ON SOME CLASS NUMBER RELATIONS OF ALGEBRAIC TORI

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1. INTRODUCTION

A formula has long been known for expressing the class number h of a finite algebraic number field k in terms of other arithmetic invariants; namely

$$h = \frac{w|d|^{1/2}\rho}{2^r(2\pi)^t R}, \tag{1}$$

where w is the number of roots of unity contained in k, d is the discriminant of k, ρ is the residue at $s = 1$ of the Dedekind zeta function of k, R is the regulator of k, and r (resp. 2t) is the number of real (resp. complex) imbeddings of k into $\mathbb{C}$. This is equivalent to the fact that the volume of the compact group $k_A^1/k^\times$ is equal to ρ, where k_A^1 and $k^\times$ are the group of k-idele with volume 1 and the group of nonzero elements of k, respectively (*cf.* [9]). Or, in the language of algebraic groups, (1) is equivalent to the fact that the Tamagawa number of the algebraic torus $R_{k/\mathbb{Q}}(G_m)$ over $\mathbb{Q}$ is 1 (*cf.* [6]), $R_{k/\mathbb{Q}}$ being the Weil functor of restricting the field of definition from k to $\mathbb{Q}$ (*cf.* [10]). In view of the above interpretation, we shall generalize (1) to a formula for the class number of an algebraic torus T defined over $\mathbb{Q}$ which has the same form as (1) except a factor involving the Tamagawa number of T. From this generalized class number formula, we obtain a relation which expresses the relative class number of two isogenous tori in terms of their Tamagawa numbers and certain indices of the maps induced naturally by an isogeny between them. We shall also indicate how the above-mentioned class number relation can be applied to the studies of totally positive binary quadratic forms over totally real algebraic number fields and norm forms of algebraic number fields. The details will be discussed elsewhere.

In this paper, we shall use the following standard notations: $\mathbb{Z}$ for the ring of rational integers; $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the fields of rational, real, and complex numbers; $\mathbb{R}_+^\times$ for the multiplicative group of positive real numbers; Ω for a universal domain containing $\mathbb{Q}$; $R^\times$ for the multiplicative group of invertible elements of a ring R; [G] for the order of a group G; and Ker α, Im α, and Cok α for the kernel, image, and cokernel of a homomorphism α.

Let G, G' be commutative groups, α a homomorphism $G \to G'$. If Ker α, Cok α are both finite, we define the q-symbol of α by $q(\alpha) = [\text{Cok}\ \alpha]/[\text{Ker}\ \alpha]$.

2. PRELIMINARIES

We shall start by recalling some basic definitions and results on algebraic tori. We refer to [5], [6] for the details. Let T be an algebraic torus defined over a field k. We denote $\hat{T}$ by $\mathbb{Z}$-module Hom(T, G_m) of rational characters of T. An extension K of k is called a splitting field of T if $\hat{T} = (\hat{T})_K$, where $(\hat{T})_K$ denotes the

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submodule of $\hat{T}$ consisting of rational characters defined over K. We say that T splits over K or T is split by K if K is a splitting field of T. The torus T always splits over a finite separable extension of k. Let K be a finite Galois extension of k such that T splits over K. Then the Galois group G of K over k acts on $\hat{T}$. Hence, $\hat{T}$ becomes a $\mathbb{Z}$-free G-module of rank equal to the dimension of T. It is well known that the map $T \to \hat{T}$ defines an isomorphism between the category $\mathscr{C}(K/k)$ of tori defined over k and split by K, and the dual of the category $\hat{\mathscr{C}}(K/k)$ of finitely generated $\mathbb{Z}$-free G-modules. Moreover, for $T, T' \in \mathscr{C}(K/k)$, T is isogenous to T' over k if and only if $\hat{T} \otimes \mathbb{Q}$ is isomorphic to $\hat{T}' \otimes \mathbb{Q}$ as G-modules with G acting trivially on $\mathbb{Q}$.

Let k/k_0 be a finite separable field extension, and K/k a finite extension such that K/k_0 is Galois, then the Weil functor R_{k/k_0} of restricting the field of definition from k to k_0 maps $\mathscr{C}(K/k)$ into $\mathscr{C}(K/k_0)$. Moreover, we have the following identifications;

$$\widehat{R_{k/k_0}(T)} = \mathbb{Z}[G_0] \otimes_{\mathbb{Z}[G]} \hat{T};$$

G_0, G being the Galois groups of K/k_0 and K/k respectively, and

$$R_{k/k_0}(T)_{k_0} = T_k \quad \text{and} \quad (\hat{T})_k = (\widehat{R_{k/k_0}(T)})_{k_0}.$$

For $T, T' \in \mathscr{C}(K/k)$, let $\lambda: T \to T'$ be an isogeny defined over k. Then $R_{k/k_0}(\lambda)$ is an isogeny of $R_{k/k_0}(T) \to R_{k/k_0}(T')$ defined over k_0, and

$$\hat{\lambda}((\hat{T}')_k) = \widehat{R_{k/k_0}(\lambda)}\,(\widehat{R_{k/k_0}(T')_{k_0}})$$

under the above identification.

A sequence of homomorphisms of tori:

$$1 \to T' \xrightarrow{f} T \xrightarrow{g} T'' \to 1 \tag{2}$$

is called an exact sequence if it is exact in the usual sense and f, g are separable. We say that the sequence (2) is exact over k if it is exact and T, T', T'', f, g are all defined over k. Let the sequence (2) be exact over k. Then, $T \in \mathscr{C}(K/k)$ if and only if $T', T'' \in \mathscr{C}(K/k)$, and T is isogenous to $T' \times T''$ over k.

