INTERPOLATING SEQUENCES FOR HARDY AND BERGMAN CLASSES IN POLYDISKS

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let U^n denote the unit polydisk in n dimensional complex space, $\mathbb{C}^n$. For $(E, |\cdot|)$ a non-trivial complex Banach space, $1 \leq p < \infty$, and $\alpha \geq 0$, we define the following Banach spaces: $\ell^\infty(E)$ is the space of bounded, E-valued sequences; $\ell^p(E)$ is the space of E-valued sequences satisfying $(\|(e_i)_{i=1}^\infty\|_p)^p \equiv \sum |e_i|^p < \infty$; $H^\infty(U^n, E)$ is the space of bounded analytic E-valued functions on U^n; $H^p(U^n, E)$ is the space of analytic E-valued functions on U^n satisfying

$$(\|f\|_p)^p \equiv \sup_{r<1} (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f(re^{i\theta_1}, \cdots, re^{i\theta_n})|^p d\theta_1 \cdots d\theta_n < \infty;$$

and $A^{p,\alpha}(U^n, E)$ is the space of analytic E-valued functions satisfying

$$(\|f\|_{A^{p,\alpha}})^p = ((\alpha+1)/\pi)^n \int_{U^n} |f(z)|^p \prod_{k=1}^{n} (1 - |z_k|^2)^\alpha d\nu(z) < \infty,$$

where $z = (z_1, z_2, \cdots, z_n)$, and $d\nu(z)$ is Lebesgue measure on U^n. When $E = \mathbb{C}$, these are the familiar sequence, Hardy, and Bergman spaces. (*Cf.* [6], [10], and [11].)

If $a = (a_1, \cdots, a_n) \in U^n$, and f is a function on U^n, define $T_a^\infty f = f(a)$; $T_a^p f = \left(\prod_{k=1}^n (1 - |a_k|^2)\right)^{1/p} f(a)$; and $T_a^{p,\alpha} f = \left(\prod_{k=1}^n (1 - |a_k|^2)\right)^{(\alpha+2)/p} f(a)$. The operators, T_a^∞, T_a^p, and $T_a^{p,\alpha}$ are the normalized point evaluation operators on $H^\infty(U^n, E)$, $H^p(U^n, E)$, and $A^{p,\alpha}(U^n, E)$ respectively. If

$$\mathscr{A} = (a_i)_{i=1}^\infty = ((a_{i1}, \cdots, a_{in}))_{i=1}^\infty$$

is a sequence in U^n define $T_{\mathscr{A}}^p f = (T_{a_i}^p f)_{i=1}^\infty$, for $1 \leq p \leq \infty$; and

$$T_{\mathscr{A}}^{p,\alpha} f = (T_{a_i}^{p,\alpha} f)_{i=1}^\infty, \quad \text{for } 1 \leq p < \infty \text{ and } \alpha \geq 0.$$

The fundamental questions of this paper are: When is $T_{\mathscr{A}}^p(H^p(U^n, E)) = \ell^p(E)$? When is $T_{\mathscr{A}}^{p,\alpha}(A^p(U^n, E)) = \ell^p(E)$?

The sequence $\mathscr{A}$ is said to be $H^p(U^n, E)$ or $A^{p,\alpha}(U^n, E)$ interpolating if $T_{\mathscr{A}}^p(H^p(U^n, E)) \supseteq \ell^p(E)$ or $T_{\mathscr{A}}^{p,\alpha}(A^{p,\alpha}(U^n, E)) \supseteq \ell^p(E)$. We remark, first, that if a

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sequence is interpolating for one of the spaces of vector valued functions, it is clearly interpolating for the corresponding class of complex valued functions. Also, we note that when $p < \infty$, there is no *a priori* reason to expect that if $\mathscr{A}$ is $H^p(U^n, E)$ interpolating, then $T^p_{\mathscr{A}}(H^p(U^n, E)) = \ell^p(E)$.

Throughout this paper, $\rho(z, w)$ shall denote the pseudo-hyperbolic metric on U^n. Of course,

$$\begin{aligned} \rho(z, w) &= \operatorname{Max}\{|z_i - w_i|/|1 - z_i \bar{w}_i| : i = 1, 2, \cdots, n\} \\ &= \sup\{|f(w)| : f \in H^\infty(U^n, \mathbf{C}),\ \|f\|_\infty \leq 1,\ f(z) = 0\}. \end{aligned}$$

The sequence $\mathscr{A}$ is said to be separated if there exists $\delta > 0$ such that

$$\rho(a_i, a_j) \geq \delta \qquad \text{whenever } i \neq j.$$

The constant δ is called a *separation bound* for $\mathscr{A}$. The sequence $\mathscr{A}$ is said to be uniformly separated (U.S.) if there exists a constant K and functions $f_i \in H^\infty(U^n, \mathbf{C})$, $i = 1, 2, \cdots$, such that $f_i(a_i) = 1$, $f_i(a_j) = 0$ if $i \neq j$, and $\|f_i\|_\infty \leq K$ for all i. The constant K is called a *uniform separation bound* for $\mathscr{A}$.

A wedge W, in U^1 is the region inside U^1 lying between two distinct circles, γ_1 and γ_2 such that $\gamma_1 \cap \gamma_2$ is contained in the boundary of U^1. W is said to be tangential or non-tangential depending on whether $\gamma_1 \cap \gamma_2$ consists of one or two points.

A wedge in U^n is the Cartesian product of n one dimensional wedges. A near-wedge in U^n is the Cartesian product of one copy of U^1 and $n - 1$ one dimensional wedges. We leave to the reader the definition of tangential and non-tangential wedges and near-wedges in U^n.

The central results of this paper are

THEOREM 1.1. *Let* $\mathscr{A} = ((a_{i1}, \cdots, a_{in}))_{i=1}^\infty$ *be a sequence contained in a finite union of near-wedges in* U^n. *Let* $1 \leq p \leq \infty$, *and let* E *be a non-trivial Banach space. Then each of the following statements implies the other three.*

i) $\mathscr{A}$ *is* U.S.

ii) $\mathscr{A}$ *is* $H^p(U^n, E)$ *interpolating.*

iii) $T^p_{\mathscr{A}}(H^p(U^n, E)) = \ell^p(E)$.

iv) $\mathscr{A}$ *is separated, and there exists a constant* M *such that for all* i,

$$\sum_{j=1}^{\infty} \prod_{k=1}^{n} [(1 - |a_{ik}|^2)(1 - |a_{jk}|^2)/|1 - \bar{a}_{ik} a_{jk}|^2] \leq M.$$

THEOREM 1.2. *Let* Ω *be a finite union of non-tangential wedges in* U^n. *Then the* $H^p(U^n, E)$ *and* $A^{p,\alpha}(U^n, E)$ *interpolating sequences lying in* Ω *are the same for all* $p \in [1, \infty)$ *and all* $\alpha \in [0, \infty)$, *and all non-trivial* E; *and are, in fact, just the separated sequences lying in* Ω.

THEOREM 1.3. *Let* $\mathscr{A}$ *be a sequence in* U^n. ($\mathscr{A}$ *need not lie in a wedge or near-wedge.*) *Then there exists* $\alpha \geq 0$ *such that* $\mathscr{A}$ *is* $A^{2,\alpha}(U^n, \mathbf{C})$ *interpolating if and only if* $\mathscr{A}$ *is separated.*

The $n = 1$, $E = \mathbb{C}$ case of Theorem 1.1 is due to Carleson [4] for $p = \infty$, and to Shapiro and Shields [13] for $1 \leq p < \infty$. In particular, for $n = 1$, $H^p(U^1, \mathbb{C})$ interpolating sequences are the same for all p. For $n > 1$, the $p = \infty$ case (at least when E is a uniform algebra, and only non-tangential wedges are considered) can be found in [7]. The case $n = 1$, E arbitrary is the main result of [2]. Statement (iv) gives an explicit characterization of U.S. sequences in near-wedges, which, when $n = 1$, is essentially the familiar condition involving Blaschke products.

For $n = 1$, results somewhat weaker than Theorems 1.2 and 1.3 can be found in [13]. An immediate corollary to Theorem 1.3 is the existence of $H^2(U^5, \mathbb{C})$ interpolating sequences which are not U.S.

The results of this paper were announced in June, 1976 [8], and preprints were circulated in July. In November, 1976, we received a preprint titled "Interpolation dans le polydisque de $\mathbb{C}^n$" by Eric Amar, who, using different techniques, obtained results which overlap with some of those in this paper.

