A representation σ: Rat $(\bar{R}) \to \mathscr{L}(\mathscr{H})$ is said to be completely contractive if the homomorphism $\sigma \otimes 1$: Rat $(\bar{R}) \otimes M_k \to \mathscr{L}(\mathscr{H}) \otimes M_k$ is contractive for $1 \leq k < \infty$, where M_k is the C*-algebra of $k \times k$ matrices. By results of [3], it can be shown that any c.c.u. representation of class C_{00} can be represented in the form described above, for some inner $\Omega \in H^{\infty}_{\alpha,\beta^*}(R)$. When σ and Ω are related in this way, we say that Ω is the *characteristic function* of a model for σ, and that

$$\mathscr{H} = H^2_\alpha(R) \ominus \Omega\, H^2_\beta(R)$$

is a model space for σ. The model is said to be *minimal* if M_z on $L^2_{\mathscr{X}}(R)$ has no proper reducing subspaces containing the model space $\mathscr{H}$. A minimal model for σ can always be arranged, and hence we will assume all models are minimal. When this is the case, the dimension of $\mathscr{K}$ is referred to as the *rank* of the model.

Two c.c.u. representations σ_i : Rat $(\bar{R}) \to \mathscr{L}(\mathscr{H}_i)$ $(i = 1, 2)$ are said to be *unitarily equivalent* if there is a unitary operator $V: \mathscr{H}_1 \to \mathscr{H}_2$ such that

$$\sigma_2(f)\, V = V \sigma_1(f) \qquad \text{for all f in Rat}(\bar{R}).$$

If $\alpha_i, \beta_i \in \mathscr{U}(\mathscr{K}_i)^n$ and $\Omega_i \in H^{\infty}_{\alpha_i,\beta_i^*}(R)$ is the characteristic function for the model space $\mathscr{H}_i = H^2_{\alpha_i}(R) \ominus \Omega_i H^2_{\beta_i}(R)$, defining the representation σ_i $(i = 1, 2)$, the models induced by Ω_1 and Ω_2 are said to be *unitarily equivalent* if there is a unitary operator $W: H^2_{\alpha_1}(R) \to H^2_{\alpha_2}(R)$ such that $Wf(S_{\alpha_1}) = f(S_{\alpha_2})W$ for every f in Rat $(\bar{R})$, and $W|\mathscr{H}_1$ implements a unitary equivalence between the representations σ_1 and σ_2. Thus unitary equivalence for the representations involves a Hilbert space isomorphism between the representation spaces, while unitary equivalence for the models involves such a Hilbert space isomorphism having additional properties involving the dilation. By results of [3], any unitary operator W: $H^2_{\alpha_1}(R) \to H^2_{\alpha_2}(R)$ implementing a unitary equivalence of the models arises via multiplication by a unitary transformation (also called W) from $\mathscr{K}_1$ onto $\mathscr{K}_2$, and $\alpha_2 = W\alpha_1 W^*$. When the above situation is expressed in terms of the characteristic functions Ω_1 and Ω_2, the uniqueness part of Theorem 1.4 implies that there is a unitary constant operator Ψ: $H^2_{\beta_1}(R) \to H^2_{\beta_2}(R)$ such that $W\Omega_1 = \Omega_2 \Psi$; that is, Ω_1 and Ω_2 *coincide* in the sense of Sz.-Nagy and Foiaş. Conversely, when Ω_1 and Ω_2 coincide in the above sense, the models induced by Ω_1 and Ω_2 are unitarily equivalent.

Let us say that two inner characteristic functions Ω_1 and Ω_2 are *weakly equivalent* if and only if the c.c.u. representations of class C_{00} which they define are unitarily equivalent (but the models they define are not necessarily unitarily equivalent, and thus *a priori* Ω_1 and Ω_2 need not coincide). Trivially, coincidence implies weak equivalence. It is well known to those familiar with the Sz.-Nagy-Foiaş theory that, for the case $R = D$, the converse also holds. As was pointed out by Abrahamse and Douglas, for the general case, the converse fails: Ω_1 and Ω_2 can induce unitarily equivalent representations without coinciding. In this section we study the nature of the nonuniqueness in a special finite-dimensional setting.