From now on, k denotes a finite algebraic number field. We denote by v a prime of k. To each prime v of k, let $|\ |_v$ denote the corresponding normalized valuation of v, k_v the completion of k with respect to $|\ |_v$, and U_v the group $\{x \in k_v: |x|_v = 1\}$. Then, for each prime v, we have a locally compact group T_{k_v} consisting of points in T which are rational over k_v. For the sake of simplicity, we shall denote by T_{k_v}, $(\hat{T})_{k_v}$ by T_v, $(\hat{T})_v$ respectively. We define

$$T_v^c = \{x \in T_v: \chi(x) \in U_v \text{ for all } \chi \in (\hat{T})_v\}.$$

Then T_v^c is the unique maximal compact subgroup of T_v. For a finite set S of primes of k containing the set S_∞ of all infinite primes of k, we set

$$T_A(S) = \prod_{v\in S} T_v \times \prod_{v\notin S} T_v^c,$$

with the product topology. Let T_A be the inductive limit of $T_A(S)$ with respect to S. Then T_A is a locally compact abelian group which is called the adele group of T over k. By imbedding T_k into T_A diagonally, we identify T_k with a discrete subgroup of T_A. We put $T_k(S) = T_k \cap T_A(S)$ and call it the S-unit group of T over k. Then $T_k(S)$ is the direct product of the finite group $T_k \cap T_A^c$ and of a group isomorphic to $\mathbb{Z}^{r(S)-r}$, where $T_A^c = \prod_v T_v^c$ with v ranging over all primes of k, $r = \operatorname{rank}(\hat{T})_k$, and $r(S) = \sum_{v\in S} \operatorname{rank}(\hat{T})_v$ (*cf.* [8]). Let k_A^1 denote the group of k-idele with volume 1. We define $T_A^1 = \{x \in T_A \colon \chi(x) \in k_A^1 \text{ for all } \chi \in (\hat{T})_k\}$. By Artin's product formula, T_k is actually contained in T_A^1. It is well known that the quotient group T_A^1/T_k is compact. This implies that the index

$$h_T(S) = [T_A \colon T_k \cdot T_A(S)]$$

is finite. The number $h_T(S)$ is called the S-class number of T over k. When $S = S_\infty$, $h_T(S)$ is called simply the class number of T over k and denoted by h_T.

Let k_0 be a subfield of k. Then we have the following canonical isomorphisms:

$$(R_{k/k_0}(T))_{v_0} \simeq \prod_{v/v_0} T_v, \quad \text{and} \quad (R_{k/k_0}(T))_A \simeq T_A,$$

where v_0 is a prime of k_0, and the product runs over all primes v of k lying above v_0. Moreover, by the maximal compactness, we have $R_{k/k_0}(T)_{v_0}^c \simeq \prod_{v/v_0} T_v^c$. Let T, T' be tori defined over k, and $\lambda\colon T \to T'$ an isogeny defined over k. The isogeny $\lambda\colon T \to T'$ induces naturally the following maps:

$$\lambda_v\colon T_v \to T'_v, \qquad \lambda_v^c\colon T_v^c \to T_v'^c,$$

$$\lambda_k(S)\colon T_k(S) \to T'_k(S), \qquad (\hat{\lambda})_k\colon (\hat{T}')_k \to (\hat{T})_k,$$

$$(\hat{\lambda})_v\colon (\hat{T}')_v \to (\hat{T})_v.$$

Then, the q-symbols of λ_v, λ_v^c, $\lambda_k(S)$, $(\hat{\lambda})_k$, $(\hat{\lambda})_v$ are defined for any prime v of k and any S. Moreover, we have the following equalities:

$$q(R_{k/k_0}(\lambda)_{v_0}) = \prod_{v/v_0} q(\lambda_v),$$

$$q(R_{k/k_0}(\lambda)_{v_0}^c) = \prod_{v/v_0} q(\lambda_v^c),$$

$$q(R_{k/k_0}(\lambda)_{k_0}(S_0)) = q(\lambda_k(S)),$$

$$q(\widehat{R_{k/k_0}(\lambda)_{k_0}}) = q((\hat{\lambda})_k),$$

where S is the set of all primes v lying above v_0, $v_0 \in S_0$. If T, T' $\in \mathscr{C}(K/k)$, then $q(\lambda_v^c) = 1$ if v is a finite prime unramified relative to K/k and the residue class characteristic of v is prime to the degree of λ.

3. A CLASS NUMBER RELATION FOR ALGEBRAIC TORI

In the present section, we shall consider algebraic tori defined over $\mathbb{Q}$ only. Let χ_i, $1 \leq i \leq r = \mathrm{rank}(\hat{T})_{\mathbb{Q}}$, be a $\mathbb{Z}$-basis of $(\hat{T})_{\mathbb{Q}}$. Define a map

$$\Lambda: T_A \to (\mathbb{R}_+^\times)^r$$

by $\Lambda(x) = (\|\chi_i(x)\|_{\mathbb{Q}})_{1 \leq i \leq r}$, $x \in T_A$, where $\| \, \|_{\mathbb{Q}}$ denotes the idele volume of $\mathbb{Q}$. Then Λ induces the following isomorphism:

$$\tilde{\Lambda}: T_A / T_A^1 \xrightarrow{\sim} (\mathbb{R}_+^\times)^r .$$

Let $d\tilde{t}$ be the pullback of the product measure $\Lambda_{i=1}^r t_i^{-1} dt_i$ on $(\mathbb{R}_+^\times)^r$ by means of $\tilde{\Lambda}$, $dT_{\mathbb{Q}}$ the canonical discrete measure on the discrete group $T_{\mathbb{Q}}$, and $d(T_A^1/T_{\mathbb{Q}})$ the normalized Haar measure on the compact group $T_A^1/T_{\mathbb{Q}}$. Matching $d\tilde{t}$, $d(T_A^1/T_{\mathbb{Q}})$, $dT_{\mathbb{Q}}$ together topologically (*cf.* [10]), we obtain a Haar measure μ_T on T_A.

