CHAINS OF PRIMES IN $R\langle x\rangle$

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INTRODUCTION

Let R be a commutative semilocal (Noetherian) domain with 1, and let $R\langle x\rangle$ denote the quotient ring of $R[x]$ with respect to the multiplicative system

$$R[x] - \bigcup \{Q: Q \text{ is a maximal ideal of } R[x] \text{ with } Q \cap R \text{ maximal in } R\}$$

(here, x is an indeterminate). As is pointed out in [6], when studying chain conditions, $R\langle x\rangle$ is essentially $R[x]$ with all maximal ideals of $R[x]$ whose intersection with R is nonmaximal having been discarded. The main result of [6] is that $R\langle x\rangle$ is taut-level (height P + depth P = altitude $R\langle x\rangle$ for all primes P of $R\langle x\rangle$) if and only if $R\langle x\rangle$ satisfies the first chain condition (every maximal chain of prime ideals of $R\langle x\rangle$ has length equal to altitude $R\langle x\rangle$). This work represents a further study of chains of prime ideals in $R\langle x\rangle$.

The paper is divided into two sections. In section I our main result is as follows. If K is prime in $R\langle x\rangle$ with $K \cap R = 0$ and depth $K = d > 1$, then there are infinitely many primes p in R with height p = 1 and depth p = d - 1. A number of consequences of this result are then derived. For instance, we show that if i is a positive integer, then R is a D_i ring (every depth i prime ideal has height equal to altitude R - i) if and only if $R\langle x\rangle$ is a D_{i+1} ring. Another example is that the integral closure R' of R is shown to be an H_1 ring (every height one prime ideal of R' has depth equal to altitude R' - 1) if and only if $R\langle x\rangle$ is H_1. We close section I by proving a strong converse to the first result mentioned above. Specifically, if there is a height one prime ideal p in R with depth p = d, then in $R\langle x\rangle$ there are infinitely many primes K satisfying $K \cap R = 0$ and depth K = d + 1.

In the second section, we extend to $R\langle x\rangle$ a major result of Ratliff [8, Proposition 2.2]. We show that if $0 \subset P_1 \subset \cdots \subset P_h$ is a saturated chain of primes in $R\langle x\rangle$ with depth $P_h = d > 0$, then for each $j = 0, 1, \cdots, d - 1$, there are infinitely many primes P in $R\langle x\rangle$ with height P = h + j and depth P = d - j. In fact if height $P_h = h$ and N is a maximal ideal of $R\langle x\rangle$ containing P_h, then for each $j = 1, \cdots, d - 1$, we may require the primes P to satisfy $P_h \subset P \subset N$ with P contained in no other maximal ideals of $R\langle x\rangle$. Several applications of this are given to the case where $R\langle x\rangle$ is assumed to be an H_i ring. For example, if there is a saturated chain of length h from 0 to a prime P in $R\langle x\rangle$ with depth $P = d > 0$, and if h + d < altitude $R\langle x\rangle$, then either h > i or $h + d \leq i$.

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Preliminaries. Let T be a commutative semi-quasilocal integral domain with 1. If x is an indeterminate and Q is a prime ideal of T[x] with $Q \neq (Q \cap T)T[x]$, then we shall call Q an *upper* to $Q \cap T$. A maximal ideal Q of T[x] is said to be a *type I maximal* if $Q \cap T$ is maximal in T; otherwise Q is called a *type II maximal.* Following [6] we shall use $T\langle x \rangle$ to denote the quotient ring of T[x] with respect to the multiplicative system $T[x] - \bigcup\{Q\text{: } Q \text{ is a type I maximal ideal of } T[x]\}$. A discussion of facts about $T\langle x \rangle$ is given in [6]. In particular it is pointed out that if $M_1, \cdots, M_n$ denote the maximal ideals of T, then a maximal ideal Q of T[x] is a type I maximal if and only if $Q \supset M_1T[x] \cap \cdots \cap M_nT[x]$. From this and [1, Lemma 3], it follows that the maximal ideals of $T\langle x \rangle$ are precisely the ideals $QT\langle x \rangle$ where Q is a type I maximal ideal of T[x]. As $T\langle x \rangle$ is essentially T[x] with the type II maximals discarded, we shall frequently refer to a prime P of $T\langle x \rangle$ as being an upper to $P \cap T$. Our meaning is that the prime $P \cap T[x]$ of T[x] is an upper to $(P \cap T[x]) \cap T$. Throughout the paper we shall freely use facts about uppers in $T\langle x \rangle$; these facts follow easily from [4, Section 1-5] and from elementary knowledge of quotient rings.

The terminology in this paper is fairly standard, and definitions of unfamiliar terms may be found in [2], [4], or [8]. We do wish to point out that when using the terms "local" and "semilocal", we are assuming "Noetherian" also. Also, we shall use T' to denote the integral closure of a domain T.

SECTION 1

We begin with a result which will quite often be needed in what follows and for which we know of no reference (although it follows quite readily from known results).

LEMMA 1.0. *Let* T *be an integral extension domain of a semilocal domain* R, *and let* $0 \subset Q_1 \subset \cdots \subset Q_n$ $(n \geq 1)$ *be a saturated chain of prime ideals in* T. *If* depth $Q_n = d > 0$, *then there exist infinitely many prime ideals* q *in* R *with* height q = n *and* depth q = d.