Suppose $\mathscr{H} = H^2_\alpha(R) \ominus \Omega H^2_\beta(R)$ is a C_{00} model space of finite dimension k, and $\sigma(z)^* = P_{\mathscr{H}} M_z^* | \mathscr{H}$ has k distinct eigenvalues $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k$ in the image of R under complex conjugation. Since $\sigma(z)^* = S_\alpha^* | \mathscr{H}$, it follows from Lemma 1.3 that there are k vectors $x_1, \ldots, x_k$ in $\mathscr{H}$ such that $\mathscr{H} = \vee \{k^\alpha_{w_i} x_i : i = 1, \ldots, k\}$. In the next theorem we consider the question of when the characteristic functions of two such models coincide, and when they are weakly equivalent.

THEOREM 3.1. *Let* α_i *and* β_i *be elements of* $\mathscr{U}(\mathscr{K}_i)^n$, *and* Ω_i *be an inner function in* $H^\infty_{\alpha_i,\beta_i}(R)$ $(i = 1,2)$ *such that* $\mathscr{H}_1 = H^2_{\alpha_1}(R) \ominus \Omega_1 H^2_{\beta_1}(R)$ *is spanned by* k *elements of the form* $\{k^{\alpha_i}_{w_i} x_i : i = 1, \ldots, k\}$, *where* $w_1, \ldots, w_n$ *are distinct points in* R *and* $x_1, \ldots, x_n$ *are unit vectors in* $\mathscr{K}_1$, *and* $\mathscr{H}_2 = H^2_{\alpha_2}(R) \ominus \Omega_2 H^2_{\beta_2}(R)$ *is similarly spanned by* k *elements* $\{k^{\alpha_2}_{\eta_i} y_i : i = 1, \ldots, k\}$, *where* $\eta_1, \ldots, \eta_k$ *are distinct points in* R *and* $y_1, \ldots, y_k$ *are unit vectors in* $\mathscr{K}_2$. *Then:*

(a) Ω_1 *and* Ω_2 *coincide if and only if, possibly after a renumbering,* $w_i = \eta_i$ *for* $i = 1, \ldots, k$, *and there is a unitary operator* V *from* $\mathscr{K}_1$ *onto* $\mathscr{K}_2$ *such that*

(3.1) $V \alpha_1 V^* = \alpha_2$, *and*

(3.2) $V x_i = \omega_i y_i$, *where* ω_i *is a complex number of modulus* 1, $i = 1, \ldots, k$.

(b) Ω_1 *and* Ω_2 *are weakly equivalent if and only if, possibly after a renumbering,* $w_i = \eta_i$ *for* $i = 1, \ldots, k$, *and*

$$(3.3)\quad \begin{aligned} &\{\|k^{\alpha_1}_{w_i} x_i\| \, \|k^{\alpha_1}_{w_j} x_j\|\}^{-1} \langle k^{\alpha_1}_{w_i}(w_j) x_i, x_j \rangle_{\mathscr{K}_1} \\ &\qquad = \omega_i \bar{\omega}_j \{\|k^{\alpha_2}_{w_i} y_i\| \, \|k^{\alpha_2}_{w_j} y_j\|\}^{-1} \langle k^{\alpha_2}_{w_i}(w_j) y_i, y_j \rangle_{\mathscr{K}_2}, \end{aligned}$$

where ω_i *is a complex number of modulus* 1, $i,j = 1, \ldots, k$.