For each prime v of $\mathbb{Q}$, let ξ_i, $1 \leq i \leq r_v = \mathrm{rank}(\hat{T})_v$, be a $\mathbb{Z}$-basis of $(\hat{T})_v$. Define a map $\Lambda_v: T_v \to (\mathbb{R}_+^\times)^{r_v}$ by $\Lambda_v(x) = (|\xi_i(x)|_v)_{1 \leq i \leq r_v}$, $x \in T_v$. Then Λ_v induces the following isomorphism:

$$\tilde{\Lambda}_v: T_v / T_v^c \simeq \begin{cases} (\mathbb{R}_+^\times)^{r_v} & \text{if } v = \infty \\ \mathbb{Z}^{r_v} & \text{if } v = p . \end{cases}$$

We denote by dt_∞ (resp. dt_p) the pullback of the product measure $\Lambda_{i=1}^{r_\infty} t_i^{-1} dt_i$ on $(\mathbb{R}_+^\times)^{r_\infty}$ (resp. the canonical discrete measure on $\mathbb{Z}^{r_p}$) by means of $\tilde{\Lambda}_v$, and by dT_v^c the normalized Haar measure on the compact group T_v^c. Matching dt_v and dT_v^c together topologically, we obtain a measure ν_v on T_v. Clearly, $\prod \nu_p(T_p^c)$ is absolutely convergent, and hence $\prod_v \nu_v$ defines a Haar measure on T_A which will be denoted by ν_T.

Now, let ω be an invariang gauge form on T defined over $\mathbb{Q}$. Then, ω induces canonically a Haar measure ω_v on T_v (*cf.* [10]). Since $\prod_p \omega_p(T_p^c)$ is not absolutely convergent, we shall introduce a set of canonical convergence factors for $\{\omega_v\}_v$ in the following way. Let K be a finite Galois extension of $\mathbb{Q}$ such that T splits over K. Then, $\hat{T}$ can be converted as a G-module, G being the Galois group of $K/\mathbb{Q}$. Let L(s) be the Artin L-function $L(s, \chi_T : K/\mathbb{Q})$ defined by the G-module $\hat{T}$ (*cf.* [1], [2]), and $L(s) = \prod_p L_p(S)$ be its Euler product. We can choose a sufficiently large finite set S of primes of $\mathbb{Q}$ containing the infinite prime ∞ such that $L_p(1)\,\omega_p(T_p^c) = 1$ for all $p \notin S$ (*cf.* [5]). Then $\omega_\infty \prod_p L_p(1)\,\omega_p$ defines a Haar measure on T_A which is independent of the choice of a splitting field for T. We call this measure the Tamagawa measure on T_A and denote it by ω_T. For a torus T' in $\mathscr{C}(k/\mathbb{Q})$ isogenous to T over $\mathbb{Q}$, we have recalled in Section 1 that the G-module $\hat{T}' \otimes \mathbb{Q}$ is equivalent to the G-module $\hat{T} \otimes \mathbb{Q}$. Therefore, T and T' have the same set of canonical convergence factors.

The character χ_T of the G-module $\hat{T}$ (in the sense of group representation) can be expressed as $\chi_T = \sum_{j=1}^{h} m_j \chi_j$, $m_j \in \mathbb{Z}$, $m_j \geq 0$, $m_1 = r$, where χ_j, $1 \leq j \leq h$, are irreducible characters of G with χ_1 the principle character. By the properties of Artin L-functions (*cf.* [1], [2]), we have $L(s) = \zeta(s)^r \prod_{j=2}^{h} L(s, \chi_j)^{m_j}$, $\zeta(s)$ being the Riemann ζ-function. Since $L(1, \chi_j)$, $j \geq 2$, is nonzero (*cf.* [3]), the limit

$$\rho_T = \lim_{s \to 1} (s-1)^r L(s) = \prod_{j=2}^{h} L(1, \chi_j)^{m_j}$$

is nonzero and finite. Moreover, ρ_T is independent of the choice of a splitting field for T (*cf.* [5]). We shall call ρ_T the quasi-residue of T over $\mathbb{Q}$. The Tamagawa number τ_T of T over $\mathbb{Q}$ is defined to be the total measure on the compact group $T_A^1/T_{\mathbb{Q}}$ with respect to the measure m on $T_A^1/T_{\mathbb{Q}}$ such that $d\tilde{t}$, m, $dT_{\mathbb{Q}}$ give the measure $\rho_T^{-1}\,\omega_T$ on T_A when matched together topologically. By the uniqueness of Haar measures on a locally compact group, there is a positive constant c_T, depending on T and $\mathbb{Q}$ only, such that $\omega_T = c_T\,\nu_T$. The number $D_T = 1/c_T^2$ will be called the quasi-discriminant of T over $\mathbb{Q}$.

In order to find a relation among the arithmetic invariants of T over $\mathbb{Q}$, we shall compare the three Haar measures μ_T, ν_T, ω_T. This will be done by evaluating these three measures on a fundamental domain for $T_A^1/T_{\mathbb{Q}}$. Put

$$T_A^1(\infty) = T_A^1 \cap T_A(\infty).$$

Then, $T_A^1(\infty) = T_\infty^1 \times \prod_p T_p^c$, where $T_\infty^1 = \{x \in T_\infty : |\chi(x)|_\infty = 1 \text{ for all } \chi \in (\hat{T})_{\mathbb{Q}}\}$. Let ξ_i, $1 \leq i \leq r_\infty = \operatorname{rank}(\hat{T})_\infty$, be a $\mathbb{Z}$-basis of $(\hat{T})_\infty$ such that ξ_i,

$$1 \leq i \leq r = \operatorname{rank}(\hat{T})_{\mathbb{Q}},$$

form a $\mathbb{Z}$-basis of $(\hat{T})_{\mathbb{Q}}$ (note that the existence of such a $\mathbb{Z}$-basis for $(\hat{T})_\infty$ is guaranteed; *cf.* [5], p. 130, Footnote 23). Let Φ_0 be a map $T_\infty \to \mathbb{R}^{r_\infty}$ defined by $\Phi_0(x) = (\log|\xi_i(x)|_\infty)_{1 \leq i \leq r_\infty}$. Then Φ_0 is surjective with kernel T_∞^c and we have

$$\Phi_0(T_\infty^1) = \underbrace{\{0\} \times \cdots \times \{0\}}_{r} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{r} \simeq \mathbb{R}^{r_\infty - r}.$$