Proof. There exists one such q by [3, Theorem 1,11]. If n = 1 then the conclusion holds by [5, Proposition 2]. If n > 1 then we may choose a prime ideal p in R such that $p \subset q$, height p = n - 1, and height q/p = 1. Then, applying [5, Proposition 2] to R/p yields infinitely many primes $q' \supset p$ with height $q'/p = 1$ and depth q' = d. For all but finitely many of the q', we have by [5, Theorem 1] that height q' = 1 + height p = n, and the proof is complete.

PROPOSITION 1.1. *In the semilocal domain* R, *let* M *be a maximal ideal, and let* Q *be an upper to* M *in* R[x]. *Assume that there is an upper* K *to* 0 *in* R[x] *with* $K \subset Q$, height Q/K = d > 1, *and* $K \not\subset MR[x]$. *Then there are infinitely many primes* p *in* R *with* height p = 1 *and* depth p = d - 1.

Proof. Since $K \not\subset MR[x]$, by [3, Theorem 1.3], there is a prime N in R' with $N \cap R = M$ and height N = d. Since R is semilocal, R' has only finitely many maximal ideals $N_1 = N, N_2, \cdots, N_k$ and we may choose $u \in N - N_2 \cup \cdots \cup N_k$. If $N' = N \cap R[u]$ then height N' = d since $R[u] \subset R'$ is an integral extension and N is the only prime of R' lying over N'. By the principal ideal theorem, we may shrink N' to a prime P' minimal over u with height N'/P' = d - 1. Of course height P' = 1. Since $u \in P'$, N is the only maximal of R[u] containing P', so that depth P' = d - 1. The conclusion now follows from Lemma 1.0.

Remark. In Theorem 1.3 we shall remove the requirement $K \not\subset MR[x]$ from Proposition 1.1. On the other hand, we shall have to assume that depth $K = d$ rather than merely requiring height $(Q/K) = d$.

LEMMA 1.2. *If* R *is a domain and the maximal ideal* Q *of* $R[x]$ *contains a monic polynomial, then* $Q \cap R$ *is maximal in* R.

Proof. Since Q contains a monic polynomial, the extension $R/Q \cap R \subset R[x]/Q$ is an integral extension. However, $R[x]/Q$ is a field, whence $R/Q \cap R$ is a field [4, Theorem 16]. Thus $Q \cap R$ is maximal.

THEOREM 1.3 (*cf.* [5, Theorem 3]). *If* R *is a semilocal domain and there is an upper* K_1 *to* 0 *in* $R\langle x\rangle$ *with* depth $K_1 = d > 1$, *then in* R *there are infinitely many primes* p *with* height $p = 1$ *and* depth $p = d - 1$.

Proof. Choose a maximal ideal Q_1 in $R\langle x\rangle$ with $Q_1 \supset K_1$ and

$$\text{height } Q_1/K_1 = d\,.$$

Let $Q = Q_1 \cap R[x]$ and $K = K_1 \cap R[x]$. Also, let $M = Q \cap R$. Then M is maximal in R and $Q = (M, f)R[x]$ for some monic polynomial f irreducible modulo M. By Proposition 1.1 we may as well assume $K \subset MR[x]$, so that $f \notin K$. (Alternatively, if $f \in K$ then $R[x]/K$ is an integral extension of R of altitude d, in which case R has altitude d, and the theorem is trivial.) Use the principal ideal theorem to shrink Q to a prime Q_0 minimal over (K, f) with height $Q/Q_0 = d - 1$ and height $Q_0/K = 1$. If N is any type I maximal of $R[x]$ with $Q_0 \subset N$, then

$$\text{height } N/Q_0 \leq \text{height } N/K - 1 \leq d - 1\,.$$

By Lemma 1.2, Q_0 lies in no type II maximals, whence depth $Q_0 = d - 1$. Let $P_0 = Q_0 \cap R$, then $R/P_0 \subset R[x]/Q_0$ is an integral extension. Thus

$$\text{depth } P_0 = \text{altitude } R/P_0 = \text{altitude } R[x]/Q_0 = d - 1\,.$$

Moreover, since height $Q_0/K = 1$, $K \not\subset P_0R[x]$, so that by [3, Theorem 1.3] there is a height one prime P_0' of R' with $P_0' \cap R = P_0$. Of course,

$$\text{depth } P_0' = \text{depth } P_0 = d - 1\,.$$

Therefore, by Lemma 1.0 there are infinitely many primes p of R with

$$\text{height } p = 1 \quad \text{and} \quad \text{depth } p = d - 1\,,$$

as desired.

Remark. Let R be a semilocal domain with maximal ideals $M_1, M_2, \cdots, M_n$. In $R[x]$ let s denote the multiplicatively closed set

$$R[x] - \bigcup \{(M_i, x)R[x]: i = 1, \cdots, n\}\,.$$

Then Theorem 1.3 remains true if we replace $R\langle x\rangle$ by $R[x]_s$. In the proof of the theorem, one need only take $f(x) = x$.

As a first application of Theorem 1.3 (and the remark above), we generalize the domain case of [2, Theorem 12].