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2. FACTORIZATION FOR $H^p(U^n, E)$

We begin by introducing some notation. If $a \in U^n$ and $\delta > 0$ let

$$\Delta^n(a, \delta) = \{z \in U^n: \rho(a, z) < \delta\}.$$

If $a \in \mathbb{C}^n$ and $r = (r_1, r_2, \cdots, r_n) \in \mathbb{R}^n$ $(r_j > 0,\ j = 1, 2, \cdots, n)$, let

$$U^n(a, r) = \{z \in \mathbb{C}^n: |z_j - a_j| < r,\ j = 1, 2, \cdots, n\}.$$

For $n = 1$, the disks $\Delta^1(a, r)$ and $U^1(a, r)$ are related as follows: if $0 < \delta < 1$ and $a \in U^1$ then

$$U^1(a, \delta(1 - |a|^2)/2) \subset \Delta^1(a, \delta); \quad \Delta^1(a, \delta/2) \subset U^1(a, \delta(1 - |a|^2)). \tag{1}$$

We shall denote the distinguished boundary of $U^n(a, r)$ by $C^n(a, r)$, and the distinguished boundary of U^n by C^n. We shall denote Lebesgue measure on $\mathbb{C}^n$ by $d\nu$, and normalized Haar measure on $C^n(0, r)$ by $(2\pi)^{-n} d\theta$. Algebraic operations on n-tuples of real and complex numbers will be carried out coordinate-wise, and frequently coordinate subscripts will be suppressed. Thus, for example, if $r, R \in \mathbb{R}^n$, $r < R$ means $r_j < R_j$ for $j = 1, 2, \cdots, n$; and if $a, b, s \in \mathbb{C}^n$,

$$(b - a)/s = ((b_1 - a_1)/s_1, \cdots, (b_n - a_n)/s_n),$$

and $(1 - |a|^2)^{1/p} = ((1 - |a_1|^2)^{1/p}, \cdots, (1 - |a_n|^2)^{1/p})$. (Fortunately, we shall not need the Euclidean norm on $\mathbb{C}^n$, so no confusion need arise.) Finally,

$$\Pi (1 - |a|^2)^{1/p} = \prod_{j=1}^{n} (1 - |a_j|^2)^{1/p}.$$

It is well known that a function $f \in H^p(U^n, \mathbb{C})$ cannot always be factored into the product of an inner function and an outer function [11]. Nevertheless, a very useful weak factorization theorem is valid.

THEOREM 2.1. *Let* $1 \leq p < \infty$. *Let* $f \in H^p(U^n, E)$, *and* $f \neq 0$. *The function* f *possesses the factorization* $f(z_1, \cdots, z_n) = G(z_1, \cdots, z_n) F(z_n)$, *where* F *is an outer function in* $H^p(U^1, \mathbb{C})$, $\|F\|_p = \|f\|_p$, *and* G *has the property that for each* $z_n \in U^1$, *the function* $\hat{G}_{z_n}(z_1, \cdots, z_{n-1})$, *defined by* $\hat{G}_{z_n}(z_1, \cdots, z_{n-1}) = G(z_1, \cdots, z_{n-1}, z_n)$, *is in* $H^p(U^{n-1}, E)$, *and* $\operatorname{Sup}\{\|\hat{G}_{z_n}\|_p : z_n \in U^1\} = 1$.

Note that $H^p(U^n, E)$ is isometrically isomorphic to $H^p(U^k, H^p(U^{n-k}, E))$ via the natural isomorphism

$$[(z_1, \cdots, z_n) \to f(z_1, \cdots, z_n)] \xrightarrow{\phi} [(z_1, \cdots, z_k) \to ((z_{k+1}, \cdots, z_n) \to f(z_1, \cdots, z_n))].$$

Consequently, Theorem 2.1 is an immediate corollary of the following one-variable vector-valued factorization theorem.

THEOREM 2.2. *Let* $1 \leq p < \infty$. *Let* $f \in H^p(U^1, E)$, *and* $f \neq 0$. *Then* $f = GF$, *where* $G \in H^\infty(U^1, E)$, $\|G\|_\infty = 1$, F *is an outer function in* $H^p(U^1, \mathbb{C})$, *and* $\|F\|_p = \|f\|_p$.

Proof. Since $f \in H^p(U^1, E)$, $|f|^p$ has a least harmonic majorant H. Also $\log|f|$ is subharmonic, so $\log|f|$ has a least harmonic majorant h. Let h_* denote the harmonic conjugate of h vanishing at 0. Then $|f| \leq |\exp(h + ih_*)|$. Set $F_1 = \exp(h + ih_*)$ and $G_1 = f/F_1$. It is clear that $\|G_1\|_\infty \leq 1$. For $r \in (0, 1)$, if h_r is the Poisson integral of $\log|f|$ around $C_1(0, r)$, then $h_r \nearrow h$ is $r \to 1$, and clearly $\exp(ph_r) \leq H$ inside $U^1(0, r)$. Consequently, $\exp(ph) \leq H$ in U^1. Hence

$$F_1 \in H^p(U^1, \mathbb{C})$$

and $\|F_1\|_p \leq \|f\|_p$. Since $f = G_1 F_1$, it is clear that $\|F_1\|_p = \|f\|_p$ and $\|G_1\|_\infty = 1$. Let $F_1 = IF$ be the inner-outer factorization of F_1, and set $G = IG_1$ to obtain the required factorization of f.

Although we shall not need the fact, it is worth noting that G really is inner in the sense that $\log|G|$ is the sum of a singular harmonic function and a linear combination of Green's functions. (*Cf.* [10, especially Theorem 2.2, p. 17] and [9, chapter 2].) From these comments, we see that a version of Theorem 2.2 holds for $H^p(R, E)$, where R is an admissible Riemann surface in Neville's sense [9, chapter 5].

3. VECTOR VALUED H^p INTERPOLATION IN U^1

We begin with a formal statement of a result of Shapiro and Shields.

PROPOSITION 3.1 (*cf.* [13, p. 521]). *Let* $1 \leq p \leq \infty$. *If* $\mathscr{A}$ *is an* $H^p(U^n, E)$ *interpolating sequence, then there is a constant* K *such that for every* c *in* $\ell^p(E)$ *there is an* f *in* $H^p(U^n, E)$ *with the property that* $T^p_{\mathscr{A}} f = c$ *and* $\|f\|_p \leq K\|c\|_p$. *A similar statement holds if* $\mathscr{A}$ *is* $A^{p,\alpha}(U^n, E)$ *interpolating.*

The constant K in the above Proposition will be called an $H^p(U^n, E)$ or $A^{p,\alpha}(U^n, E)$ interpolating bound for $\mathscr{A}$.

Aron, Globevnik, and Schottenloher [2] have recently proved an interpolation theorem for $H^\infty(U^1, E)$. The theorem is

THEOREM 3.2. *Let* $\mathscr{A}$ *be a uniformly separated sequence in* U^1 *with uniform separation bound* m. *Then* $\mathscr{A}$ *is* $H^\infty(U^1, E)$ *interpolating with* $H^\infty(U^1, E)$ *interpolation bound* $\kappa_\infty(m)$, *depending only on* m.

(This theorem may also be obtained from P. Beurling's result on the existence of absolutely converging sequences of 0 - 1 interpolating functions in $H^\infty(U^1, \mathbb{C})$ [5]).

Theorem 3.2 may easily be generalized to $H^p(U^1, E)$.

THEOREM 3.3. *Let* $\mathscr{A}$ *be as in Theorem* 3.2. *Let* $1 \leq p < \infty$. *Then* $\mathscr{A}$ *is* $H^p(U^1, E)$ *interpolating with interpolation bound* $\kappa_p(m)$ *depending only on* m.

Proof. Let $c = (c_i)_{i=1}^\infty \in \ell^p(E)$, and let $e_i = c_i|c_i|^{-1}$ if $c_i \neq 0$, $e_i = 0$, otherwise. Let $d_i = |c_i|$. Then $e = (e_i)_{i=1}^\infty \in \ell^\infty(E)$ and $d = (d_i)_{i=1}^\infty \in \ell^p$. Furthermore, $\|e\|_\infty \leq 1$ and $\|d\|_p = \|c\|_p$. Let $G \in H^\infty(U^1, E)$ be such that $T^\infty_{\mathscr{A}}(G) = e$ and $\|G\|_\infty \leq \kappa_\infty(m)$. Let $g \in H^p(U^1, \mathbb{C})$ be such that $T^p_{\mathscr{A}}(g) = d$ and $\|g\|_p \leq c_p(m)$, where $c_p(m)$ depends only on p and m. The existence of G is guaranteed by Theorem 3.2, and the existence of g is guaranteed by Theorem 2 of [13] (*cf.* also [6, pp. 152-153]). Set $f = Gg$. Then $T^p_{\mathscr{A}}(f) = c$ and $\|f\|_p \leq \kappa_p(m)$.

We may easily obtain an interpolation theorem for the Cartesian product of interpolating sequences, by applying Theorem 3.3 to the Banach space $H^p(U^{n-1}, E)$ to obtain an $H^p(U^1, H^p(U^{n-1}, E)) = H^p(U^n, E)$ interpolation theorem. More generally, let $\mathscr{A} = (a_i)_{i=1}^\infty$ be as in Theorem 3.3, and for each i let $B_i = (b_{ij})_{j=1}^\infty$ be an $H^p(U^{n-1}, E)$ interpolating sequence in U^{n-1} with $H^p(U^{n-1}, E)$ interpolation bound K. Let $C = (c_i)_{i=1}^\infty$ be a univalent enumeration of the double sequence $((a_i, b_{ij}))_{i,j=1}^\infty$. We immediately have

COROLLARY 3.4. C *is* $H^p(U^n, E)$ *interpolating with interpolation bound* $\kappa_p(m)K$.

By induction, it follows that the Cartesian product of n U.S. sequences in U^1 is $H^p(U^n, E)$ interpolating.