We have remarked in Section 1 that the unit group $T_{\mathbb{Q}}(\infty)$ of T over $\mathbb{Q}$ has the following decomposition:

$$T_{\mathbb{Q}}(\infty) = (T_{\mathbb{Q}} \cap T_A^c) \times E,$$

where E is $\mathbb{Z}$-free of rank $r_\infty - r$. Let e_j, $r+1 \le j \le r_\infty$, be a $\mathbb{Z}$-basis of E. Then we have $r_\infty - r$ linearly independent vectors

$$\Phi_0(e_j) = (\underbrace{0, \cdots, 0}_{r}, \log|\xi_{r+1}(e_j)|_\infty, \cdots, \log|\xi_{r_\infty}(e_j)|_\infty), \qquad r+1 \le j \le r_\infty \text{ in } \mathbb{R}^{r_\infty}$$

(*cf.* [5]). The number $R_T = |\det(\log|\xi_i(e_j)|_\infty)_{r+1 \le i,j \le r_\infty}|$, which is independent of the choice of ξ_i and e_j, is called the regulator of T over $\mathbb{Q}$. In $\mathbb{R}^{r_\infty - r}$, we have a parallelotope $P_0 = \left\{ \sum_{j=r+1}^{r_\infty} \lambda_j \Phi_0(e_j) : 0 \le \lambda_j < 1 \right\}$. We shall extend P_0 to a parallelotope P in $\mathbb{R}^{r_\infty}$. First, let X_j, $1 \le j \le r$, be the elements in the Lie algebra of $T_{\mathbb{Q}}$ (in the sense of Chevalley [4]) defined by the condition $d\xi_i(X_j) = \delta_{ij}$, $1 \le i, j \le r$. Since $\exp X_j$, $1 \le j \le r$, are contained in the identity component of T_∞, we have

$$\Phi_0(\exp X_j) = (0, \cdots, \overset{j}{1}, 0, \cdots, 0, d\xi_{r+1}(X_j), \cdots, d\xi_{r_\infty}(X_j))$$

(*cf.* [5]). We define the parallelotope P in $\mathbb{R}^\infty$ by

$$P = \left\{ \sum_{i=1}^{r} \lambda_i \Phi_0(\exp X_i) + \sum_{j=r+1}^{r_\infty} \lambda_j \Phi_0(e_j) : 0 \le \lambda_i, \lambda_j < 1 \right\}.$$

It is easily seen that the Euclidean volume of P in $\mathbb{R}^{r_\infty}$ is R_T.

LEMMA 1. *If we extend the map* $\Phi_0 : T_\infty \to \mathbb{R}^{r_\infty}$ *to a map* $\Phi : T_A(\infty) \to \mathbb{R}^{r_\infty}$ *by putting* $\Phi(x) = \Phi_0(x_\infty)$, *then the set* $\bigcup_{\ell=1}^{h_T} x_\ell E_0$ *is a fundamental domain of* $T_A^1/T_{\mathbb{Q}}$, *where* $\{x_\ell\}_{1 \le \ell \le h_T}$ (*resp.* E_0) *is a complete set of representatives for* T_A^1 (*resp.* $\Phi^{-1}(P_0)$) *modulo* $T_{\mathbb{Q}} \cdot T_A^1(\infty)$ (*resp.* $T_A^c \cap T_{\mathbb{Q}}$). *Moreover, we have*

$$\mu_T(\Phi^{-1}(P)) = w_T/h_T$$

with $w_T = [T_A^c \cap T_{\mathbb{Q}}]$.

Proof. Cf. [5].

LEMMA 2. *Let* $\Phi : T_A(\infty) \to \mathbb{R}^{r_\infty}$ *be the map defined in Lemma* 1. *Then,*

$$\nu_T(\Phi^{-1}(P)) = R_T, \quad \omega_T(\Phi^{-1}(P)) = cR_T = R_T/D_T^{1/2}, \quad \textit{and}$$

$$c = \omega_\infty(\Phi_0^{-1}(I)) \prod_p L_p(1)\,\omega_p(T_p^c).$$

Proof. Let I be the parallelotope in $\mathbb{R}^{r_\infty}$ spanned by the r_∞ unit vectors $(1, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0)$, $\cdots$, $(0, \cdots, 0, 1)$. Since Φ is a homomorphism, we have $\nu_T(\Phi^{-1}(P))/\nu_T(\Phi^{-1}(I))$ = the Euclidean volume of P = the regulator R_T of T. From the definition of Φ, we see easily that $\Phi^{-1}(I) = \Phi_0^{-1}(I) \times \prod_p T_p^c$, where

$\Phi_0^{-1}(I) = \{x_\infty \in T_\infty : 0 \le \log|\xi_i(x_\infty)|_\infty < 1,\ 1 \le i \le r_\infty\}$; ξ_i, $1 \le i \le r_\infty$, being the $\mathbb{Z}$-basis of $(\hat{T})_\infty$ we chose to define the map Φ_0. Hence, $\nu_T(\Phi^{-1}(I)) = 1$. This proves the first two equalities. The number c is given by

$$\omega_T(\Phi^{-1}(I))/\nu_T(\Phi^{-1}(I)) = \omega_\infty(\Phi_0^{-1}(I)) \prod_p L_p(1)\,\omega_p(T_p^c).$$

THEOREM 1. *Let* T *be an algebraic torus defined over* $\mathbb{Q}$, h_T, R_T, τ_T, ρ_T, D_T, w_T *the arithmetic invariants of* T *over* $\mathbb{Q}$ *defined above. Then,*

$$h_T = \tau_T \rho_T w_T (D_T)^{1/2}/R_T.$$

Proof. From the definition of τ_T, we have $\omega_T = \rho_T \tau_T \mu_T$. Evaluating both sides on $\Phi^{-1}(P)$, we get $R_T/D_T^{1/2} = \rho_T \tau_T w_T/h_T$ by Lemma 1 and Lemma 2.