PROPOSITION 1.4. *Let* R *be a semilocal domain, and let* i *be a positive integer. Then the following statements are equivalent:*

1) R *is* D_i

2) $R\langle x\rangle$ *is* D_{i+1}

3) $R[x]_s$ *is* D_{i+1}

Proof. The proofs of 2) implies 1) and 3) implies 1) are routine and therefore omitted. We shall prove 1) implies 2); the proof that 1) implies 3) is similar. Let P be a prime of $R\langle x\rangle$ with height P = h and depth P = i + 1. If $P = (P \cap R)R\langle x\rangle$ then height $(P \cap R) = h$ and depth $(P \cap R)$ = depth P - 1 = i. Since R is D_i we have $h + (i + 1) = (h + i) + 1$ = altitude R + 1 = altitude $R\langle x\rangle$. Hence we assume $P \neq (P \cap R)R\langle x\rangle$. Since $i + 1 > 1$ we may apply Theorem 1.3 to

$$R\langle x\rangle/(P \cap R)R\langle x\rangle = (R/P \cap R)\langle x\rangle.$$

This yields infinitely many primes $p \supset P \cap R$ in R with height $(p/P \cap R) = 1$ and depth p = i. By [5, Theorem 1] there is such a p with

$$\text{height } p = \text{height } (P \cap R) + 1 = \text{height } P = h.$$

Again, since R is D_i, we have $h + i$ = altitude R and $h + i + 1$ = altitude $R\langle x\rangle$.

By considering the integral closure R' of R, we can add several equivalences to the above.

PROPOSITION 1.5. *Let* R *be a semilocal domain, and let* i *be a positive integer. Then the following statements are equivalent:*

1) R *is* D_i

2) R' *is* D_i

3) $R'\langle x\rangle$ *is* D_{i+1}

Proof. Assume that R is D_i and let P be prime in R' with height P = h and depth P = i. Since $i > 0$ there is by [3, Theorem 1.11] a prime in R of height h and depth i, whence $h + i$ = altitude R = altitude R'. Hence 1) implies 2). That 2) implies 1) follows easily from the going up and incomparability theorems.

We next prove that 2) implies 3). Accordingly, let P be a prime of R'[x] with height $PR'\langle x\rangle$ = height P = h and depth $PR'\langle x\rangle = i + 1$. Let $P_0 = P \cap R'$ and pick $u \in P_0$ with u in no other prime of R' which lies over $P_0 \cap R$. Let $Q_0 = P_0 \cap R[u]$ and $Q = P \cap R[u][x]$. Now if $P = P_0R'[x]$ then $Q = Q_0R[u][x]$ and height P_0 = height Q_0 = height $Q \geq$ height P = h. Thus height P_0 = h. Of course depth P_0 = depth Q_0 = depth $QR\langle x\rangle$ - 1 = depth $PR\langle x\rangle$ - 1 = i, and the result follows from the fact that R' is D_i. Thus we assume that $P \neq P_0R'[x]$, whence $Q \neq Q_0R[u][x]$. Thus Q is an upper to Q_0 and depth $QR[u]\langle x\rangle = i + 1 > 1$. Applying Theorem 1.3 to $R[u]/Q_0$ we find a prime q in R[u] with $q \supset Q_0$, height $(q/Q_0) = 1$, and depth q = i. Since height Q_0 = h - 1 there is by [5, Theorem 8] a prime p of R with height p = h and depth p = i. Therefore, since R is D_i we have $h + i$ = altitude R and $h + i + 1$ = altitude $R'\langle x\rangle$.

Finally, 3) implies that $R\langle x\rangle$ is D_{i+1} by integral dependence, whence R is D_i by Proposition 1.4.

We now turn to what can be said concerning the H_i condition in $R\langle x\rangle$.

PROPOSITION 1.6. *If* R *is a semilocal domain, and* $R\langle x\rangle$ *is* H_i, *then* R *is both* H_i *and* H_{i-1}.

Proof. If P is a height i prime of R, then height $PR\langle x\rangle = i$, whence $i + \text{depth } P = i + \text{depth } PR\langle x\rangle - 1 = \text{altitude } R\langle x\rangle - 1 = \text{altitude } R$, as desired. On the other hand, if Q is a height $i - 1$ prime of R and depth $Q = d$, we must show that $i - 1 + d = \text{altitude } R$. If $d = 0$ then Q is maximal and $(Q, x)R\langle x\rangle$ is a height i maximal ideal of $R\langle x\rangle$, whence altitude $R\langle x\rangle = i$ and altitude $R = i - 1$. We therefore assume $d > 0$. Let $M \supset Q$ be a maximal ideal of R with

$$\text{height } (M/Q) = d .$$

Then height $((M, x)R\langle x\rangle/QR\langle x\rangle) = d + 1 > 1$ and by [8, Proposition 2.2], together with [5, Theorem 1], there is a prime $Q' \supset QR\langle x\rangle$ of $R\langle x\rangle$ with height $Q' = i$ and height $((M, x)R\langle x\rangle/Q') = d$. Since depth $QR\langle x\rangle = 1 + \text{depth } Q = d + 1$ we have depth $Q' = d$. Thus, since $R\langle x\rangle$ is H_i,

$$i - 1 + d = \text{altitude } R\langle x\rangle - 1 = \text{altitude } R .$$

This completes the proof.

PROPOSITION 1.7. (*cf.* [7, Theorem 2.6]). *Let* R *be a semilocal domain and assume that* $i < \text{altitude } R$. *Then* $R\langle x\rangle$ *is* H_i *if and only if* R *is* H_i *and there are no maximal chains of primes of length* $i + 1$ *in* $R\langle x\rangle$.