4. UNIFORM PSEUDO-HYPERBOLIC CONTINUITY

If $\mathscr{A} = (a_i)_{i=1}^\infty$ and $\mathscr{B} = (b_i)_{i=1}^\infty$ are sequences in U^n, and if $\sup_i [\rho(a_i, b_i)] \leq \varepsilon$, it follows from the definition of ρ that $\|T^\infty_{\mathscr{A}} f - T^\infty_{\mathscr{B}} f\| \leq 2\varepsilon \|f\|_\infty$, for every f in $H^\infty(U^n, E)$. This inequality can be used to show that if $\mathscr{A}$ is $H^\infty(U^n, E)$ interpolating, and $\mathscr{B}$ is sufficiently close to $\mathscr{A}$ in the pseudo-hyperbolic metric, then $\mathscr{B}$ is also $H^\infty(U^n, E)$ interpolating [7, Corollary 3.4]. The following uniform pseudo-hyperbolic continuity result can be used to obtain an analagous result for $p < \infty$.

Let $\sigma: \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the map $\sigma(z) = (z_1, \cdots, z_{n-1})$.

THEOREM 4.1. *Let* $1 \leq p < \infty$ *and suppose* $\mathscr{A} = (a_i)_{i=1}^\infty$ *is a* U.S. *sequence in* U^n *with* U.S. *bound* M, *and that* $\sigma(\mathscr{A})$ *may be rearranged so that it is a subsequence of a multi-sequence of the form* $\mathscr{R}_1 \times \mathscr{R}_2 \times \cdots \times \mathscr{R}_{n-1}$, *where each* $\mathscr{R}_i$ *is*

U.S. *with bound* m. *Then there are constants* $C(n, m, M, p)$ *and* $\Gamma(n, m, M, p)$ (*depending only on* n, m, M, *and* p) *such that for all* f *in* $H^p(U^n, E)$:

i) $T^p_{\mathscr{A}} f \in \ell^p(E)$; *in fact* $\|T^p_{\mathscr{A}} f\| \leq C(n, m, M, p)\|f\|_p$.

ii) *If* $\mathscr{B} = (b_i)_{i=1}^{\infty}$ *is a sequence in* U^n *satisfying* $b_{in} = a_{in}$ *and*

$$\rho(\sigma(a_i), \sigma(b_i)) \leq \delta \leq 1/(36m) \quad \textit{for all } i,$$

then $\|T^p_{\mathscr{A}} f - T^p_{\mathscr{B}} f\| \leq \delta\Gamma(n, m, M, p)\|f\|_p$.

In proving Theorem 4.1, we lose no generality in assuming that $\sigma(\mathscr{A})$ is identified with all of $\mathscr{R}_1 \times \mathscr{R}_2 \times \cdots \times \mathscr{R}_{n-1}$.

Before proving this theorem, we must recall a few facts about Carleson measures. If $0 < h < 1$, we will say an h-sector is a region in U^1 of the form $S = \{z = re^{i\theta}: 1 - h \leq r < 1, \ \theta_0 \leq \theta \leq \theta_0 + h\}$. A Carleson measure on U^1 is a finite positive measure μ with the property that $\mu(S) \leq A_\mu h$ for every h-sector, S, where A_μ is a positive constant associated with μ. The fundamental property of Carleson measures is the following:

PROPOSITION 4.2. (*cf.* [6, Theorem 9.3]). *The measure* μ *is a Carleson measure if and only if there exists a constant* C_μ *such that*

$$\left(\int_{U^1} |f(z)|^p \, d\mu(z)\right)^{1/p} \leq C_\mu^{1/p}\|f\|_p \quad \textit{for all } f \in H^p(U^1), \ p \in [1, \infty). \tag{2}$$

Moreover, if μ *is a Carleson measure,* $A_\mu \leq 5C_\mu$ *and* $C_\mu \leq 16(80)^8 A_\mu^4$.

By letting f = FG as in the statement of Theorem 2.2, it is clear that if μ is a Carleson measure, (2) also holds, with the same constant, for all $f \in H^p(U^1, E)$, and all $p \in [1, \infty)$. Similarly, if $\mu_1, \mu_2, \cdots, \mu_n$ are Carleson measures on U^1 and μ is the product measure of $\mu_1, \mu_2, \cdots, \mu_n$ on U^n, then an elementary induction argument, using Theorem 2.1 and the canonical isomorphism between $H^p(U^n, E)$ and $H^p(U^1, H^p(U^{n-1}, E))$, shows

$$\left(\int_{U^n} |f(z)|^p \, d\mu(z)\right)^{1/p} \leq \left(\prod_{i=1}^{n} C_{\mu_i}\right)^{1/p} \|f\|_p \quad \text{for all } f \in H^p(U^n, E), \ p \in [1, \infty).$$

Now, if $\mathscr{A} = \{a_i\}_{i=1}^{\infty}$ is a U.S. sequence in U^1 with U.S. bound m, and δ_z is the unit point mass at z then it is well known that $\mu = \sum_{i=1}^{\infty} (1 - |z_i|^2)\delta_{a_i}$ satisfies (2). These observations immediately yield the following lemma and its useful corollary.

LEMMA 4.3. *Let* $(a_i)_{i=1}^{\infty}$ *be a* U.S. *sequence in* U^1 *with* U.S. *bound* m; *let* γ_i *be the circle* $\gamma_i = C^1(a_i, r_i)$ *where* $r_i \leq (1 - |a_i|^2)/(6m)$; *and let* μ *be arc length measure around* $\gamma = \bigcup_{i=1}^{\infty} \gamma_i$. *Then* μ *is a Carleson measure and* (2) *holds with constant* $C_\mu = C_m^*$ *depending only on* m.

COROLLARY 4.4. *Suppose* $\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_N$ *are* U.S. *sequences in* U^1, *each with* U.S. *bound* m. *For each point* a *in the multi-sequence* $\mathscr{R}_1 \times \mathscr{R}_2 \times \cdots \times \mathscr{R}_N$, *let* γ_a *be the poly-circle* $C^N(a, r_a)$ *where* $r_a \leq (1 - |a|^2)/(6m)$, *and let* μ *be surface area measure on the* (*disjoint*) *union of all the* γ_a's. *Then*

$$\left(\int_{U^N} |f(z)|^p \, d\mu(z)\right)^{1/p} \leq (C_m^*)^{N/p}\|f\|_p \quad \textit{for all } f \in H^p(U^N, E), \ p \in [1, \infty),$$

where C_m^* *is the constant in Lemma* 4.3.

Proof of Theorem 4.1. The proof of part (i) is essentially the same, but easier than that of part (ii), so we will only verify (ii). Before doing so, we record the following elementary fact.

LEMMA 4.5. *If* $\lambda_1, \cdots, \lambda_N \in \mathbb{C}$ *satisfy* $|1 - \lambda_i| \leq \varepsilon$ for $i = 1, \cdots, N$, *then* $|1 - \prod_{i=1}^N \lambda_i| \leq N(1+\varepsilon)^{N-1}\varepsilon$.

Proof. $|1 - \prod_{i=1}^N \lambda_i| \leq \sum_{i=1}^N |1-\lambda_i| \prod_{k=1}^{i-1} |\lambda_i|$.

If $\alpha \in U^{n-1}$, let r_α be the poly-radius, $(1/6m)(1 - |\alpha|^2)$; let

$$U_\alpha = U^{n-1}(\alpha, r_\alpha);$$

let $\gamma_\alpha = C^{n-1}(\alpha, r_\alpha)$; let μ_α be the surface area measure on γ_α; and let $S_\alpha = \int d\mu_\alpha = \left((2\pi)^{n-1} \prod r_\alpha\right)$. For $1 \leq p < \infty$, define $H^p(U_\alpha, E)$ to be the space of holomorphic E-valued functions g on U_α satisfying

$$(\|g\|_p)^p = \sup_{r < r_\alpha} \left\{ (1/S_\alpha) \int_{\gamma_\alpha} |g(rz)|^p \, d\mu_\alpha(z) \right\} < \infty .$$

Clearly if g is in fact holomorphic in a polydisk containing $\overline{U}_\alpha$, then

$$(\|g\|_p)^p = (1/S_\alpha) \int_{\gamma_\alpha} |g(z)|^p \, d\mu_\alpha(z) .$$

Recall that for $a_i, a_j \in \mathscr{A}$, if $\sigma(a_i) \neq \sigma(a_j)$, then $\rho(\sigma(a_i), \sigma(a_j)) \geq 1/m$. Note that the r_α's are defined so that the above inequality along with (1) implies that $\overline{U}_{\sigma(a_i)} \cap \overline{U}_{\sigma(a_j)} = \emptyset$ whenever $\sigma(a_i) \neq \sigma(a_j)$.

Fix a $\sigma(a_i)$, say $\sigma(a_i) = \alpha$. If $f \in H^p(U^n, E)$, let $f_\alpha = f|_{U_\alpha \times U^1}$ and consider f_α as an element of $H^p(U^1, H^p(U_\alpha, E))$. By applying the factorization Theorems 2.1 and 2.2, we have $f_\alpha = F_\alpha G_\alpha$ where $F_\alpha \in H^p(U^1, \mathbb{C})$, $\|F\|_p = \|f\|_p$,

$$G_\alpha \in H^\infty(U^1, H^p(U_\alpha, E)),$$

and $\|G_\alpha\|_\infty = 1$. We shall write $G_\alpha(z_1, \cdots, z_n)$ for $(G_\alpha(z_n)(z_1, \cdots, z_{n-1})$. As in Section 2, we shall identify $H^p(U^n, E)$ with $H^p(U^{n-1}, H^p(U^1, E))$, so that if $g \in H^p(U^n, E)$, and $z \in U^{n-1}$, then $(\|g(z, \cdot)\|_p)^p = \sup_{r<1} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z, re^{i\theta})|^p \, d\theta \right]$.