Remark. Let k be a finite algebraic number field, and K a finite Galois extension over $\mathbb{Q}$ containing k. Let G, H denote the Galois groups of $K/\mathbb{Q}$, K/k respectively. Put $T = R_{k/\mathbb{Q}}(G_m)$. Then T is an algebraic torus defined over $\mathbb{Q}$ and split by K. Since the G-module $\hat{T}$ is the induced module by the trivial H-module $\hat{G}_m = \mathbb{Z}$ with respect to G/H, the character χ_T of G defined by the G-module $\hat{T}$ is the induced character 1_H^G induced by the principal character 1_H of H. Thus

$$L(s, \chi_T; K/\mathbb{Q}) = L(s, 1_H^G; K/\mathbb{Q}) = L(s, 1_H; K/k)$$
$$= \text{the Dedekind zeta function } \zeta_k(s) \text{ of } k,$$

and $\rho_T = \mathrm{Res}_{s=1}\, \zeta_k(s) = \rho_k$. It is easily seen that $w_T = w_k$ = the number of roots of unity contained in k, $R_T = R_k$ = the regulator of k, $h_T = h_k$ = the class number of k, and $D_T = |d|_k/(2^r(2\pi)^t)^2$. These clarify our nomenclature. Moreover, since the Tamagawa number τ_T is 1 (*cf.* [6]), Theorem 1, applied to the torus $T = R_{k/\mathbb{Q}}(G_m)$, gives the following well-known class number relation for the algebraic number field k:

$$h_k = \frac{w_k |d_k|^{1/2} \rho_k}{2^r (2\pi)^t R_k}.$$

4. THE RELATIVE CLASS NUMBER OF ISOGENOUS TORI

Let K be a finite Galois extension over $\mathbb{Q}$, and G the Galois group of $K/\mathbb{Q}$. Let $\lambda: T \to \tilde{T}$ be an isogeny defined over $\mathbb{Q}$ of tori T, $\tilde{T}$ in $\mathscr{C}(K/\mathbb{Q})$. We shall apply Theorem 1 to the study of the relative class number $h_T/h_{\tilde{T}}$ of T and $\tilde{T}$.

Let $\tilde{\omega}$ be an invariant gauge form on $\tilde{T}$ defined over $\mathbb{Q}$, and ω the gauge form on T obtained from $\tilde{\omega}$ by λ. Then ω is also invariant and defined over $\mathbb{Q}$. Let $\tilde{\Phi}_0 : \tilde{T}_\infty \to \mathbb{R}^{r_\infty}$ be the map to $\tilde{T}$ what the map $\Phi_0 : T_\infty \to \mathbb{R}^{r_\infty}$ is to T. Then we have

$$\tilde{\omega}_\infty(\tilde{\Phi}_0^{-1}(I)) = \frac{q(\lambda_\infty)}{q((\hat{\lambda})_\infty)}\, \omega_\infty(\Phi_0^{-1}(I)) \tag{3}$$

and

$$\tilde{\omega}_p(\tilde{T}_p^c) = q(\lambda_p^c)\,\omega_p(T_p^c) \tag{4}$$

for every finite prime p of $\mathbb{Q}$ (*cf.* [5]).

THEOREM 2. *Let* $\lambda: T \to \tilde{T}$ *be an isogeny defined over* $\mathbb{Q}$ *of tori* $T, \tilde{T}$ *in* $\mathscr{C}(K/\mathbb{Q})$. *Then the relative class number of* T *and* $\tilde{T}$ *can be expressed as*

$$\frac{h_T}{h_{\tilde{T}}} = \frac{\tau_T}{\tau_{\tilde{T}}} \cdot \frac{q(\lambda_\infty)}{q(\lambda_{\mathbb{Q}}(\infty))\,q((\hat{\lambda})_{\mathbb{Q}})} \cdot \prod_p q(\lambda_p^c)\,.$$

Remark. The product $\prod_p q(\lambda_p^c)$ is well-defined since we have remarked at the end of Section 1 that $q(\lambda_p^c) = 1$ for almost all p. The number

$$\frac{q(\lambda_\infty)}{q(\lambda_{\mathbb{Q}}(\infty)\,q((\hat{\lambda})_{\mathbb{Q}})} \cdot \prod_p q(\lambda_p^c)$$

being equal to $h_T\,\tau_{\tilde{T}}/h_{\tilde{T}}\,\tau_T$, is independent of the isogeny λ.

Proof. From Lemma 1, Lemma 2, and Theorem 1, we get

$$\frac{h_T}{h_{\tilde{T}}} = \frac{\rho_T\,\tau_T\,w_{\tilde{T}}R_{\tilde{T}}}{\rho_{\tilde{T}}\,\tau_{\tilde{T}}\,w_T R_T} \cdot \frac{\tilde{\omega}_\infty(\tilde{\Phi}_0^{-1}(I)) \prod_p L_p(1, \chi_{\tilde{T}}; K/\mathbb{Q})\,\tilde{\omega}_p(\tilde{T}_p^c)}{\omega_\infty(\Phi^{-1}(I)) \prod_p L_p(1, \chi_T; K/\mathbb{Q})\,\omega_p(T_p^c)}\,. \tag{5}$$

As we remarked before, $T, \tilde{T}$ have the same L-function and $\mathrm{rank}(\hat{T})_{\mathbb{Q}} = \mathrm{rank}(\hat{\tilde{T}})_{\mathbb{Q}}$. Hence, $L_p(1, \chi_T; K/\mathbb{Q}) = L_p(1, \chi_{\tilde{T}}; K/\mathbb{Q})$ and $\rho_T = \rho_{\tilde{T}}$. Therefore, by (3) and (4), (5) can be simplified as

$$\frac{h_T}{h_{\tilde{T}}} = \frac{\tau_T\,w_{\tilde{T}}R_{\tilde{T}}}{\tau_{\tilde{T}}\,w_T R_T} \cdot \frac{q(\lambda_\infty)}{q((\hat{\lambda})_\infty)} \cdot \prod_p q(\lambda_p^c)\,. \tag{6}$$