Proof. Assume that $R\langle x\rangle$ is H_i. By Proposition 1.6 R is also H_i. Let $0 \subset P_1 \subset \cdots \subset P_{i+1}$ be a maximal chain of primes in $R[x]$ with $P_{i+1} \cap R$ a maximal ideal of R. By [3, Corollary 1.5] we may assume that height $P_i = i$ and that P_i is an upper to $P_i \cap R$. By [6, Theorem 3] applied to $R/P_i \cap R$ we may assume that P_{i+1} is the only type I maximal ideal containing P_i. Hence depth $P_iR\langle x\rangle = 1$. Of course, height $P_iR\langle x\rangle = i$ so that $i + 1 = \text{altitude } R\langle x\rangle$ and altitude $R = i$, a contradiction.

Conversely, assume that R is H_i and that there are no maximal chains of primes of length $i + 1$ in $R\langle x\rangle$. In $R\langle x\rangle$ let P be prime with height $P = i$ and depth $P = d$. Since R is H_i we may assume that P is upper to $P \cap R$. Also we have $d > 1$ by hypothesis. Thus by Theorem 1.3 applied to $R/P \cap R$ there are infinitely many primes p of R with $P \cap R \subset p$, height $(p/P \cap R) = 1$, and

$$\text{depth } p = d - 1 .$$

Hence by [5, Theorem 1], there is such a prime p with

$$\text{height } p = \text{height } (P \cap R) + 1 = \text{height } P = i .$$

Since R is H_i we have $i + d - 1 = \text{altitude } R$ and $i + d = \text{altitude } R\langle x\rangle$.

PROPOSITION 1.8. *Let* R *be a semilocal domain. Then* $R\langle x\rangle$ *is* H_i *if and only if* $R[x]_S$ *is* H_i.

Proof. Assume that $R\langle x\rangle$ is H_i and let P be a prime of $R[x]$ with height $P = i$ and depth $PR[x]_S = d$. We must show that $i + d = \text{altitude } R[x]_S$. The result being trivial if $d = 0$, we assume $d > 0$. Also, if $P = (P \cap R)R[x]$ then

$$d = \text{depth } PR[x]_s = \text{depth } PR\langle x \rangle,$$

whence $i + d = \text{altitude } R\langle x \rangle = \text{altitude } R[x]_s$. Consequently, we take P to be an upper to $P \cap R$. Choose M maximal in R so that height $((M, x)R[x]/P) = d$. If $d = 1$ we may use [6, Theorem 3] to replace P by an upper P' to $P \cap R$ with $P' \subset (M, x)R[x]$, height $((M, x)R[x]/P') = 1$, and P' contained in no other type I maximal ideal of $R[x]$. Thus

$$\text{height } P'R\langle x \rangle = \text{height } (P \cap R)R\langle x \rangle + 1 = \text{height } P = i$$

and depth $P'R\langle x \rangle = 1$. Since $R\langle x \rangle$ is H_i we have

$$i + d = i + 1 = \text{altitude } R\langle x \rangle = \text{altitude } R[x]_s.$$

Finally, if $d > 1$ we may use the remark following Theorem 1.3 (applied to $R/P \cap R$), together with [5, Theorem 1], to produce a prime $p \supset P \cap R$ in R with height $p = i$ and depth $p = d - 1$. Thus, since R is H_i, we have that

$$i + d = \text{altitude } R + 1 = \text{altitude } R[x]_s.$$

Conversely, assume that $R[x]_s$ is H_i. Let Q be a prime of $R[x]$ with height $Q = i$ and depth $QR\langle x \rangle = d$. Let $Q_0 = Q \cap R$; if $Q = Q_0R[x]$ then height $Q_0 = i$ and depth $Q_0 = d - 1$. Thus height $QR[x]_s = i$ and

$$\text{depth } QR[x]_s = \text{depth } Q_0R[x]_s = d,$$

whence $i + d = \text{altitude } R[x]_s = \text{altitude } R\langle x \rangle$. Hence we may as well assume that Q is an upper to Q_0. The result is trivial if $d = 0$. If $d = 1$ there is by [9, Theorem 5.1 and Proposition 5.3] a maximal ideal N of R and an upper Q' to Q_0 with $Q' \subset (N, x)R[x]$ and height $((N, x)R[x]/Q') = 1$. By [6, Theorem 3] we may assume Q' to be contained in no other type I maximal of $R[x]$, whence depth $Q'R[x]_s = 1$ and the result follows. Now assume $d > 1$. Applying Theorem 1.3, together with [5, Theorem 1], to R/Q_0, we find a prime $q \supset Q_0$ in R with height $q = \text{height } Q_0 + 1 = i$ and depth $q = d - 1$. Therefore, height $qR[x]_s = i$, depth $qR[x]_s = d$, and the conclusion follows easily.

We would now like to prove the H_i analogues of Propositions 1.4 and 1.5. As we have already seen in Proposition 1.7, R and $R\langle x \rangle$ do not necessarily share the H_i property. In fact the situation is even less tractable, for in Theorem 1.9 we are forced to settle for the H_1 case.

THEOREM 1.9. *The following statements are equivalent for a semilocal domain* R:

1) $R[x]_s$ *is* H_1.
2) R' *is* H_1.
3) $R\langle x \rangle$ *is* H_1.
4) $R'\langle x \rangle$ *is* H_1.