Proof. The computation, for $g \in H^2_{\beta_1}(R)$,

$$0 = \langle k^{\alpha_1}_{w_i} x_i, \Omega_1 g \rangle = \langle x_i, \Omega_1(w_i) g(w_i) \rangle = \langle \Omega_1(w_i)^* x_i, g(w_i) \rangle,$$

shows that $\langle x_i \rangle = \ker \Omega_1(w_i)^*$, and that $\ker \Omega_1(w)^*$ is trivial for any w in R not one of $w_1, \ldots, w_k$. Similarly, $\langle y_i \rangle = \ker \Omega_2(\eta_i)^*$, and $\ker \Omega_2(\eta)^*$ is trivial for any η not one of the $\eta_1, \ldots, \eta_k$. If Ω_1 and Ω_2 coincide, say $V\Omega_1 = \Omega_2 U$ for unitary constant operators U and V, then for any w in R, $V \ker \Omega_1(w)^* = \ker \Omega_2(w)^*$. Hence we must have $w_i = \eta_i$ for some enumeration, $i = 1, \ldots, k$, and the unitary operator V maps the unit vector x_i to a unit vector in $\ker \Omega_2(w_i)^*$. The condition $V \alpha_1 V^* = \alpha_2$ is part of our definition of coincidence.

Conversely, if $w_i = \eta_i$ and there is such a unitary operator V, then V maps $H^2_{\alpha_1}(R)$ onto $H^2_{\alpha_2}(R)$ and a simple computation gives

$$V k^{\alpha_1}_{w_i} x_i = \omega_i k^{\alpha_2}_{\eta_i} y_i, \qquad i = 1, \ldots, k.$$

Thus the models $\mathscr{H}_1$ and $\mathscr{H}_2$ are unitarily equivalent, and Ω_1 and Ω_2 must coincide, and (a) follows.

If Ω_1 and Ω_2 are weakly equivalent, then there is a unitary operator $U : \mathscr{H}_1 \to \mathscr{H}_2$ such that $\sigma_2(f) U = U \sigma_1(f)$ for all f in $\mathrm{Rat}(\bar{R})$, where σ_1 and σ_2 are the representations defined by Ω_1 and Ω_2. Since $\bar{w}_1, \ldots, \bar{w}_n$ are the eigenvalues of $\sigma_1(z)^*$ and $\bar{\eta}_1, \ldots, \bar{\eta}_n$ are the eigenvalues of $\sigma_2(z)^*$, we must have $w_i = \eta_i$ for some enumeration, $i = 1, \ldots, k$. Since U must send a unit eigenvector for $\sigma_1(z)^*$ with corresponding eigenvalue $\bar{w}_i$ to a corresponding quantity for $\sigma_2(z)^*$, we must have $U: \{\|k_{w_i}^{\alpha_1} x_i\|\}^{-1} k_{w_i}^{\alpha_i} x_i \to \omega_i \{\|k_{w_i}^{\alpha_2} y_i\|\}^{-1} k_{w_i}^{\alpha_2} y_i$ for some number ω_i of modulus 1, $i = 1, \ldots, k$. Equation (3.3) is simply the statement that U is unitary when considered on these special vectors.

Conversely, if $w_i = \eta_i$ for $i = 1, \ldots, k$ and (3.3) is satisfied, the operator $U: k_{w_i}^{\alpha_1} x_i \to \omega_i k_{\eta_i}^{\alpha_2} y_i$, $i = 1, \ldots, k$, extends by linearity to be a unitary operator of $\mathscr{H}_1$ onto $\mathscr{H}_2$ establishing the unitary equivalence between σ_1 and σ_2.

We now illustrate Theorem 3.1 with several examples.

Example 1. Let R be equal to the unit disc D ($n = 0$). Then the n-tuples α_i, β_i ($i = 1,2$) are vacuous. The kernel function for $H^2_{\mathscr{X}}(D)$ is of the form

$$k_w(z) = (1 - z\bar{w})^{-1} I_{\mathscr{X}},$$

and for any x in $\mathscr{X}$, $\|k_w x\| = (1 - |w|^2)^{-1/2}\|x\|$. It follows easily that condition (3.3) is equivalent to conditions (3.1) and (3.2), and thus we recover the Sz.-Nagy-Foiaş uniqueness theorem for this special situation. We see that the nonuniqueness in the general situation is partly due to the plethora of different kernel functions.