Now, let $T_{\mathbb{Q}}(\infty) = E \times F$ be a decomposition of the unit group of T over $\mathbb{Q}$ with E torsion-free and F finite. By choosing a maximal torsion-free subgroup $\tilde{E}$ of $\tilde{T}_{\mathbb{Q}}(\infty)$ such that $\lambda(E) \subset \tilde{E}$, we get a similar decomposition $\tilde{T}_{\mathbb{Q}}(\infty) = \tilde{E} \times \tilde{F}$ for $\tilde{T}_{\mathbb{Q}}(\infty)$ with $\tilde{F}$ finite. Let e_j (resp. $\tilde{e}_j$), $r + 1 \leq j \leq r_\infty$, $r = \mathrm{rank}(\hat{T})_{\mathbb{Q}}$,

$$r_\infty = \mathrm{rank}(\hat{T})_\infty = \mathrm{rank}(\hat{\tilde{T}})_\infty\,,$$

be a $\mathbb{Z}$-basis of E (resp. $\tilde{E}$), and ξ_i (resp. $\tilde{\xi}_i$), $1 \leq i \leq r_\infty$, a $\mathbb{Z}$-basis of $(\hat{T})_\infty$ (resp. $(\hat{\tilde{T}})_\infty$) such that ξ_i (resp. $\tilde{\xi}_i$), $1 \leq i \leq r$, form a $\mathbb{Z}$-basis of $(\hat{T})_{\mathbb{Q}}$ (resp. $(\hat{\tilde{T}})_{\mathbb{Q}}$). By definition, we have

$$R_T = \left|\det(\log|\xi_i(e_j)|_\infty)_{r+1 \leq i,j \leq r_\infty}\right|, \quad \text{and} \quad R_{\tilde{T}} = \left|\det(\log|\tilde{\xi}_i(\tilde{e}_j)|_\infty)_{r+1 \leq i,j \leq r_\infty}\right|.$$

Denote by $M = (m_{ij})_{r+1 \leq i,j \leq r_\infty}$ the matrix defined by

$$\lambda(e_j) = \prod_{i=r+1}^{r_\infty} \tilde{e}_j^{\,m_{ij}}, \qquad r + 1 \leq j \leq r_\infty\,. \tag{7}$$

Then, we have

$$|\det M| = q(\lambda_E), \tag{8}$$

λ_E being the restriction of λ to E. Denote by $N = (n_{k\ell})_{1 \leq k, \ell \leq r_\infty}$ the matrix defined by

$$(\hat{\lambda})_\infty(\tilde{\xi}_\ell) = \sum_{k=1}^{r_\infty} n_{k\ell}\, \xi_k, \qquad 1 \leq \ell \leq r_\infty . \tag{9}$$

By our choice of ξ_i, $\tilde{\xi}_i$, the matrix N is of the form

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

with $N_1 = (n_{k\ell})_{1 \leq k, \ell \leq r}$, and $N_2 = (n_{k\ell})_{r+1 \leq k, \ell \leq r_\infty}$. Moreover, we have

$$|\det N| = q((\hat{\lambda})_\infty), \quad \text{and} \quad |\det N_1| = q((\hat{\lambda})_{\mathbb{Q}}). \tag{10}$$

From (7), (9), we get $\prod_{k=1}^{r_\infty} (\xi_k(e_j))^{n_{k\ell}} = (\hat{\lambda})_\infty(\tilde{\xi}_\ell)(e_j) = \tilde{\xi}_\ell(\lambda(e_j)) = \prod_{i=r+1}^{r_\infty} \tilde{\xi}_\ell(\tilde{e}_i)^{m_{ij}}$. Since $T_{\mathbb{Q}}(\infty) \subset T_A^1(\infty)$, $|\xi_k(e_j)|_\infty = 1$, $1 \leq k \leq r$. Hence, we have

$$\prod_{k=r+1}^{r_\infty} |\xi_k(e_j)|_\infty^{n_{k\ell}} = \prod_{i=r+1}^{r_\infty} |\tilde{\xi}_\ell(\tilde{e}_i)|_\infty^{m_{ij}}. \tag{11}$$

By taking logarithms on both sides of (11), we obtain

$$\sum_{k=r+1}^{r_\infty} n_{k\ell} \log|\xi_k(e_j)|_\infty = \sum_{i=r+1}^{r_\infty} m_{ij} \log|\tilde{\xi}_\ell(\tilde{e}_i)|_\infty .$$

This implies that $|\det N_2| \cdot R_T = |\det M| \cdot R_{\tilde{T}}$. Since $\det N = \det N_1 \cdot \det N_2$, it follows from (8), (10) that

$$\frac{R_{\tilde{T}}}{R_T} = \frac{q((\hat{\lambda})_\infty)}{q(\lambda_E)\, q((\hat{\lambda})_{\mathbb{Q}})} . \tag{12}$$

Denote $\lambda_F \colon F \to \tilde{F}$ the restriction of λ to F. Since F, $\tilde{F}$ are finite,

$$q(\lambda_F) = \frac{[\tilde{F}]}{[F]} = \frac{w_{\tilde{T}}}{w_T}. \tag{13}$$

It is clear that

$$q(\lambda_{\mathbb{Q}}(\infty)) = q(\lambda_F)\, q(\lambda_F) . \tag{14}$$

Thus, our assertion follows from (6), (12), (13), and (14).