Now let J be the set of integers, j, such that $\sigma(a_j) = \alpha$. Since $a_{jn} = b_{jn}$,

$$\sum_{j \in J} |T^p_{a_j}(f) - T^p_{b_j}(f)|^p = \sum_{j\in J} (1 - |a_{jn}|^2) |F_\alpha(a_{jn})|^p \tag{3}$$

$$\cdot \left| \prod (1 - |\alpha|^2)^{1/p} G_\alpha(a_j) - \prod (1 - |\sigma(b_j)|^2)^{1/p} G_\alpha(b_j) \right|^p .$$

For j fixed in J, and $1 \leq k \leq n - 1$, by applying inequality (1) and the fact that $\rho(a_{jk}, b_{jk}) \leq \delta$, we obtain $|1 - (1 - |b_{jk}|^2)/(1 - |a_{jk}|^2)| \leq 4\delta$. An application of lemma 4.5 yields

$$(4) \qquad |\prod (1 - |\alpha|^2)^{1/p} - \prod (1 - |\sigma(b_j)|^2)^{1/p}| \leq 2^n(n-1)\delta \prod (1 - |\alpha|^2)^{1/p}.$$

We claim

$$(5) \qquad |G_\alpha(a_j) - G_\alpha(b_j)| \leq 7(n-1)4^n m\delta, \quad \text{and}$$

$$(6) \qquad |G_\alpha(a_j)| \leq 1,$$

so that $|G_\alpha(b_j)| \leq n4^n$. By combining this with (4) and (5), we have

$$(7) \qquad \begin{aligned} &|\prod (1 - |\alpha|^2)^{1/p} G_\alpha(a_j) - \prod (1 - |\sigma(b_j)|^2)^{1/p} G_\alpha(b_j)| \\ &\qquad \leq C_1(n, m)\,\delta \prod (1 - |\alpha|^2)^{1/p}. \end{aligned}$$

Finally

$$(8) \qquad \|F_\alpha\|_p^p = \|f_\alpha\|_p^p = (1/S_\alpha) \int_{\gamma_\alpha} \|f(z, \cdot)\|_p^p \, d\mu_\alpha(z).$$

Since $\mathscr{A}$ is U.S. with bound M, $\{a_{jn}\}_{j \in J}$ is U.S. in U^1 with bound M. Thus

$$(9) \qquad \sum_{j \in J} (1 - |a_{jn}|^2) |F_\alpha(a_{jn})|^p \leq C_2(M) \|F_\alpha\|_p^p.$$

From (3), (7), (8), and (9) we see that the right hand side of (3) is dominated by

$$(10) \qquad C_3(m, M, p)\,\delta^p \int_{\gamma_\alpha} \|f(z, \cdot)\|_p^p \, d\mu_\alpha(z).$$

By summing (10) over all $\alpha \in \sigma(\mathscr{A})$ and using corollary 4.4, we get the desired bound on $\|T^p_{\mathscr{A}} f - T^p_{\mathscr{B}} f\|$.

It remains to establish (5) and (6). To do this we note that if

$$P_{s,a}(z, e^{i\theta}) = (s^2 - |z - a|^2)/|se^{i\theta} - (z - a)|^2$$

is the Poisson kernel for the disk $U^1(a, s)$, then

$$|1 - P_{s,a}(z, e^{i\theta})| \leq 4\left(\frac{|z-a|}{s}\right) \Big/ \left(1 - \frac{|z-a|}{s}\right)^2.$$

In particular, if $s = (1 - |a|^2)/(6m)$ and $|z - a| < 2\delta(1 - |a|^2) < (1 - |a|^2)/(18m)$, then

$$(11) \qquad |1 - P_{s,a}(z, e^{i\theta})| \leq (108)\, m\delta.$$

Hence if $P_\alpha(z, t)$ is the Poisson kernel for the $(n-1)$-harmonic functions in the polydisk U_α, Lemma 4.5 shows that for $z \in U$ and $\rho(z, \alpha) < \delta$,

$$|1 - P_\alpha(z, t)| < 7(n-1)4^n m\delta .$$

Now $\|G_\alpha(\cdot, a_{jn})\|_p \leq 1$, so by Hölder's Inequality,

$$(2\pi)^{1-n} \int_{\gamma_\alpha} |G_\alpha(t, a_{jn})| \, d\theta^{(n-1)}(t) \leq 1$$

where $(2\pi)^{1-n} d\theta^{(n-1)}$ is $n - 1$ dimensional normalized surface area measure on γ_α. Inequality (6) follows from the above inequality and the fact that $|G_\alpha(\cdot, a_{jn})|$ is $n - 1$ subharmonic. Inequality (5) is obtained by applying (11), and the above inequality, to the observation that since $a_{jn} = b_{jn}$,

$$G_\alpha(a_j) - G_\alpha(b_j) = (2\pi)^{1-n} \int_{\gamma_\alpha} G_\alpha(t, a_{jn})(1 - P_\alpha(\sigma(b_j), t)) \, d\theta^{n-1}(t).$$

COROLLARY 4.6. *Suppose* $\mathscr{A}$ *is a* U.S. *sequence in* U^n *satisfying the conditions of Theorem* 4.1.

Then $T^p_{\mathscr{A}}(H^p(U^n, E)) = \ell^p(E)$.

Proof. Apply Corollary 3.4, f.f., and Theorem 4.1.

COROLLARY 4.7. *Let* m, M, *and* $\mathscr{A}$ *be as in Theorem* 4.1, *and* $p \in [1, \infty]$. *Let* $\delta < \min\{[2\Gamma(n, m, M, p)\kappa_p(m)^{n-1}\kappa_p(M)]^{-1}, (36m)^{-1}\}$, *and suppose* $\mathscr{B}$ *is a sequence in* U^n *with the property that* $b_{jn} = a_{jn}$ *and* $\rho(a_j, b_j) < \delta$ *for* $j = 1, 2, \cdots$. *Then* $\mathscr{B}$ *is* $H^p(U^n, E)$ *interpolating with bound* $2(\kappa_p(m))^{n-1}\kappa_p(M)$.

Proof. The $p = \infty$ case of corollary 4.7 (with $\Gamma(n, m, M, \infty) = 2$) is a special case of Theorem 3.3 of [7]. The $p < \infty$ case is similar: if $\lambda \in \ell^p(E)$, there is an f in $H^p(U^n, E)$ such that $\|f\| \leq \kappa_p(m)^{n-1}\kappa_p(M)\|\lambda\|$, and $T^p_{\mathscr{A}} f = \lambda$. By Theorem 4.1, $\|T^p_{\mathscr{B}} f - \lambda\| \leq \|\lambda\|/2$. Corollary 4.7 now follows from the following well known fact.

LEMMA 4.8 (*cf.* [3, Theorem 1.2]). *Suppose* H *and* ℓ *are Banach spaces and* T *is a bounded linear map from* H *to* ℓ. *Suppose there are constants* $0 < k < 1$ *and* $K < \infty$ *such that for every* $\lambda \in \ell$ *there is a* f *in* H *for which* $\|Tf - \lambda\| \leq k\|\lambda\|$ *and* $\|f\| \leq K\|\lambda\|$. *Then for every* $\lambda \in \ell$ *there is an* $f \in H$ *for which* $Tf = \lambda$ *and* $\|f\| \leq K\|\lambda\|/(1-k)$.

5. $H^p(U^n, E)$ INTERPOLATING SEQUENCES

We are now ready to prove Theorem 1.1. The $p = \infty$ theorem is essentially the main result of [7], and it is also an important tool in proving the $p < \infty$ case, so we will state it separately.

THEOREM 5.1. *If* $\mathscr{A}$ *is a sequence contained in a near-wedge,* Ω, *in* U^n, *then* $\mathscr{A}$ *is* $H^\infty(U^n, E)$ *interpolating if and only if it is* U.S.

Proof. The necessity of U.S. is obvious. If Ω is non-tangential, and E is a uniform algebra, then the $H^\infty(U^n, E)$ case is Theorem 5.9 of [7]. If E is an arbitrary Banach space, apply Theorem 3.2 of the present paper to the arguments of [7]. To remove the restriction to non-tangential near-wedges, one replaces Lemma 5.10 of [7] with the following lemma and proceeds with the proof of Theorem 5.9 of [7].

LEMMA 5.2. *Let* W_0 *be a wedge in* U^{n-1}. *Let* $\varepsilon > 0$. *There exist constants* K_0 *and* m (*independent of* ε *and depending only on* W_0) *and finitely many sequences,*

$C_1, \cdots, C_N$, *in* W_0 *such that* $W_0 \subseteq \bigcup\{\Delta^{n-1}(C_j, \varepsilon): j = 1, 2, \cdots, N\}$. *Furthermore, each* C_j *is* $H^\infty(U^{n-1}, E)$ *interpolating, with interpolation bound* K_0, *and each* C_j *is of the form* $C_j = R_1 \times \cdots \times R_{n-1}$, *where each* R_i *is* U.S. *in* U^1 *with* U.S. *bound* m.