Example 2. Choose R with connectivity $n \geq 1$, let $\alpha_i \in T^n$, and let $\mathscr{H}_i \subset H^2_{\alpha_i}(R)$ be spanned by the single vector $k_w^{\alpha_i}$ for some w in R ($i = 1,2$). Then trivially the representations σ_1 and σ_2 are unitarily equivalent, but the models are not unitarily equivalent unless the added condition (3.1) is satisfied. However, the models are similar; that is, there is a similarity mapping W of $H^2_{\alpha_1}(R)$ onto $H^2_{\alpha_2}(R)$ such that $Wf(S_{\alpha_1})^* = f(S_{\alpha_2})^* W$ for all f in Rat $(\bar{R})$, and $W|H_1$ implements a unitary equivalence between σ_1 and σ_2. (This example was pointed out to the author by Bruce Abrahamse.)

Example 3. Choose R with connectivity $n \geq 1$, and then choose α_1 in T^n and two points w_1 and w_2 in R so that $k_{w_1}^{\alpha_1}(w_2) = 0$ as in Theorem 1.2. Let $\alpha_2 = \alpha_1 \oplus \alpha_1$ (acting on $\mathbb{C} \oplus \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers), $x_1 = x_2 = 1$, and $y_1 = 1 \oplus 0$, while $y_2 = 0 \oplus 1$. Then it is easy to check that (3.3) is satisfied, so that for this case, Ω_1 and Ω_2 are weakly equivalent. However, (3.2) fails, so Ω_1 and Ω_2 cannot coincide. In fact, the rank of Ω_1 is one, while the rank of Ω_2 is two, so the associated models are not even similar. This gives a negative answer to Questions 1 and 2 of Abrahamse and Douglas [4].

The referee pointed out that these negative examples serve to illustrate how the *-commutant of a representation (that is, the commutant of the von Neumann algebra generated by the representation) can fail to lift, or can lift in essentially different ways, to the *-commutant of a dilation of the representation, despite previous results of Arveson (Theorem 1.3.1 and its corollaries in [6]) suggesting that the *-commutant does lift, and uniquely as well. The analogous phenomenon for the commutant of a representation of the type under consideration here has been analyzed by Abrahamse [1].

4. AN ESTIMATE OF THE RANK OF A C_{00} REPRESENTATION

In this section we balance the negative results of the previous section with a positive result.

THEOREM 4.1 *Let* R *be a region of connectivity* n. *Let* Ω *be an inner characteristic function which defines the* c.c.u. C_{00} *representation* σ *of* $\mathrm{Rat}(\bar{R})$. *Let* ψ *be a complex-valued inner function on* R *with precisely* $n+1$ *zeros in* R, *as in Theorem* 1.5. *Then*

$$\frac{1}{n+1}\operatorname{rank}[I-\sigma(\psi)\sigma(\psi)^*] \le \operatorname{rank}\Omega \le \operatorname{rank}[I-\sigma(\psi)\sigma(\psi)^*]\,. \tag{4.1}$$

Proof. To establish notation, let $\sigma\colon \mathrm{Rat}(\bar{R}) \to \mathscr{L}(\mathscr{H})$, where

$$\mathscr{H} = H^2_\alpha(R) \ominus \Omega\, H^2_\beta(R)$$

have a minimal ∂R-dilation $\tau\colon C(\partial R) \to \mathscr{L}(\mathscr{M})$, where $\mathscr{M} = L^2_{\mathscr{K}}(m)$ and

$$\dim \mathscr{K} = \operatorname{rank}\Omega\,.$$

(Abusing notation slightly, we identify $\mathscr{H}$ via boundary functions as a subspace of $L^2_{\mathscr{K}}(m)$.) If $N = \tau(z)$ $(=M_z$ on $L^2_{\mathscr{K}}(m))$, an alternative description of the rank of Ω is the cardinality of a minimal set of vectors $\Gamma \in \mathscr{H}$ such that

$$\vee\{N^i N^{*j}x\colon x \in \Gamma,\ i, j = 0, 1, 2, \dots\} = \mathscr{M}\,. \tag{4.2}$$