Let $0 \to T' \to T \to T'' $ be an exact sequence of algebraic tori in $\mathscr{C}(K/\mathbb{Q})$. Then, as we have recalled in Section 1, T is isogenous to $T' \times T''$ over $\mathbb{Q}$. Let $\lambda: T \to T' \times T''$ be an isogeny defined over $\mathbb{Q}$. From Theorem 2, we obtain the following:

THEOREM 3. *For any exact sequence* $1 \to T' \to T \to T'' \to 1$ *in* $\mathscr{C}(K/\mathbb{Q})$, *we have*

$$\frac{h_T}{h_{T'} h_{T''}} = \frac{\tau_T}{\tau_{T'} \tau_{T''}} \frac{q(\lambda_\infty)}{q(\lambda_{\mathbb{Q}}(\infty))\, q((\hat{\lambda})_{\mathbb{Q}})} \prod_p q(\lambda_p^c),$$

where the isogeny $\lambda: T \to T' \times T''$ *can be chosen arbitrarily.*

Proof. This is an immediate consequence of Theorem 2 together with the facts that $h_{T' \times T''} = h_{T'} h_{T''}$, $\tau_{T' \times T''} = \tau_{T'} \tau_{T''}$ (*cf.* [6]).

5. APPLICATIONS

Let k be a finite algebraic number field, and K a finite Galois extension of k. Denote by G the Galois group of K/k. Put $T_0 = R_{K/k}(G_m)$, $T_0'' = G_m$. Let N denote the norm map $N: T_0 \to T_0''$, and T_0' the kernel of N. We get an exact sequence

(N) $$1 \to T_0' \xrightarrow{i} T_0 \xrightarrow{N} T_0'' \to 1$$

of tori in $\mathscr{C}(K/k)$, i being the canonical inclusion. The dual of the sequence (N) is given by

($\hat{N}$) $$0 \to \mathbb{Z} \xrightarrow{\hat{N}} \mathbb{Z}[G] \xrightarrow{\hat{i}} \mathbb{Z}[G]/\mathbb{Z}s \to 0,$$

where $s = \sum_{g \in G} g$, and $\hat{N}$, $\hat{i}$ are defined by $\hat{N}(z) = zs$, $\hat{i}(\gamma) = \gamma \bmod \mathbb{Z}s$. Define a map $\hat{\lambda}_0: \mathbb{Z} \times (\mathbb{Z}[G]/\mathbb{Z}s) \to \mathbb{Z}[G]$ by

(15) $$\hat{\lambda}_0(z, \gamma \bmod \mathbb{Z}s) = zs + (m\gamma - S(\gamma)s)$$

where $m = [G]$, and $S(\gamma) = \sum_{g \in G} z_g$ if $\gamma = \sum_{g \in G} z_g g$. Also define a map $\hat{\lambda}_0': \mathbb{Z}[G] \to \mathbb{Z} \times (\mathbb{Z}[G]/\mathbb{Z}s)$ by $\hat{\lambda}_0'(\gamma) = (S(\gamma), \gamma \bmod \mathbb{Z}s)$. Then, $\hat{\lambda}_0 \circ \hat{\lambda}_0' = m$, *i.e.*, the map $\gamma \to m\gamma$, $\gamma \in \mathbb{Z}[G]$. Dualizing the above situation, we obtain a commutative diagram of isogenies of algebraic tori in $\mathscr{C}(K/k)$:

$$\begin{array}{ccc} T_0 & \xrightarrow{\quad m \quad} & T_0 \\ & {\scriptstyle \lambda_0} \searrow \quad \nearrow {\scriptstyle \lambda_0'} & \\ & T_0' \times T_0'' & \end{array}$$

where m is the map $x \to x^m$, $x \in T_0$, and λ_0, λ_0' are the isogenies defined over k by $\lambda_0(x) = (x^m N(x)^{-1}, N(x))$, and $\lambda_0'(x', x'') = x'x''$. Now, applying the Weil functor $R_{k/\mathbb{Q}}$ to the isogeny $\lambda_0: T_0 \to T_0' \times T_0''$, we get the isogeny $\lambda: T \to T' \times T''$ defined over $\mathbb{Q}$, λ, T, T', T'' being $R_{k/\mathbb{Q}}(\lambda_0)$, $R_{k/\mathbb{Q}}(T_0)$, $R_{k/\mathbb{Q}}(T_0')$, $R_{k/\mathbb{Q}}(T_0'')$ respectively.

Since $\tau_T = \tau_{T''} = 1$, $h_T = h_K$, and $h_{T''} = h_k$, from Theorem 3 we obtain the following:

THEOREM 4. *Let* k, K, λ, T' *be stated as above. Then*

$$\frac{h_K}{h_k} = \frac{h_{T'}}{\tau_{T'}} \cdot \frac{q(\lambda_\infty)}{q(\lambda_{\mathbb{Q}}(\infty))\, q((\hat{\lambda})_{\mathbb{Q}})} \prod_p q(\lambda_p^c).$$

Now, let k be a totally real number field, and $q(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ a totally positive binary quadratic form over k; *i.e.*, $\alpha, \beta, \gamma \in k$, and the discriminant $\delta = \beta^2 - 4\alpha\gamma$ is a totally negative number in k. The study of the binary quadratic form q(x, y) is equivalent to that of the norm form N of the totally imaginary quadratic extension $K = k(\sqrt{\delta})$ over k. Let L be an $\mathcal{O}_k$-free lattice in K, $\mathcal{O}_k$ being the ring of algebraic integers in k. Since T' is nothing but the special orthogonal group SO(N), $h_{T'}$ can be interpreted as the number h(L) of classes in the genus of L with respect to N. In this case, the q-symbols appeared in Theorem 4 can be explicitly determined using results in algebraic number theory. Theorem 4 then gives the following simple class number relation:

$$h(L) = \frac{h_K}{h_k} \cdot \frac{1}{2^{u-1}\,[U_K^0 : U_k^0]}, \tag{16}$$

where u is the number of distinct prime factors of the relative discriminant $d_{K/k}$ of K/k, and U_K^0 (resp. U_k^0) is the torsion-free part of the unit group U_K (resp. U_k) of K (resp. k). If $k = \mathbb{Q}$ and K is an imaginary quadratic field, then $[U_K^0 : U_k^0] = 1$. We have $h_K/h_L = 2^{u-1}$; this is the well-known result due to Gauss on the genera of primitive integral binary quadratic forms. If K is the ℓ-th cyclotomic field, ℓ being an odd rational prime, and k is the maximal totally real subfield of K, then $u = 1$ and $[U_K^0 : U_k^0] = 1$. The relation (16) gives an interpretation for the first factor in the well-known factorization of h_K due to Kummer.