In stating this lemma, we have used some notation and conventions which we shall use throughout this section. First, if A is a subset of U^n,

$$\Delta^n(A, \varepsilon) = \bigcup\{\Delta^n(a, \varepsilon): a \in A\}.$$

(Recall $\Delta^n(a, \varepsilon) = \{z: \rho(z, a) < \varepsilon\}$.) Second, we shall sometimes identify sequences with the point sets comprised of their elements. Thus we shall feel free to write $\Delta^n(C_j, \varepsilon)$, or even $C_j \subseteq \bigcup_{j=1}^N C_j$. Third, for $\alpha \in U^{n-1}$ and $Q \subset U^{n-1}$,

$$D^n(\alpha, \varepsilon) = \Delta^{n-1}(\alpha, \varepsilon) \times U^1$$

and $D^n(Q, \varepsilon) = \Delta^{n-1}(Q, \varepsilon) \times U^1$. Finally, for convenience, we have adopted a notation which makes all near-wedges appear to be of the form

$$W_1 \times W_2 \times \cdots \times W_{n-1} \times U^1,$$

where each W_j is a wedge in U^1.

Proof of Lemma 5.2. The proof is essentially the same as that of Lemma 5.10 of [7]. For tangential wedges, replace $\sigma(z)$ in that argument with

$$\tau(z) = (\beta - iz)/(1 - i\bar{\beta}z),$$

where $\beta = (1 + i)/2$.

We also need the following three lemmas from [7]. The first is Corollary 3.6 of [7]. The second two are consequences of Proposition 4.5, Lemma 4.6 and Lemma 5.11 of [7], and Lemma 5.2 above. We will let $\sigma: U^n \to U^{n-1}$ be the projection onto the first n - 1 coordinates as in section 4, and $\pi: U^n \to U^1$ be projection onto the nth coordinate. Let κ_∞ and κ_p be as in Theorems 3.2 and 3.3.

LEMMA 5.3. *Let* $a \in U^n$ *and let* $\mathscr{A}$ *be a* U.S. *sequence with* U.S. *bound* K. *If* $a_i \neq a_j$, *and both lie in* $D^n(\sigma(a), \varepsilon_0) \cap \mathscr{A}$, *where* $\varepsilon_0 = (4K\kappa_\infty(4K))^{-1}$, *then* $\pi(a_i) \neq \pi(a_j)$. *Furthermore,* $\pi(D^n(\sigma(a), \varepsilon_0) \cap \mathscr{A})$ *is* U.S. *with* U.S. *bound* 4K.

LEMMA 5.4. *Let* $\Omega = W_0 \times U^1$, *where* W_0 *is a wedge in* U^{n-1}. *Let* $\mathscr{A}$ *be a sequence in* Ω. *Let* m *be as in Lemma* 5.2. *Let* $0 < \eta < m/2$, *and let* $0 < \varepsilon < \eta^3/2$. *Let* $\mathscr{C}_1, \mathscr{C}_1, \cdots, \mathscr{C}_N$ *be as in Lemma* 5.2. *In particular, the* $D^n(\mathscr{C}_i, \varepsilon)$, $i = 1, 2, \cdots, N$, *cover* Ω. *Suppose* $\mathscr{A} \cap D^n(\mathscr{C}_i, \eta)$ *is* $H^\infty(U^n, \mathbb{C})$ *interpolating for each* $i = 1, 2, \cdots, N$. *Then* $\mathscr{A}$ *is* $H^\infty(U^n, \mathbb{C})$ *interpolating.*

LEMMA 5.5. *Suppose* $\mathscr{A}$ *is a sequence in* U^n *such that*

$$\mathscr{A} \subset \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s,$$

where each $\Omega_i = W_i \times U^1$, *and each* W_i *is a wedge in* U^{n-1}. *For each* $i = 1, 2, \cdots, s$ *let* m_i *be the constant corresponding to the constant* m *in Lemma* 5.2 *and let* $m = \mathrm{Max}_i\{m_i\}$. *Let* $\hat{\Omega}_i = \Delta^n(\Omega_i, 1/(2m))$, *and suppose* $\mathscr{A} \cap \hat{\Omega}_i$ *is* U.S. *for each* i, $i = 1, 2, \cdots, s$. *Then* $\mathscr{A}$ *is* $H^\infty(U^n, E)$ *interpolating.*

Lemma 5.5 and Theorem 5.1 have the following three corollaries.

COROLLARY 5.6. *If* $\mathscr{A}$ *is a sequence contained in a finite union of near-wedges in* U^n *then* $T^{\infty}_{\mathscr{A}}(H^{\infty}(U^n, E)) = \ell^{\infty}(E)$ *if and only if* $\mathscr{A}$ *is* U.S. (*This is the* $p = \infty$ *case of Theorem* 1.1.)

COROLLARY 5.7. *Fix* p, $1 \leq p < \infty$. *If* $\mathscr{A} \subset \Omega$, *where* Ω *is a finite union of near wedges, and* $\mathscr{A}$ *is* U.S., *and* $\mathscr{A} = \bigcup_{i=1}^{N} \mathscr{A}_i$ *where each* $\mathscr{A}_i$ *is* $H^p(U^n, E)$ *interpolating, then* $\mathscr{A}$ *is* $H^p(U^n, E)$ *interpolating.*

Proof. Write $\mathscr{A}$ as a *disjoint* union, $\mathscr{A} = \bigcup_{i=1}^{N} \mathscr{A}_i'$, where $\mathscr{A}_i' \subseteq \mathscr{A}_i$ for each i. Let $f_i \in H^{\infty}(U^n, \mathbb{C})$ be such that $f_i = 1$ on $\mathscr{A}_i'$ and $f_i = 0$ on $\mathscr{A}_j'$ for $i \neq j$, $i, j = 1, \cdots, N$. The proof is now obvious.

COROLLARY 5.8. *Suppose* $\mathscr{A}$ *is a sequence in* $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$ *where each* Ω_i *is a near-wedge in* U^n. *Suppose* $\mathscr{A}$ *has the property that for every near-wedge,* Ω, $\mathscr{A} \cap \Omega$ *is* U.S. *Then* $\mathscr{A}$ *is* U.S.

Proof. For $i = 1, 2, \cdots, s$. Let $\hat{\Omega}_i$ be as in the statement of Lemma 5.5. Each $\hat{\Omega}_i$ is itself contained in a near-wedge, so by hypothesis $\mathscr{A} \cap \hat{\Omega}_i$ is U.S. By Lemma 5.5 $\mathscr{A}$ is $H^{\infty}(U^n, E)$ interpolating and therefore U.S.

As a result of the above corollaries, it will be sufficient to prove Theorem 1.1 for the case where $1 \leq p < \infty$ and $\mathscr{A}$ is contained in a single near-wedge. Throughout the remainder of this section $\mathscr{A} \subset \Omega$, where $\Omega = W_0 \times U^1$ and W_0 is a wedge in U^{n-1}.

Proof of Theorem 1.1. We first show (i) implies (ii). Let K be the U.S. bound for $\mathscr{A}$. Let m be as in Lemma 5.2. Let $M = \kappa_{\infty}(m)^{n-1} \kappa_{\infty}(4K)$. Let

$$0 < \varepsilon < \min\{[2\Gamma(n, m, M, p)\, \kappa_p(m)^{n-1} \kappa_p(M)]^{-1}, [4K\kappa_{\infty}(4K)]^{-1}, (36m)^{-1}\},$$

where $\Gamma(n, m, M, p)$ is as in Theorem 4.1. Let $\mathscr{C}_1, \cdots, \mathscr{C}_N$ be as in Lemma 5.2. In particular, $W_0 \subseteq \bigcup_{j=1}^{N} \Delta^{n-1}(\mathscr{C}_j, \varepsilon)$. For each i, let

$$\mathscr{A}_i = \bigcup \{\{c_j\} \times \pi(D^n(c_j, \varepsilon) \cap \mathscr{A}) : c_j \in \mathscr{C}_i\}.$$

By Lemma 5.3 and Corollary 3.4 f.f., $\mathscr{A}_i$ is $H^{\infty}(U^n, \mathbb{C})$ interpolating with interpolation, and hence U.S., bound M. Similarly, $\mathscr{A}_i$ is $H^p(U^n, \mathbb{C})$ interpolating. Clearly, $\sigma(\mathscr{A}_i) \subseteq \mathscr{C}_i$, so $\mathscr{A}_i$ satisfies the hypotheses of Corollary 4.7.

For each i let $\mathscr{B}_i = \mathscr{A} \cap D^n(\mathscr{C}_i, \varepsilon)$. Clearly, the sequences $\mathscr{A}_i$ and $\mathscr{B}_i$ can be rearranged so that if $(a_j)_{j=1}^{\infty} = \mathscr{A}_i$ and $(b_j)_{j=1}^{\infty} = \mathscr{B}_i$, then $\rho(\sigma(a_j), \sigma(b_j)) < \varepsilon$ and $\pi(a_j) = \pi(b_j)$ for all j. By corollary 4.7, $\mathscr{B}_i$ is $H^p(U^n, E)$ interpolating for each i. By Corollary 5.7, $\mathscr{A} = \bigcup_{i=1}^{N} \mathscr{B}_i$ is $H^p(U^n, E)$ interpolating. Therefore, (i) implies (ii).

We note that for each i, $T^p_{\mathscr{B}_i}(H^p(U^n, E)) \subseteq \ell^p(E)$ by the proof of Corollary 4.7. Hence $T^p_{\mathscr{A}}(H^p(U^n, E)) \subseteq \ell^p(E)$. Consequently, we have also shown that (i) implies (iii).

Since (ii) clearly follows from (iii), we will be done if we can show (ii) implies (i) and (i) holds if and only if (iv) does. To do this we need the notion of local uniform separation.