Proof. Assume that $R[x]_s$ is H_1 and let P' be prime in R' with height $P' = 1$ and depth $P' = d$. If $d = 0$ choose u in P', such that u lies in no other maximal ideal of R'. Let K be the kernel of the natural map from $R[x]$ to $R[u]$. Then if $P' \cap R = M$ we have that $(M, x)R[x]_s$ is the only maximal ideal of $R[x]_s$ which contains K_s (since $P' \cap R[u]$ is the only maximal ideal of $R[u]$ which contains u). Thus in $R[x]_s$ height $K_s = 1$ and depth $K_s = 1$. Since $R[x]_s$ is H_1, this implies that altitude R' = altitude $R[x]_s - 1 = 1$, as required. Thus assume $d > 0$. By [3, Theorem 1.11], there is a prime p of R with height $p = 1$ and depth $p = d$. Since R is obviously H_1 we get $1 + d$ = altitude R = altitude R'. Thus (1) implies (2).

We next show that 2) implies 1). Accordingly, let K be a height one prime of $R[x]$ with depth $K_s = d$. The result being easy if $K = (K \cap R)R[x]$, we assume that K is an upper to 0. If $d = 0$, then R is a field and there is nothing to prove. If $d = 1$, then by [3, Theorem 1.3] there is a height one maximal ideal in R', and the result follows since R' is H_1. We therefore assume $d > 1$ and choose a maximal M of R such that height $((M, x)R[x]/K) = d$. If $x \in K$ then, since K is the only height one prime of $R[x]$ containing x, we have height $(M, x)R[x] = d + 1$ by the principal ideal theorem. It follows easily that altitude $R[x]_s = d + 1$. If $x \notin K$ use the principal ideal theorem to shrink $(M, x)R[x]$ to a prime Q minimal over $(K, x)R[x]$ with height $((M, x)R[x]/Q) = d - 1$. Clearly depth $Q = d - 1$, and if $P = Q \cap R$, then, since $R/P = R[x]/Q$, we have depth $P = d - 1$. Again, by [3, Theorem 1.3] since height $K = 1$ and height $(Q/K) = 1$, we find a height one prime P' in R' with $P' \cap R = P$. Of course, depth P' = depth $P = d - 1$. Thus since R' is H_1, altitude $R[x]_s$ = altitude $R' + 1 = 1 + d - 1 + 1 = d + 1$, as desired.

The equivalence of 1) and 3) was established in Proposition 1.8. Thus we need only show the equivalence of 3) and 4). As 4) implies 3) by integral dependence, we confine ourselves to showing that 3) implies 4). Accordingly, let K' be prime in $R'[x]$ with height $K' = 1$ and depth $K'R'\langle x\rangle = d$. As usual we distinguish two cases. If $K' = (K' \cap R')R'[x]$ then $K' \cap R'$ is a height one prime of R' whose depth is $d - 1$. If $d = 1$ then $K' \cap R'$ is maximal and the result follows easily, since 3) implies 2). Thus assume $d > 1$ so that $d - 1 > 0$. By [3, Theorem 1.11], this yields in R a height one prime q with depth $q = d - 1$. In this case the result follows from the fact that R is H_1. On the other hand, if K' is an upper to 0 in $R'[x]$, then $K = K' \cap R[x]$ is an upper to 0 in $R[x]$, whence height $K = 1$. Of course

$$\text{depth } KR\langle x\rangle = \text{depth } K'R'\langle x\rangle = d.$$

Thus $d + 1$ = altitude $R\langle x\rangle$ = altitude $R'\langle x\rangle$. This completes the proof.

Remark. If in Proposition 1.9 "H_1" is replaced by "H_i", then, as noted in Proposition 1.8, 1) and 3) are equivalent. Moreover, it is not difficult to show that 4) implies 3) and that 1) implies 2). The author does not know whether these implications can be reversed.

We now wish to prove a strong converse to Theorem 1.3. For this we shall need a generalization of [5, Proposition 2]. As a proof of Proposition 1.10 requires only trivial modifications in the proof of [5, Proposition 2], we omit it and merely state the result.

PROPOSITION 1.10. *Let* R *be a Noetherian domain with primes* $P \subset P_0$ *such that* height $P = 1$ *and* depth P = depth $P_0 + 1 = d$. *Assume that* P *does not contain the Jacobson radical of* R. *Finally, let* $A = \{M\colon M$ *is a maximal ideal of* R *with* $P \not\subset M\}$, *and assume that* $P \not\subset \bigcup \{M\colon M \in A\}$. *Then there are infinitely many height one primes* $p \subset P_0$ *with* depth $p = d$.

THEOREM 1.11 (*cf.* [5, Theorem 4]). *In the semilocal domain* R, *let* P *be a height one prime with* depth $P = d \geq 0$. *Then in* $R\langle x\rangle$ *there are infinitely many uppers* K *to* 0 *with* depth $K = d + 1$.