Let ψ be an inner function as described in Theorem 1.5. Since ψ is unimodular on ∂R, $\tau(\psi) = \psi(N)$ is unitary. Since ψ wraps each component of ∂R once around the unit circle and N has uniform spectral multiplicity equal to rank Ω, it follows that $\psi(N)$ has uniform spectral multiplicity equal to $(n+1)$ rank Ω. It follows as in the proof of Theorem 2.2 that $\sigma(\psi) = P_{\mathscr{H}}\psi(N)|\mathscr{H}$ is a C_{00} contraction operator, and since $\mathscr{H}$ is semiinvariant for $\psi(N)$, $\psi(N)$ is a unitary dilation of $\sigma(\psi)$. Since the multiplicity of the minimal unitary dilation of $\sigma(\psi)$ is rank $[I-\sigma(\psi)\sigma(\psi)^*]$ (see [11]) and the multiplicity of the minimal unitary dilation must be less than the multiplicity of any other unitary dilation, it follows that

$$\operatorname{rank}[I-\sigma(\psi)\sigma(\psi)^*] \le (n+1)\operatorname{rank}\Omega.$$

Hence we have half of (4.1).

An alternate expression for the multiplicity of the minimal unitary dilation of $\sigma(\psi)$ is the cardinality of a minimal set of vectors $\Gamma \subseteq \mathscr{H}$ such that

$$\begin{aligned}&\vee\{\psi(N)^{*j}\psi(N)^i x : x \text{ in } \Gamma,\ i, j = 0, 1, 2, \dots\}\\ &= \vee\{\psi(N)^{*j}\psi(N)^i x\colon x \text{ in } H,\ i, j = 0, 1, 2, \dots\}\,.\end{aligned} \tag{4.3}$$

Since $\psi(N)$ can be approximated uniformly by polynomials in N and N^*, it follows that any set Γ satisfying (4.3) must also satisfy (4.2). Hence

$$\text{rank}\,\Omega \leq \text{rank}\,[I - \sigma(\psi)\sigma(\psi)^*]\ ,$$

giving the other half of (4.1).

COROLLARY 4.2. *If* Ω_1 *and* Ω_2 *are weakly equivalent characteristic inner functions and* rank $\Omega_1 = \infty$, *then also* rank $\Omega_2 = \infty$.

COROLLARY 4.3. *If* Ω_1 *and* Ω_2 *are weakly equivalent characteristic inner functions for a region* R *of connectivity* n *and* rank $\Omega_1 = 1$, *then* rank $\Omega_2 \leq n + 1$.

Proof. To establish notation, let $\Omega_1 \in H^\infty_{\alpha_1,\beta_1}(R)$ where $\alpha_1 \in T^n$. Then the induced representation σ_1 is given by $\sigma_1(f) = P_{\mathscr{H}_1} f(S_{\alpha_1})|\mathscr{H}_1$, where

$$\mathscr{H}_1 = H^2_{\alpha_1}(R) \ominus \Omega H^2_{\beta_1}(R),$$

and hence $I - \sigma_1(\psi)\sigma_1(\psi)^* = P_{\mathscr{H}_1}(I - \psi(S_{\alpha_1})\psi(S_{\alpha_1})^*)|\mathscr{H}_1$. It is not difficult to see that rank $[I - \psi(S_{\alpha_1})\psi(S_{\alpha_1})^*] = n + 1$, and hence

$$\text{rank}\,[I - \sigma_1(\psi)\sigma_1(\psi)^*] \leq n + 1.$$

If σ_2 is the representation induced by Ω_2, then

$$\text{rank}\,[I - \sigma_2(\psi)\sigma_2(\psi)^*] = \text{rank}\,[I - \sigma_1(\psi)\sigma_1(\psi)^*],$$

since σ_1 and σ_2 are unitarily equivalent. The result now follows from the second half of (4.1).