Definition. Let $\mathscr{D}$ be a sequence in U^n. $\mathscr{D}$ is *locally uniformly separated* (L.U.S.) if for some $\varepsilon > 0$ and $K < \infty$, the sequence $D^n(\alpha, \varepsilon) \cap \mathscr{D}$ is U.S. in $D^n(\alpha, \varepsilon)$ with U.S. bound K, for all $\alpha \in U^{n-1}$. That is, for each $\alpha \in U^{n-1}$ and $d_i \in D^n(\alpha, \varepsilon) \cap \mathscr{D}$, there exists $f_i \in H^\infty(D^n(\alpha, \varepsilon), \mathbb{C})$ such that

$$\|f\|_\infty \leq K, \quad f_i(d_i) = 1,$$

and $f_i(d_j) = 0$ for all $d_j \neq d_i$ such that $d_j \in D^n(\alpha, \varepsilon) \cap \mathscr{D}$. The constant K is called an ε-L.U.S. bound for $\mathscr{D}$.

Clearly, every L.U.S. sequence, with ε-L.U.S. bound K, is separated with separation bound ε/K. This follows from the same conformal mapping argument used to prove

LEMMA 5.9. *Let* $a \in U^n$, *and let* $\mathscr{D}$ *be an* L.U.S. *sequence in* U^n *with* ε-L.U.S. *bound* K. *If* $d_i \neq d_j$ *and both lie in* $D^n(\sigma(a), \varepsilon\varepsilon_0) \cap \mathscr{D}$, *where*

$$\varepsilon_0 = [4K\kappa_\infty(4K)]^{-1},$$

then $\pi(d_i) \neq \pi(d_j)$. *Furthermore,* $\pi(D^n(\sigma(a), \varepsilon\varepsilon_0) \cap \mathscr{D})$ *is* U.S. *with* U.S. *bound* 4K.

Proof. Map $D^n(\sigma(a), \varepsilon)$ conformally onto U^n via the obvious map ϕ, note that $\phi(D^n(\sigma(a), \varepsilon) \cap \mathscr{D})$ is a U.S. sequence in U^n with U.S. bound K, and apply Lemma 5.3.

LEMMA 5.10. $\mathscr{A}$ *is* U.S. *if and only if* $\mathscr{A}$ *is* L.U.S. [*Remember* $\mathscr{A} \subset \Omega$.]

Proof. The only if direction is obvious. So assume $\mathscr{A}$ is L.U.S. Let m be as in Lemma 5.2. Let K be a δ-L.U.S. bound for $\mathscr{A}$, for some $\delta > 0$. Let

$$M = \kappa_\infty(m)^{n-1}\kappa_\infty(4K).$$

Let $0 < \eta < \min\{\delta[4K\kappa_\infty(4K)]^{-1}, [4M]^{-1}, [36m]^{-1}\}$. Let $0 < \varepsilon < \eta^3/2$. Let $\mathscr{C}_1, \cdots, \mathscr{C}_N$ be as in Lemma 5.2. In particular, $\Omega \subseteq \bigcup_{i=1}^N D^n(\mathscr{C}_i, \varepsilon)$. Construct sequences $\mathscr{A}_i$, $i = 1, \cdots, N$, and $\mathscr{B}_i$, $i = 1, \cdots, N$, as before. By Lemma 5.9 and Corollary 3.4 for each i, $\mathscr{A}_i$ is $H^\infty(U^n, \mathbb{C})$ interpolating, with interpolation bound M. Hence, by Corollary 4.7, $\mathscr{B}_i = \mathscr{A} \cap D^n(\mathscr{C}_i, \varepsilon)$ is $H^\infty(U^n, \mathbb{C})$ interpolating for each i. By Lemma 5.4, $\mathscr{A} = \bigcup_{i=1}^N \mathscr{B}_i$ is $H^\infty(U^n, \mathbb{C})$ interpolating, and hence is U.S.

In addition to Lemma 5.10, we shall need the elementary estimate

$$(1 - 2r) \leq (1 - |\zeta|^2)/(1 - |\eta|^2) \leq 1 + 2r \quad \text{for } 0 < r < 1/2 \tag{12}$$

and for all $\zeta, \eta \in U^1$ such that $|\zeta - \eta| \leq r(1 - |\eta|^2)$.

We are now ready to prove the rest of Theorem 1.1. Let $1 \leq p < \infty$, and let $\mathscr{A}$ be an $H^p(U^n, E)$ interpolating sequence in Ω. Then $\mathscr{A}$ is $H^p(U^n, \mathbb{C})$ interpolating. We must prove $\mathscr{A}$ is U.S. By Lemma 5.10, it suffices to prove $\mathscr{A}$ is L.U.S.

Let $\varepsilon = 1/6$ and let $b \in U^n$. Let $\mathscr{A} \cap D^n(\sigma(b), \varepsilon) = (a_j)_{j=1}^\infty$. By Proposition 3.1, there exists $(f_j)_{j=1}^\infty$ in $H^p(U^n, \mathbb{C})$, such that $\|f_j\| \leq M$ and

$$\Pi(1 - |a_k|^2)^{1/p} f_j(a_k) = \delta_{j,k}, \quad \text{for all } j, k.$$

By Theorem 2.1, $f_j(z_1, \cdots, z_n) = G_j(z_1, \cdots, z_n) F_j(z_n)$ where F_j is an outer function in $H^p(U^1, \mathbf{C})$ with $\|F_j\|_p = \|f_j\|_p \leq M$, and G_j has the property that for each $z_n \in U^1$, $G_j(\cdot, z_n) \in H^p(U^{n-1}, \mathbf{C})$ and $\|G(\cdot, z_n)\|_p \leq 1$.

By (1), if $z \in D^n(\sigma(b), 1/6)$ then

$$|z_j - b_j| \leq (1 - |b_j|^2)/3, \qquad \text{for } j = 1, 2, \cdots, n-1.$$

Therefore, by letting $g_j(z) = \prod_{k=1}^{n-1} (1 - |b_k|^2)^{1/p} G_j(z)$, we have by inequality (12),

$$|g_j(z)| \leq 3^{(n-1)/p}, \qquad z \in D^n(\sigma(b), \varepsilon);$$

$$g_j(a_i) = 0, \qquad j \neq i; \quad \text{and}$$

$$|g_j(a_j)| = \prod_{k=1}^{n-1} (1 - |b_k|^2)^{1/p} \prod_{k=1}^{n-1} (1 - |a_{jk}|^2)^{-1/p} |F_j(a_{jn})(1 - |a_{jn}|^2)^{1/p}|^{-1}$$

$$\geq M^{-1}(3/5)^{(n-1)/p}.$$

Hence $\mathscr{A} \cap D^n(\sigma(b), \varepsilon)$ is U.S. in $D^n(\sigma(b), \varepsilon)$ with U.S. bound $5^{(n-1)/p}M$, for all $b \in U^n$, so $\mathscr{A}$ is L.U.S. This completes the proof that (ii) implies (i).

To prove (i) implies (iv), observe that if $\mathscr{A}$ is U.S. then $\mathscr{A}$ is separated, and that the proof that (i) implies (iii) shows that $T^2_{\mathscr{A}}$ is bounded. Let $K(z, a)$ be the normalized $H^2(U^n, \mathbf{C})$ reproducing kernel, $K(z, a) = \prod ((1 - |a|^2)^{1/2}/(1 - \bar{a}z))$. Then

$$\sum_{j=1}^{\infty} \prod [(1 - |a_j|^2)(1 - |a_j|^2)/|1 - \bar{a}_i a_j|^2] = \|T^2_{\mathscr{A}}(K(z, a_i))\|^2 \leq \|T^2_{\mathscr{A}}\|^2.$$

Conversely, suppose $\mathscr{A}$ is separated with separation bound δ, and satisfies the inequality of (iv) for some $M < \infty$. Let $0 < \varepsilon < \delta/32$; let $b \in U^{n-1}$; and suppose $a_i, a_j \in D^n(b, \varepsilon) \cap \mathscr{A}$. By inequalities (1) and (12) $\rho(\sigma(a_i), \sigma(a_j)) < 16\varepsilon < \delta/2$. Since $\rho(a_i, a_j) > \delta$ for $i \neq j$, the sequence $\pi(\mathscr{A} \cap D^n(b, \varepsilon))$ is separated. Also

$$(3/4)^{n-1} \sum_{j=1}^{\infty} 1 - (\rho(\pi(a_i), \pi(a_j)))^2 \leq \sum_{j=1}^{\infty} \prod_{k=1}^{n} (1 - \rho(a_{ik}, a_{jk}))^2)$$

$$= \sum_{j=1}^{\infty} \prod [(1 - |a_i|^2)(1 - |a_j|^2)/|1 - \bar{a}_i a_j|^2] \leq M, \qquad \text{for all } i.$$

Hence $\pi(\mathscr{A} \cap D^n(b, \varepsilon))$ is U.S. in U^1 with U.S. bound K_0, depending only on M and δ. Since there is only one point in $\mathscr{A} \cap D^n(b, \varepsilon)$ with given projection $\pi(a_i)$, the sequence $\mathscr{A} \cap D^n(b, \varepsilon)$ is $H^\infty(D^n(b, \varepsilon), \mathbf{C})$ interpolating, by Corollary 3.4, with interpolation bound depending only on M and δ, but not on b. Hence $\mathscr{A}$ is L.U.S. By Lemma 5.10, $\mathscr{A}$ is U.S. This proves (iv) implies (i) and completes the proof of Theorem 1.1.