Proof. If $d = 0$, choose any upper Q to P. Then $QR\langle x\rangle$ is maximal in $R\langle x\rangle$ with height $QR\langle x\rangle = 2$. By [6, Theorem 1] there are infinitely many uppers K to 0 in R[x] with $K \subset Q$ and K contained in no other type I maximal ideal of R[x]. For each such K we have depth $KR\langle x\rangle$ = height (Q/K) = 1, as required. Now assume $d > 0$. Let $M_1, \cdots, M_k, \cdots, M_n$ denote the maximal ideals of R with $P \not\subset M_i$ for $i = 1, \cdots, k$. One verifies easily that a maximal ideal Q of $R\langle x\rangle$ satisfies $Q \not\supset PR\langle x\rangle$ if and only if $Q \supset M_1R\langle x\rangle \cap \cdots \cap M_kR\langle x\rangle$. Thus $A = \{Q\colon Q$ is maximal in $R\langle x\rangle$ and $Q \not\supset PR\langle x\rangle\}$ is closed in Spec $R\langle x\rangle$, and by [1, Lemma 3] $PR\langle x\rangle \not\subset \bigcup\{Q\colon Q \in A\}$. Also, since the Jacobson radical of $R\langle x\rangle$ is just $J = M_1R\langle x\rangle \cap \cdots \cap M_nR\langle x\rangle$, clearly $J \not\subset PR\langle x\rangle$. If $P_0 = (P, x)R\langle x\rangle$ then depth P_0 = depth $PR\langle x\rangle - 1 = d$. Therefore, the requirements of Proposition 1.10 are met and there are infinitely many primes K of $R\langle x\rangle$ with height $K = 1$, $K \subset P_0$, and depth $K = d + 1$. We assert that for the infinitely many $K \neq PR\langle x\rangle$, we have K an upper to 0. For $K \subset P_0$ implies that $K \cap R \subset P_0 \cap R = P$, whence, if $K \neq PR\langle x\rangle$, we have $K \cap R = 0$. This completes the proof.

SECTION 2

In this section we wish to extend [8, Proposition 2.2] to $R\langle x\rangle$ and to cite a few consequences of this extension. We shall require the following lemma.

LEMMA 2.1. *Let* R *be a semilocal domain with maximal ideals*

$$M_1, \cdots, M_m, \cdots, M_n,$$

and in R[x] *let* $Q_1, \cdots, Q_m$ *be uppers to* $M_1, \cdots, M_m$, *respectively. Then there is a polynomial* $f(x) \in R[x]$ *with* $f(x) \in Q_1 \cap \cdots \cap Q_m$ *but in no other type* I *maximal ideals of* R[x].

Proof. For each $j = 1, \cdots, m$, we have $Q_j = (M_j, f_j(x))R[x]$ for some monic polynomial $f_j(x)$ of R[x]. Write $f_j(x) = a_{jr}x^r + a_{j,r-1}x^{r-1} + \cdots + a_{j0}$, where $a_{ji} = 0$ for $i > \deg f_j$. For each $\ell = 1, \cdots, r$, use the Chinese Remainder Theorem to choose $b_\ell \in R$ with $b_\ell \equiv a_{j\ell} \pmod{M_j}$ for $j = 1, \cdots, m$, and $b_\ell \equiv 0 \pmod{M_j}$ for $j = m + 1, \cdots, n$. Also choose b_0 such that $b_0 \equiv a_{j0} \pmod{M_j}$ for $j = 1, \cdots, m$, and $b_0 \equiv 1 \pmod{M_j}$ for $j = m + 1, \cdots, n$. If $f(x) = b_rx^r + \cdots + b_0$, then $f(x) \equiv f_j(x) \pmod{M_j}$ for $j = 1, \cdots, m$, and $f(x) \equiv 1 \pmod{M_j}$ for $j = m + 1, \cdots, n$. Thus $f(x) \in Q_1 \cap \cdots \cap Q_m$. Clearly f(x) lies in no upper to M_j for $j = m + 1, \cdots, n$. Moreover, if $f(x) \in Q$ for some upper Q to M_i, $1 \leq i \leq m$, then $f_i(x) \in Q$ also, whence $(M_i, f_i(x))R[x] = Q_i \subset Q$ and $Q = Q_i$. This completes the proof.

LEMMA 2.2. *Let* R *be a semilocal domain, and in* $R\langle x\rangle$ *let* $0 \subset K \subset P$ *be a saturated chain of primes with* depth $P = d > 1$. *Then there are infinitely many primes* p *in* R *with* height $P = 2$ *and* depth $p = d - 1$.

Proof. Let $P_0 = P \cap R$. If $P = P_0R\langle x\rangle$ then by [5, Theorem 6], there is a saturated chain $0 \subset p_0 \subset P_0$ in R. Since depth $P_0 = d - 1$, the result follows easily from Lemma 1.0. Thus we assume that P is an upper to P_0. By [3, Theorem 1.3] there is a prime P_0' in R' with $P_0' \cap R = P_0$ and height $P_0' = 1$. Choose $u \in P_0'$ but in no other prime lying over P_0, and contract to $R[u]$. If $P_0'' = P_0' \cap R[u]$ then height $P_0'' = 1$. By going up in the integral extension $R\langle x\rangle \subset R[u]\langle x\rangle$ there is a prime P'' in $R[u]\langle x\rangle$ with height $(P''/P_0''R[u]\langle x\rangle) = 1$ and $P'' \cap R\langle x\rangle = P$. Of course P'' is an upper to P_0'' whence height $P'' = 2$. Since also

$$\text{depth } P'' = \text{depth } P = d > 1$$

we may apply Theorem 1.3 to $R[u]/P_0''$ to obtain a prime $q \supset P_0''$ in $R[u]$ with height $(q/P_0'') = 1$ and depth $q = d - 1$. By Lemma 1.0, this yields an infinitude of primes p in R with height $p = 2$ and depth $p = d - 1$.

THEOREM 2.3. *Let* R *be a semilocal domain, and in* $R\langle x\rangle$ *let* P *be a prime with* depth $P = d > 0$. *If there is a saturated chain of primes in* $R\langle x\rangle$ *of length* $h > 0$ *from* 0 *to* P, *then there are infinitely many primes* Q *in* $R\langle x\rangle$ *with* height $Q = h$ *and* depth $Q = d$.

