DISCRETE CARLESON MEASURES AND SOME INTERPOLATION PROBLEMS

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1. INTRODUCTION

The study of sequences of distinct points in the open unit disc

$$\Delta = \{z: z \in \mathbb{C},\ |z| < 1\}$$

satisfying

(1) $$\sum |f(z_n)|(1 - |z_n|) \leq M\|f\| \quad \text{for all } f \in H^1;$$

or equivalently [4, p. 152],

(2) $$\sum |f(z_n)|^2(1 - |z_n|^2) \leq M\|f\|^2 \quad \text{for all } f \in H^2,$$

has been important in the study of interpolation problems in H^∞, the algebra of bounded analytic functions on Δ. In [6], Newman proved that if the sequence $\{z_n\}$ satisfies (1) and is *uniformly separated; i.e.*,

(3) $$\prod_{i \neq j} \left|\frac{z_i - z_j}{1 - \bar{z}_i z_j}\right| \geq \delta > 0, \quad j = 1, 2, \cdots,$$

then for each $\{w_n\} \in \ell^\infty$, there exists $f \in H^\infty$ such that $f(z_n) = w_n$; that is, the sequence $\{z_n\}$ is a *universal interpolating sequence*. In [1], Carleson showed that the sequence $\{z_n\}$ is uniformly separated if and only if it is a universal interpolating sequence. A key step in Carleson's theorem was the proof that if $\{z_n\}$ is uniformly separated, then (1) is satisfied. Similarly, in another proof of Carleson's theorem by Shapiro and Shields [7], a key step was in establishing that if $\{z_n\}$ is uniformly separated, then the equivalent inequality (2) is satisfied.

In Section 2 we give a necessary and sufficient arithmetic condition on the sequence $\{z_n\}$ for (2) to be satisfied. As a corollary, we deduce a partial converse to the results of Carleson and of Shapiro and Shields. We show that if (2) holds, then the sequence $\{z_n\}$ is a finite union of uniformly separated subsequences.

The idea behind the inequalities (1) and (2) has been generalized in at least two different ways. In [2], Carleson considered general measures μ in the open unit disc and gave a necessary and sufficient geometric condition that

(4) $$\int |f(z)|^p d\mu \leq M\|f\|^p \quad \text{for all } f \in H^p.$$

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In Section 3, we restrict ourselves to the case where μ is the discrete measure $\mu(\{z_n\}) = 1 - |z_n|^2$ and give an elementary proof that if the measure μ satisfies Carleson's condition, then it satisfies (2).

A second avenue of generalization is that of Shapiro and Shields [7], where they introduce the concepts of Riesz-Fischer and Bessel sequences of normalized kernel functions in order to consider weighted interpolation problems for other Hilbert spaces of analytic functions. In Section 4, we study Bessel sequences in Hilbert spaces of analytic functions containing H^2 and give a new necessary and sufficient condition for a sequence to be a Bessel sequence in the Bergman space. In Section 5, we study Riesz-Fischer and Bessel sequences in the Dirichlet space D contained in H^2.

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2. UNIONS OF INTERPOLATING SEQUENCES

Definition. A sequence $\{z_n\}$ which satisfies

$$\sum_{j=1}^{\infty} \frac{(1-|z_i|^2)(1-|z_j|^2)}{|1-\bar{z}_i z_j|^2} \leq M, \qquad i = 1, 2, \cdots, \tag{5}$$

is said to be *uniformly square-summable.*

THEOREM 1. *A necessary and sufficient condition that the sequence* $\{z_n\}$ *satisfy* (2) *is that* $\{z_n\}$ *be uniformly square-summable.*

Proof of necessity. Note that condition (5) is merely the result of inserting the normalized kernel functions, defined by

$$k_m(z) = (1-|z_m|^2)^{1/2}(1-\bar{z}_m z)^{-1} \tag{6}$$

in place of f in (2). Thus condition (5) is clearly necessary.

The proof of sufficiency depends on two elementary lemmas.