It is easy to see that condition (iv) is a necessary condition on $\mathscr{A}$ in order that $T^2_{\mathscr{A}}(H^2(U^n, \mathbf{C})) = \ell^2$, even when $\mathscr{A}$ is not contained in a finite union of near wedges. On the other hand, the examples given at the end of section 7, show that it is not, in general, a sufficient condition for $T^{\infty}_{\mathscr{A}}(H^{\infty}(U^n, \mathbb{C})) = \ell^{\infty}$.

6. INTERPOLATION IN WEDGES FOR $A^{p,\alpha}$ SPACES

In this section we will use Theorem 1.1 to prove Theorem 1.2.

Proof of Theorem 1.2. The statement concerning $H^p(U^n, E)$ follows immediately from Theorem 1.1 and the fact that a separated sequence in a finite union of non-tangential wedges in U^n is uniformly separated [7, Theorem 5.16]. The statement that a separated sequence in Ω is $A^{p,\alpha}(U^n, E)$ interpolating will follow immediately from the $H^p(U^n, E)$ statement and the following proposition:

PROPOSITION 6.1. *Let* $1 \leq p < \infty$ *and* $0 \leq \alpha < \infty$. *The linear map*

$$S_{p,\alpha}: H^p(U^n, E) \to A^{p,\alpha}(U^n, E)$$

defined by $S_{p,\alpha}(f)|_z = f(z) \prod (1 - z^2)^{-(\alpha+1)/p}$ *is bounded.*

Suppose we grant proposition 6.1 for the moment. Let Ω_0 be a wedge in U^1 bounded by the arcs of two circles intersecting at ± 1 on C^1. Let

$$\Omega = \Omega_0 \times \Omega_0 \times \cdots \times \Omega_0 \subseteq U^n .$$

It is clearly sufficient to prove Theorem 1.2 for wedges such as Ω. There exists a constant K, depending on Ω, such that $1 - |z|^2 \leq |1 - z^2| \leq K(1 - |z|^2)$ for $z \in \Omega$. Hence if $\mathscr{A}$ is a separated sequence in Ω and $c \in \ell^p(E)$, we may easily interpolate c by an $A^{p,\alpha}(U^n, E)$ function F, by first solving a related $H^p(U^n, E)$ interpolation problem and then mapping via $S_{p,\alpha}$.

To prove Proposition 6.1, we need the following fact [*cf.*, 13 p. 530].

LEMMA 6.2. *The mapping* $S: f(z) \to (1 - z)^{-1/2} f(z)$ *is a bounded linear transformation from* $H^2(U^1, \mathbb{C})$ *onto* $A^2(U^1, \mathbb{C})$.

Proof of Proposition 6.1. First, note that the mapping

$$S: f(z) \to (1 - z^2)^{-(\alpha+1)/2} f(z)$$

is a bounded linear transformation from $H^2(U^1, \mathbb{C})$ into $A^{2,\alpha}(U^1, \mathbb{C})$, for each $0 \leq \alpha < \infty$. To show this, it suffices to show $S': f(z) \to (1 - z)^{-(\alpha+1)/2} f(z)$ is a bounded linear transformation from $H^2(U^1, \mathbb{C})$ into $A^{2,\alpha}(U^1, \mathbb{C})$. But for $f \in H^2(U^1, \mathbb{C})$,

$$\int |f(z)(1 - z)^{-(\alpha+1)/2}|^2 (1 - |z|^2)^{\alpha} \, d\nu(z) \leq 2^{\alpha} \int |f(z)(1 - z)^{-1/2}|^2 \, d\nu(z) ,$$

which is bounded by Lemma 6.2.

Now let ϕ be analytic and non-vanishing on U^1, and suppose the linear map $f \to f\phi^{1/2}$ is a bounded linear map of $H^2(U^1, \mathbb{C})$ into $A^{2,\alpha}(U^1, \mathbb{C})$. Then for each $p \in [1, \infty)$, the linear map S_p defined by $S_p f(z_1, \cdots, z_n) = f(z) \prod_{i=1}^{n} \phi(z_i)^{1/p}$ is a bounded transformation from $H^p(U^n, E)$ into $A^{p,\alpha}(U^n, E)$. This follows from the factorization theorems 2.1 and 2.2, and the observation that $A^{p,\alpha}(U^n, E)$ is canonically isometrically isomorphic to $A^{p,\alpha}(U^k, A^{p,\alpha}(U^{n-k}, E))$.

Finally, set $\phi(z) = (1 - z^2)$ to obtain Proposition 6.1.

7. INTERPOLATION IN $A^{2,\alpha}$

In this section, which was largely inspired by Shapiro and Shields' elegant application of the classical theory of quadratic forms to interpolation theory [13], we shall prove the following more specific version of Theorem 1.3.

THEOREM 7.1. *Let* $\mathscr{A} = (a_i)_{i=1}^{\infty}$ *be a sequence in* U^n. *Then there exists* $\alpha \geq 0$ *such that* $\mathscr{A}$ *is* $A^{2,\alpha}(U^n, \mathbb{C})$ *interpolating if and only if* $\mathscr{A}$ *is separated. Furthermore,*

(i) *If* $0 < \delta < 1/2$ *and the sequence of polydisks* $(U^n(a_i, (1 - |a_i|^2)\delta))_{i=1}^{\infty}$ *is pairwise disjoint, then* $\mathscr{A}$ *is* $A^{2,\alpha}(U^n, \mathbb{C})$ *interpolating for all* α *such that* $(\delta^{-2n} - 1)(1 - \delta^2)^{(\alpha-2)/2} < 1$.

(ii) *If* $0 < \delta < 1$ *and* $\mathscr{A}$ *is separated with separation bound* δ, *then* $\mathscr{A}$ *is* $A^{2,\alpha}(U^n, \mathbb{C})$ *interpolating for all* α *such that* $(4^{2n}\delta^{-2n} - 1)(1 - \delta^2)^{(\alpha-2)/2} < 1$.

The machinery for proving Theorem 7.1 comes in three parts. First we have the following theorem.

THEOREM 7.2. (i) *If* $0 < \delta < 1/2$ *and* $\mathscr{A}$ *has the property that the polydisks* $U^n(a_i, \delta(1 - |a_i|^2))$ *are pairwise disjoint, then* $T^{p,\alpha}_{\mathscr{A}}(A^{p,\alpha}(U^n, E)) \subseteq \ell^p(E)$. *In fact, if* $\alpha \in (0, \infty)$, $\|T^{p,\alpha}_{\mathscr{A}}\| \leq (3)^{\alpha n/p} \mathrm{Max}[3^{2n/p}, \delta^{-2n/p}]$. *If* $\alpha = 0$, *then*

$$\|T^{p,0}_{\mathscr{A}}\| \leq \delta^{-2n/p}.$$

(ii) *If* $0 < \delta < 1$ *and* $\mathscr{A}$ *is separated with constant* δ, *then*

$$T^{p,\alpha}_{\mathscr{A}}(A^{p,\alpha}(U^n, E) \subseteq \ell^p(E).$$

In fact, for $\alpha \in [0, \infty)$, $\|T^{p,\alpha}_{\mathscr{A}}\| \leq (3)^{\alpha n/p}(4/\delta)^{2n/p}$.

Proof. Statement (ii) follows from statement (i) and inequality (1). The proof of statement (i) is modelled on the Shapiro-Shields proof for the special case of $A^{2,0}(U^1, \mathbb{C})$ [13, p. 530].

Let $r_i = \delta(1 - |a_i|^2)$. Since $|f(z)|^p$ is n-subharmonic,

$$|T^{p,\alpha}_{a_i} f|^p \leq \delta^{-2n}\pi^{-n} \prod (1 - |a_i|^2)^{\alpha} \int_{U^n(a_i,r_i)} |f(z)|^p \, d\nu .$$

Here, $d\nu$ is Lebesgue measure on $\mathbb{C}^n$. If $\alpha > 0$, assume without loss of generality that $0 < \delta < 1/3$, and apply inequality (12) to conclude that

$$|T^{p,\alpha}_{a_i} f|^p \leq (3^{\alpha}/\delta^2 \pi)^n \int_{U^n(a_i,r_i)} |f(z)|^p \prod (1 - |z|^2)^{\alpha} \, d\nu .$$

We obtain the desired bound on $\|T^{p,\alpha}_{\mathscr{A}}\|$ by summing the above inequalities over all i.

The next step in the proof of Theorem 7.1 is:

THEOREM 7.3. *Suppose* $\mathscr{A}$ *is* $A^{p,\alpha}(U^n, E)$ *interpolating for some non-trivial Banach space* E, *some* $p \in [1, \infty)$, *and some* $\alpha \in [0, \infty)$. *Then* $\mathscr{A}$ *is separated.*

Theorems 7.2 and 7.3 show that if $T^{p,\alpha}_{\mathscr{A}}(A^{p,\alpha}(U^n, E)) \supseteq \ell^p(E)$, then in fact $T^{p,\alpha}_{\mathscr{A}}(A^{p,\alpha}(U^n, E)) = \ell^p(E)$ and $T^{p,\alpha}_{\mathscr{A}}$ is a bounded linear operator from $A^{p,\alpha}(U^n, E)$ to $\ell^p(E)$.