Proof. Let $p = P \cap R$ and first assume that $P = pR\langle x\rangle$. If $d = 1$, then p is maximal. By [5, Theorem 6] there is a saturated chain $0 \subset p_1 \subset \cdots \subset p_{h-1} \subset p$ in R and by [5, Theorem 8] we may assume that height $p_{h-1} = h - 1$. Let N be any upper to p in $R\langle x\rangle$. Then height $(N/p_{h-1}R\langle x\rangle) = 2$, and by [6, Theorem 3] applied to R/p_{h-1} there are infinitely many uppers P' to p_{h-1} in $R\langle x\rangle$ with $P' \subset N$ but in no other maximal ideal of $R\langle x\rangle$. For each such P' we have depth $P' = 1$ and height $P' = 1 + \text{height } p_{h-1} = h$.

We next assume that $d > 1$. Then in R we have depth $p = d - 1 > 0$, whence by Lemma 1.0 there are infinitely many primes q in R with height $q = h$ and depth $q = d - 1$. For each such q we have height $qR\langle x\rangle = h$ and depth $qR\langle x\rangle = d$, completing the proof in this case.

To complete the proof, we attack the case where P is upper to p. If $d = 1$ then, in the given saturated chain $0 \subset P_1 \subset \cdots \subset P_{h-1} \subset P$, we may assume by [3, Corollary 1.5] that height $P_{h-1} = h - 1$. Choose a maximal ideal $N \supset P$ in $R\langle x\rangle$; clearly height $(N/P) = 1$. Thus $P_{h-1} \subset P \subset N$ is saturated and by [8, Proposition 2.2] there are infinitely many primes P' with $P_{h-1} \subset P' \subset N$ saturated. If $P' = (P' \cap R)R\langle x\rangle$ then we have $N \cap R = P' \cap R$. Thus infinitely many of the P' are uppers. By [5, Theorem 1] we may select such a P' with

$$\text{height } P' = 1 + \text{height } P_{h-1} = h\ .$$

For this P' we apply [6, Theorem 3] to $R/P' \cap R$ to product infinitely many uppers Q to $P' \cap R$ with $Q \subset N$ but in no other maximal ideal of $R\langle x\rangle$ and

$$\text{height } (N/Q) = 1\ .$$

For each such Q we have depth $Q = 1$ and height $Q = 1 + \text{height } (P' \cap R) = h$.

Finally, assume $d > 1$. If $h = 1$ the result follows easily from Theorem 1.3. Also, if $h = 2$, the result follows from Lemma 2.2. Hence, we assume that $h > 2$. By [3, Corollary 1.5] we may pick $P_{h-1} \subset P$ with P_{h-1} upper to $p = P_{h-1} \cap R$ and height $P_{h-1} = h - 1$. Applying Lemma 2.2 to R/p we find a prime $q \supset p$ in R with height $(q/p) = 2$ and depth $q = d - 1$. Since there is a saturated chain of length height $p + 2 = h$ from 0 to q in R, we may apply Lemma 1.0 to find infinitely many primes q' in R with height $q' = h$ and depth $q' = d - 1$. For any such q', if $Q = q'R\langle x \rangle$, then height $Q = h$ and depth $Q = d$. This completes the proof.

THEOREM 2.4. *Let* R *be a semilocal domain, and in* $R\langle x \rangle$ *let* P *be prime with* depth $P = d > 1$. *If there is a saturated chain of primes in* $R\langle x \rangle$ *of length* $h > 0$ *from* 0 *to* P, *then for each* $j = 1, \cdots, d - 1$, *there are infinitely many primes* Q *in* $R\langle x \rangle$ *with* height $Q = h + j$ *and* depth $Q = d - j$. *In fact, if* height $P = h$ *and* N *is any maximal ideal of* $R\langle x \rangle$ *with* $N \supset P$ *and* height $(N/P) = d$, *then we may pick the primes* Q *in such a way that* $P \subset Q \subset N$ *and* Q *lies in no other maximal ideal of* $R\langle x \rangle$.