Theorem 4 can also be applied to the study of norm forms in the following way. Let k be a finite algebraic number field, and K a finite Galois extension of $\mathbb{Q}$ containing k. Denote by G, H the Galois groups of $K/\mathbb{Q}$, K/k respectively. Let $G = \bigcup_{i=1}^d Hg_i$ be a decomposition of G into its cosets modulo H. We can write $\mathbb{Z}[G]$ as $\mathbb{Z}[G] = \bigoplus_{i=1}^u \mathbb{Z}[H]g_i$. Define a $\mathbb{Z}[G]$-module homomorphism $\hat{N}: \mathbb{Z} \to \mathbb{Z}[G]$ by $\hat{N}(z) = zs$, $z \in \mathbb{Z}$, $s = \sum_{i=1}^d g_i$, and $\hat{i}: \mathbb{Z}[G] \to \mathbb{Z}[G]/\mathbb{Z}s$ by $\hat{i}(\gamma) = \gamma \bmod \mathbb{Z}s$. We have an exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to \mathbb{Z} \xrightarrow{\hat{N}} \mathbb{Z}[G] \xrightarrow{\hat{i}} \mathbb{Z}[G]/\mathbb{Z}s \to 0.$$

Dualizing the above situation, we get an exact sequence of tori in $\mathscr{C}(K/\mathbb{Q})$:

$$1 \to T' \xrightarrow{i} T \xrightarrow{N} T'' \to 1,$$

where T, T', T'' are the tori $R_{k/\mathbb{Q}}(G_m)$, Ker N, G_m respectively. This implies that T is isogenous to $T' \times T''$ over $\mathbb{Q}$. Let $\lambda: T \to T' \times T''$ be an isogeny defined over $\mathbb{Q}$. Then, by Theorem 3, we have

(17) $$\frac{h_k}{h_{T'}} = \frac{1}{\tau_{T'}} \cdot \frac{q(\lambda_\infty)}{q(\lambda_{\mathbb{Q}}(\infty))\, q((\hat{\lambda})_{\mathbb{Q}})} \cdot \prod_p q(\lambda_p^c)\,.$$

Let $\{\alpha_1, \cdots, \alpha_n\}$ be an integral basis of $k/\mathbb{Q}$. The form F defined by

$$F(x_1, \cdots, x_n) = N_{k/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \prod_{i=1}^{n} (\alpha_1^{(i)} x_1 + \cdots + \alpha_n^{(i)} x_n),$$

where $\alpha_j^{(i)}$, $1 \leq i, j \leq n$, denote the conjugates of α_j over $\mathbb{Q}$, is a rational form in n variables of degree n. We imbed $T = R_{k/\mathbb{Q}}(G_m)$ into $GL_n(\Omega)$ by means of the basis $\{\alpha_1, \cdots, \alpha_n\}$. Let $\mathfrak{G}$ denote the set of rational forms f in n variables such that $f = F \circ t$ with $t \in T_{\mathbb{Q}}$ and $f = F \circ t_v$ with $(t_v) \in T_A(\infty)$. The set $\mathfrak{G}$ is divided into classes by the following integral equivalence relation: $f, g \in \mathfrak{G}$, $f \sim g$ is and only if $f = g \circ t$ with $t \in T_{\mathbb{Z}} = T_{\mathbb{Q}}(\infty)$. The torus T' is nothing but the subgroup $\{t \in T;\ F = F \circ t\}$ of T. There exists an injection of the set of integral classes in $\mathfrak{G}$ into the quotient space $T'_A/T'_{\mathbb{Q}}T'_A(\infty)$. Moreover, if the class number of k is 1, this injection is surjective (*cf.* [7]). Thus, (17) gives the following result: if the class number of k is 1, the number of integral classes in $\mathfrak{G}$ is given by

(18) $$\tau_{T'} \cdot \frac{q(\lambda_{\mathbb{Q}}(\infty))\, q((\hat{\lambda})_{\mathbb{Q}})}{q(\lambda_\infty) \prod_{p \neq \infty} q(\lambda_p^c)}\,.$$

Since the quotient of q-symbols is independent of the choice of λ, we may choose λ as in (15) with slight modification and (18) can then be effectively computed for certain k.

REFERENCES

1. E. Artin, *Über eine neue Art von* L-*Reihen*. Abh. Math. Sem. Univ. Hamburg, 3 (1923), 89-108.
2. ———, *Zur Theorie der* L-*Reihen mit allgemeinen Gruppen-Charakteren*. Abh. Math. Sem. Univ. Hamburg, 8 (1931), 292-306.
3. R. Brauer, *On Artin's* L-*series with general group characters*. Ann. of Math. (2) 48 (1947), 502-514.
4. C. Chevalley, *Théorie des groupes de Lie*. Tome II. Groupes algébriques. Actualités Sci. Ind. no. 1152. Hermann & Cie., Paris, 1951.
5. T. Ono, *Arithmetic of algebraic tori*. Ann. of Math. (2) 74 (1961), 101-139.
6. ———, *On the Tamagawa number of algebraic tori*. Ann. of Math. (2) 78 (1963), 47-73.
7. J. Shyr, *On the norm form of a finite Galois extension over* $\mathbb{Q}$. Bull. Amer. Math. Soc. 81 (1975), 619-621.
8. ———, *A generalization of Dirichlet's unit theorem*. J. Number Theory 9 (1977), 213-217.

9. J. Tate, *Fourier analysis in number fields and Hecke's zeta functions*. Thesis, Princeton University, 1950.

10. A. Weil, *Adeles and algebraic groups*. Princeton, N.J., Institute for Advanced Study, 1961.

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