It would be of interest to know whether the estimate in the theorem is sharp. In particular, we pose the following

Question. If R is any region of connectivity n, does there exist a characteristic inner function Ω_1 on R such that rank $\Omega_1 = 1$, Ω_1 is weakly equivalent to a characteristic inner function Ω_2 on R, and rank $\Omega_2 = n + 1$?

5. CONSTRUCTION OF THE MODEL FROM THE REPRESENTATION

If T is a contraction operator of class C_{00} represented on $\mathscr{H} = H^2_{\mathscr{K}}(D) \ominus \Omega H^2_{\mathscr{K}}(D)$ as $T = P_{\mathscr{H}} M_z|\mathscr{H}$, it is known that there is a unitary operator

$$U: \text{Ran}\,(I - TT^*)^{1/2} \to \mathscr{K}$$

such that

$$f(w) = U(I - TT^*)^{1/2}(I - wT^*)^{-1} f \qquad \text{for } w \text{ in } D, \text{ for all } f \text{ in } \mathscr{H}, \tag{5.1}$$

and

$$U(I - TT^*)^{1/2}(I - zT^*)^{-1}(I - \bar{w}T)^{-1}(I - TT^*)^{1/2}U^* = \frac{I - \Omega(z)\Omega(w)^*}{1 - z\bar{w}}. \tag{5.2}$$

In this section, we give analogues, to the extent possible, of these formulas for a c.c.u. C_{00} representation of Rat$(\bar{R})$. The analysis sheds some light on Question 4 of Abrahamse and Douglas [4]. We only sketch some of the details.

For a region R of connectivity n as above and any n-tuple α in $\mathscr{U}(\mathscr{K})^n$, a more detailed analysis of kernel functions shows that k_w^α has an analytic continuation to a neighborhood of every boundary point of R excluding those in any of the cuts C_j $(j = 1, \ldots, n)$. Hence k_w^α is uniformly bounded in operator norm, and hence defines an element of $H^\infty_{\alpha, e_{\mathscr{K}}}(R)$. The associated operator $M_{k_w^\alpha}: H^2_{e_{\mathscr{K}}}(R) \to H^2_\alpha(R)$ intertwines $S_{e_{\mathscr{K}}}$ with S_α. Let $b(z)$ denote the Blaschke factor on R with a single zero at the point t in R (see [2]) where t is the point chosen to define the norm on the $H^2_\alpha(R)$ spaces (see Section 1), and let $\gamma \in T^n$ denote the index of b. Then $b(z)$ induces an isometry

$$M_b : H^2_{\bar{\gamma}\otimes I_{\mathscr{K}}}(R) \to H^2_{e_{\mathscr{K}}}(R)$$

via multiplication. (If $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\bar{\gamma} \otimes I_{\mathscr{K}}$ denotes $(\bar{\gamma}_1 I_{\mathscr{K}}, \ldots, \bar{\gamma}_n I_{\mathscr{K}}) \in \mathscr{U}(\mathscr{K})^n$.) The projection $I - M_b M_b^*$ projects $H^2_{e_{\mathscr{K}}}(R)$ onto $\ker(S^*_{e_{\mathscr{K}}} - \bar{t})$. The latter space consists of constant $\mathscr{K}$-valued functions, and hence can be identified with $\mathscr{K}$ in the natural way.