Proof. By Theorem 2.3 we may assume that height $P = h$. Let N be as hypothesized. By induction it suffices to produce infinitely many primes Q with $P \subset Q \subset N$, height $Q = h + 1$, depth $Q = d - 1$, and Q contained in no other maximal ideals of $R\langle x \rangle$. By Lemma 2.1 there is an element $f \in N$ with f in no other maximal ideals of $R\langle x \rangle$. If $f \in P$ the result follows easily from [8, Proposition 2.2] and [5, Theorem 1]. Thus we assume $f \notin P$. We wish to produce a prime Q_1 with $P \subset Q_1 \subset N$, height $(Q_1/P) = 1$, height $(N/Q_1) = d - 1$, and with Q_1 not contained in the union of the other maximal ideals of $R\langle x \rangle$. By the principal ideal theorem, we may shrink N to a prime Q_1 minimal over $(P, f)R\langle x \rangle$ with height $(N/Q_1) = d - 1$. Of course height $(Q_1/P) = 1$, also by the principal ideal theorem. Finally, since $f \in Q_1$, we have $Q_1 \subset N$ but not in the union of the other maximal ideals of $R\langle x \rangle$. Inductively, suppose that primes $Q_1, \cdots, Q_k$ have been chosen such that for each $i = 1, \cdots, k$, we have $P \subset Q_i \subset N$, height $(Q_i/P) = 1$, height $(N/Q_i) = d - 1$, and Q_i not contained in the union of the other maximal ideals of $R\langle x \rangle$. For each $i = 1, \cdots, k$, choose $g_i \in Q_i$ with N the only maximal ideal of $R\langle x \rangle$ containing g_i. Then $g = g_1 g_2 \cdots g_k \in Q_1 \cap \cdots \cap Q_k$ and $g \in N$ but in no other maximal ideals of $R\langle x \rangle$. Also, since for each $i = 1, \cdots, k$, height $(N/Q_i) = d - 1 > 0$ we have $MR\langle x \rangle \not\subset Q_i$ for each maximal ideal M of R. Thus the Jacobson radical J of $R\langle x \rangle$ (which is $\bigcap \{MR\langle x \rangle : M$ is a maximal ideal of $R\}$) satisfies $J \not\subset Q_i$ for each i, and by prime avoidance [4, Theorem 81], $J \not\subset Q_1 \cup \cdots \cup Q_k$. Pick $h \in J - Q_1 \cup \cdots \cup Q_k$ and note that $g + h \notin Q_1 \cup \cdots \cup Q_k$ and that $g + h \in N$ but $g + h$ lies in no other maximal ideal of $R\langle x \rangle$. Use the principal ideal theorem again to find a prime Q_{k+1} minimal over $(P, g + h)R\langle x \rangle$ with $Q_{k+1} \subset N$ and height $(N/Q_{k+1}) = d - 1$. One verifies easily that Q_{k+1} satisfies the conditions imposed on $Q_1, \cdots, Q_k$, whence by induction there are infinitely many primes Q of $R\langle x \rangle$ satisfying $P \subset Q \subset N$, height $(N/Q) = d - 1$, height $(Q/P) = 1$, and Q contained in no maximal ideal of $R\langle x \rangle$ except N. Moreover, by [5, Theorem 1] height $Q = 1 +$ height $P = h + 1$ for all but finitely many of the primes Q, so the proof is complete.

COROLLARY 2.5 (*cf.* [2, Theorem 2]). *Let* R *be a semilocal domain. In* $R\langle x \rangle$ *let* P *be a prime with* depth $P = d > 0$ *and let* h *be the length of some saturated chain of primes in* $R\langle x \rangle$ *from* 0 *to* P. *If* $R\langle x \rangle$ *is an* H_i *ring and* $h + d <$ altitude $R\langle x \rangle$, *then either* $h > i$ *or* $h + d \leq i$.

Proof. For each $j = 0, 1, \cdots, d - 1$ we may use Theorem 2.3 and 2.4 to find a prime Q with height $Q = h + j$ and depth $Q = d - j$. Suppose $h \leq i$ and $h + d > i$. If

$j = i - h$ then $0 \leq j \leq d - 1$ so that height $Q = i$ and depth $Q = d - i + h$. Since $R\langle x\rangle$ is H_i, we have altitude $R\langle x\rangle = i + d - i + h = h + d$, as desired.

COROLLARY 2.6 (*cf.* [2, Proposition 8]). *Let* R *be a semilocal domain, and assume that* $R\langle x\rangle$ *is* H_i. *If* $2i + 1 \geq$ altitude $R\langle x\rangle$, *then* $R\langle x\rangle$ *is* D_j *for* $j \geq i$. *If* $2i + 1 <$ altitude $R\langle x\rangle$, *then* $R\langle x\rangle$ *is* D_j *for* $j \geq$ altitude $R\langle x\rangle - i - 1$.

Proof. Assume $2i + 1 \geq$ altitude $R\langle x\rangle$ and let $j \geq i$. Let P be prime in $R\langle x\rangle$ with depth $P = j$ and height $P = h > 0$. If $h > i$ then

$$h + j \geq 2i + 1 \geq \text{altitude } R\langle x\rangle,$$

and we have $h + j =$ altitude $R\langle x\rangle$. Thus we assume $h \leq i$. Since $h + j > i$, we have $h + j =$ altitude $R\langle x\rangle$, by Corollary 2.5.

Now suppose that $2i + 1 < a =$ altitude $R\langle x\rangle$. Let Q be a prime with height $Q = h$ and depth $Q = j \geq a - i - 1$. If $h > i$ then $h + j > i + a - i - 1 = a - 1$, and $h + j = a$, as desired. On the other hand, if $h \leq i$, then

$$h + j \geq h + a - i - 1 > h + 2i + 1 - i - 1 = h + i > i.$$

Thus again by Corollary 2.5, we have $h + j = a$.

COROLLARY 2.7 (*cf.* [2, Proposition 10]). *Let* R *be a semilocal domain, and assume that* $R\langle x\rangle$ *is* H_i *for each* $i = i_1 = 0 < i_2 < \cdots < i_n =$ altitude $R\langle x\rangle$. *If* $j \geq i_k - i_{k-1}$ *for* $k = 1, \cdots, n$, *then* $R\langle x\rangle$ *is* D_j.

Proof. Let P be a prime in $R\langle x\rangle$ with height $P = h > 0$ and depth $P = j$. Then $i_{k-1} < h \leq i_k$ for some $k > 1$, whence

$$h + j \geq i_k - i_{k-1} + h > i_k - i_{k-1} + i_{k-1} = i_k.$$

Also $h \leq i_k$. Thus by Corollary 2.5, $h + j =$ altitude $R\langle x\rangle$.

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