To prove Theorem 7.3 we need:

LEMMA 7.4. *Suppose* $b \in U^n$ *and* $c \in U^n(b, (1 - |b|^2)\varepsilon)$ *for some* $\varepsilon \in (0, 1/4)$. *Let* $1 \leq p < \infty$, $0 \leq \alpha < \infty$, *and* $f \in A^{p,\alpha}(U^n, E)$. *Suppose* $f(b) = 0$. *Then* $|f(c)|^p \prod (1 - |c|^2)^{\alpha+2} \leq \varepsilon^p C(n, p) \|f\|^p_{A^{p,\alpha}}$, *where* $C(n, p)$ *is a constant depending only on* n *and* p.

Proof. Since $f(b) = 0$, we have

$$|f(c)| \leq \left(\prod s^{-2}\right) \pi^{-n} \int_{U^n(b,s)} |K((w - b)/s, (c - b)/s) - 1| \cdot |f(w)|\, d\nu(w),$$

where $K(\alpha, \beta) = \prod_{i=1}^n (1 - \alpha_i \bar{\beta}_i)^{-2}$ is the Bergman kernel for U^n, and

$$s = (1 - |b|^2)/2 .$$

Since $|c_j - b_j|/s_j \leq 2\varepsilon \leq 1/2$, for $j = 1, 2, \cdots, n$, an application of Lemma 4.5 gives $|K((w - b)/s, (c - b)/s) - 1| \leq 20n6^{n-1}\varepsilon = C_n\varepsilon$. Hence

$$|f(c)| \leq \left(\prod s^{-2}\right) \pi^{-n} \int_{U^n(b,s)} C_n\, \varepsilon\, |f(w)|\, d\nu(w)$$

$$\leq C_n\, \varepsilon \left[\prod (s^{-2}) \pi^{-n} \int_{U^n(b,s)} |f(w)|^p\, d\nu(w) \right]^{1/p} .$$

By substituting $s = 1/2(1 - |b|^2)$, raising both sides of this inequality to the p^{th} power, multiplying both sides of the resulting inequality by $\prod (1 - |b|^2)^{\alpha+2}$, and applying inequality (12), we obtain the desired estimate on $|f(c)|^p \prod (1 - |c|^2)^{\alpha+2}$.

Proof of theorem 7.3. Suppose $\mathscr{A}$ is $A^{p,\alpha}(U^n, E)$ interpolating, and $\mathscr{A}$ is not separated. Then $\mathscr{A}$ is $A^{p,\alpha}(U^n, \mathbb{C})$ interpolating, and by Proposition 3.1, there exists a constant M and a sequence of functions $(f_i)_{i=1}^\infty$ in $A^{p,\alpha}(U^n, \mathbb{C})$ such that $\|f_i\|_{A^{p,\alpha}} \leq M$ for all i, and $T^{p,\alpha}_{\mathscr{A}}(f_i) = (\delta_{ij})_{j=1}^\infty$ where δ_{ij} is the Kronecker δ.

Since $\mathscr{A}$ is not separated, there exist integers i and j, such that $i \neq j$ but $\rho(a_i, a_j) < \varepsilon/2$, where $\varepsilon \in (0, 1/4)$ and is so small that $\varepsilon^p C(n, p) M^p < 1/2$, where $C(n, p)$ is as in Lemma 7.4. But, by inequality (1), $a_i \in U^n(a_j, (1 - |a_j|^2)\varepsilon)$, so by Lemma 7.4, $|f_i(a_i)|^p \prod (1 - |a_i|^2)^{\alpha+2} < 1/2$. This is a contradiction.

The third part of the machine needed to prove Theorem 7.1 is the following result of J. Schur:

Let $A = (a_{ij})$ be an infinite Hermitian matrix. Schur proves that if

$$\sum_j |a_{ij}| \leq M \qquad \text{for all } i,$$

then A defines a bounded operator on ℓ^2 and $\|A\| \leq M$ [12, p. 6].

We also need to observe that $A^{2,\alpha}(U^n, \mathbb{C})$ is a Hilbert space under the inner product $\langle f, g \rangle = (\alpha+1)^n \pi^{-n} \int_{U^n} f(z)\bar{g}(z) \prod (1-|z|^2)^\alpha d\nu$. The reproducing kernel for $A^{2,\alpha}(U^n, \mathbb{C})$ is $K_\alpha(z, w) = \prod (1 - z\bar{w})^{-(\alpha+2)}$.

Proof of Theorem 7.1. The *only if* part of Theorem 7.1 follows from Theorem 7.3. The *if* part will be disposed of by proving statements (i) and (ii). Let us first consider statement (i). Since

$$U^n(a_i, (1-|a_i|^2)\delta) \cap U^n(a_j, (1-|a_j|^2)\delta) = \emptyset \qquad \text{if } i \neq j,$$

$\|T^{2,0}_{\mathscr{A}}\| \leq \delta^{-n}$ by Theorem 7.2.

For each i, let $\sigma_i(z) = \prod (1-|a_i|^2)/(1-\bar{a}_i z)^2$, and let

$$\sigma_{ij} = \prod (1-|a_j|^2)\sigma_i(a_j) = \prod ((1-|a_i|^2)(1-|a_j|^2))/(1-\bar{a}_i a_j)^2).$$

Then $\sigma_i \in A^{2,0}$; $\|\sigma_i\|_{A^{2,0}} = 1$; and $\sigma_{ii} = 1$.

Consequently, for all i,

$$\sum_{\substack{j \neq i \\ j=1}}^{\infty} |\sigma_{ij}|^2 = \|T^{2,0}_{\mathscr{A}}(\sigma_i)\|^2 - 1 \leq \|T^{2,0}_{\mathscr{A}}\|^2 \, \|\sigma_i\|^2 - 1 \leq \delta^{-2n} - 1.$$

By inequality (1), $\rho(a_i, a_j) \geq \delta$ if $i \neq j$. Thus for each $i \neq j$,

$$|\sigma_{ij}| \leq 1 - (\rho(a_i, a_j))^2 \leq 1 - \delta^2.$$

Now let α be so large that $(\delta^{-2n} - 1)(1-\delta^2)^{(\alpha-2)/2} < 1$. Then for all i,

$$(13) \quad \sum_{\substack{j \neq i \\ j=1}}^{\infty} |\sigma_{ij}|^{(\alpha+2)/2} = \sum_{\substack{j \neq i \\ j=1}}^{\infty} |\sigma_{ij}|^2 |\sigma_{ij}|^{(\alpha-2)/2} \leq (\delta^{-2n} - 1)(1-\delta^2)^{(\alpha-2)/2} < 1.$$

Let B be the inner product matrix of normalized $A^{2,\alpha}(U^n, \mathbb{C})$ reproducing kernels, $B = ((\sigma_{ij})^{(\alpha+2)/2})_{ij}$. Then $\|I - B\| < 1$ by inequality (13) and Schur's result quoted earlier, so B is invertible. Consequently, $T^{2,\alpha}_{\mathscr{A}}$ carries $A^{2,\alpha}(U^n, \mathbb{C})$ onto ℓ^2 (*cf.*, [13, Theorem 3]). (This follows easily from the fact that B is the matrix of $T^{2,\alpha}_{\mathscr{A}} T^{2,\alpha *}_{\mathscr{A}}$.)

Statement (ii) follows from the same argument, since $\|T^{2,0}_{\mathscr{A}}\| = 4^n/\delta^n$, and $\rho(a_i, a_j) > \delta$ if $i \neq j$. This concludes the proof of Theorem 7.1.

It is easy to show that the minimal norm interpolating function in $A^{2,\alpha}(U^n, \mathbb{C})$ is unique and is a sum of reproducing kernels. From this we can show that minimal norm interpolation can be done linearly.

An immediate corollary to Theorem 1.3 is the following counterexample which should be compared to Theorem 1.1.

COROLLARY 7.5. *There is a sequence which is* $H^2(U^5, \mathbb{C})$ *interpolating, but not uniformly separated.*

Proof. Let $1/2 < \delta < 1$ be such that $63(1 - \delta^2)^{1/2} < 1$. Construct a sequence $\mathscr{A}_1 = (a_i)_{i=1}^{\infty} \subset U^1$ which is separated, with separation bound δ, but not uniformly separated. Theorem 7.1 and the choice of δ guarantees that $\mathscr{A}_1$ is $A^{2,3}(U^1, \mathbb{C})$ interpolating.

Now let $\mathscr{A} = ((a_i, a_i, a_i, a_i, a_i))_{i=1}^{\infty} \subset U^5$. If f is a function defined on U^n and $z \in U^1$, let $(\mathscr{D}f)(z) = f(z, z, \cdots, z)$. Then $\mathscr{D}$ maps $H^2(U^n, \mathbb{C})$ onto

$$A^{2,n-2}(U^1, \mathbb{C})$$

[11]. Clearly as maps from $H^2(U^5, \mathbb{C})$ to ℓ^2, $T^{2,3}_{\mathscr{A}} \cdot \mathscr{D} = T^2_{\mathscr{A}}$. Since $T^{2,3}_{\mathscr{A}}$ and $\mathscr{D}$ are surjective, $\mathscr{A}$ is $H^2(U^5, \mathbb{C})$ interpolating. Since $\mathscr{A}_1$ is not U.S. in U^1, $\mathscr{A}$ is not U.S. in U^5.

The above corollary is a weaker result than that in [1], in which an $H^2(U^2, \mathbb{C})$ interpolating sequence that is not U.S. is exhibited.

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