LEMMA 5.1. *With notation as above, for any* f *in* $H^2_\alpha(R)$,

$$f(w) = (I - M_b M_b^*)(M_{k_w^\alpha})^*(f). \tag{5.3}$$

Proof. The formula follows for elements f of the form $f = k_\eta^\alpha x$ (η in R, x in $\mathscr{K}$) by direct computation. Since such elements span a dense set in $H^2_\alpha(R)$, the result for a general f follows by an approximation argument.

To avoid unwanted complications, we now suppose that $\alpha = e_{\mathscr{K}}$ and that $\mathscr{H}$ is a C_{00}-model space of the form $\mathscr{H} = H^2_{\mathscr{K}}(R) \ominus \Omega H^2_\beta(R)$ for some inner Ω in $H^\infty_{e_{\mathscr{K}},\beta^*}(R)$. Associated with $\mathscr{H}$ is a model space $\mathscr{H}' = H^2_{\bar{\gamma}\otimes I_K}(R) \ominus \Omega H^2_{\beta(\bar{\gamma}\otimes I_K)}(R)$, and since M_b maps $\Omega H^2_{\beta(\bar{\gamma}\otimes I_{\mathscr{K}})}(R)$ into $\Omega H^2_\beta(R)$, it follows that $(M_b)^*$ maps $\mathscr{H}$ into $\mathscr{H}'$. Define $T_b: \mathscr{H}' \to \mathscr{H}$ by $T_b = P_{\mathscr{H}} M_b | \mathscr{H}'$; then $(T_b)^* = (M_b)^* | \mathscr{H}$.

LEMMA 5.2. *There is a unitary map* U *mapping* $\operatorname{Ran}(I - T_b T_b^*)^{1/2}$ *onto*

$$\operatorname{Ran}[(I - M_b M_b^*)|\mathscr{H}].$$

Proof. Since $(T_b)^* = (M_b)^* | \mathscr{H}$ for h in $\mathscr{H}$,

$$\|(I - T_b T_b^*)^{1/2} h\|^2 = \langle (I - T_b T_b^*) h, h\rangle =$$
$$\langle h, h\rangle - \langle M_b^* h, M_b^* h\rangle = \|(I - M_b M_b^*) h\|^2 .$$

If $k_w(z)$ is the kernel function for $H^2(R)$, the operator kernel function for $H^2_{\mathscr{K}}(R)$ is $k_w(z) I_{\mathscr{K}}$. By previous remarks, $k_w \in H^\infty(R)$ and hence $T_{k_w} \equiv P_{\mathscr{H}} M_{k_w} | \mathscr{H}$ is given by $\sigma(k_w)$, where σ is the c.c.u. representation of Rat$(\bar{R})$ defined by Ω.

THEOREM 5.3. *If* $\mathscr{H} = H^2_{\mathscr{K}}(R) \ominus \Omega H^2_\beta(R)$, *and notation is as above, then for* f *in* $\mathscr{H}$ *and* w *in* R,

$$f(w) = U(I - T_b T_b^*)^{1/2} \sigma(k_w)^*(f), \tag{5.4}$$

where U *is a Lemma* 5.2, *and*

$$U(I - T_b T_b^*)^{1/2} \sigma(k_z)^* \sigma(k_w)(I - T_b T_b^*)^{1/2} U^* = k_w(z) I - \Omega(z) k^\beta_w(z) \Omega(w)^*. \tag{5.5}$$

Proof. Formula (5.4) follows by combining Lemmas 5.1 and 5.2. Equation (5.5) follows by computing the operator kernel function for the space $\mathscr{H}$ in two ways.

It is easily seen that, if $R = D$, (5.4) and (5.5) specialize to (5.1) and (5.2) respectively, where $T = \sigma(z)$. In this case the quantity $T_b = \sigma(z)$ is completely determined by the representation, while in the general case, it is only defined in the context of a model; hence (5.5) does not quite give the characteristic function Ω completely in terms of the representation σ which it defines. This is the problem posed by Question 4 of Abrahamse and Douglas.

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