Definition. We shall call a sequence $\{z_n\}$ satisfying

$$(1-|z_n|^2)(1-|z_m|^2)|1-\bar{z}_m z_n|^{-2} < \alpha < 1 \qquad \text{for all } m, n \tag{7}$$

a *weakly separated sequence.*

LEMMA 1. *If condition* (5) *is satisfied, and in addition the sequence* $\{z_n\}$ *is weakly separated, then the sequence is uniformly separated.*

Proof. We use the well-known fact [5, p. 201] that if $0 < a_i < 1 - \delta^2$ and $\sum a_i < M$, then $\prod (1 - a_i) > e^{-M/\delta^2}$. Using (7) and

$$\delta^2 \leq \left|\frac{z_i - z_j}{1-\bar{z}_i z_j}\right|^2 = 1 - \frac{(1-|z_i|^2)(1-|z_j|^2)}{|1-\bar{z}_i z_j|^2},$$

we deduce that

$$\prod_{i \neq j} \left|\frac{z_i - z_j}{1-\bar{z}_i z_j}\right|^2 = \prod_{i \neq j} \left\{1 - \frac{(1-|z_i|^2)(1-|z_j|^2)}{|1-\bar{z}_i z_j|^2}\right\} \geq e^{-M_2/\delta^2} \qquad \text{for all } j.$$

Therefore, the sequence $\{z_n\}$ is uniformly separated.

LEMMA 2. *If condition* (5) *is satisfied, then* $\{z_n\}$ *can be expressed as a finite union of subsequences* $\{z_{n,k}\}$ *which are both weakly separated and uniformly square-summable.*

Proof. The proof is by a simple enumeration technique. Fix α, $0 < \alpha < 1$, and choose $N > M_2/\alpha$, where M_2 is given in (5). Let $z_{n,1} = z_n$ for $1 \leq n \leq N$. We now have the first term in each of the N subsequences. Assume that we have partitioned $\{z_i\}_{i \leq J}$ into N subsequences so that within each subsequence, we have

$$(1 - |z_{n,i}|^2)(1 - |z_{n,j}|^2)|1 - z_{n,i} z_{n,j}|^{-2} < \alpha . \tag{8}$$

We must now allocate z_{J+1} to one of the subsequences. By virtue of the inequality (5), and by our choice of N, we know that (8) may be violated for at most N - 1 values of i if j = J + 1. Thus z_{J+1} can be made the next term of at least one of the subsequences in such a way that each subsequence still obeys (8). By induction, the proof of Lemma 2 is complete.

Proof of sufficiency in Theorem 1. By [7, pp. 519-520], any uniformly separated sequence $\{z_n\}$ satisfies (2). Therefore, by Lemma 2, we may split any uniformly square-summable sequence $\{z_n\}$ into a finite union of sequences which are also weakly separated. By Lemma 1, these subsequences are uniformly separated. Obviously, if (2) is satisfied by each of a finite number of sequences, it is satisfied by the union of those sequences. This concludes the proof of the theorem.

3. CARLESON MEASURES

Following [4, p. 157], we define a *Carleson measure* μ to be a finite measure on Δ such that for some constant A,

$$\mu(S) \leq Ah \tag{9}$$

for every set S of the form $S = \{z = re^{i\theta}: 1 - h \leq r < 1,\ \theta_0 \leq \theta \leq \theta_0 + h\}$. Measures of this type were considered by Carleson en route to his proof of the corona theorem [2]. He proved

THEOREM A. *In order that there exist a constant* C *such that*

$$\int_{|z|<1} |f(z)|^2 d\mu(z) \leq C\|f\|^2 \quad \text{for all } f \in H^2, \tag{10}$$

it is necessary and sufficient that μ *be a Carleson measure.*

In this section we restrict ourselves to discrete measures of the type

$$\mu(z_n) = 1 - |z_n|^2 \tag{11}$$

and give an elementary proof that if μ is a Carleson measure, then the sequence $\{z_n\}$ is uniformly square-summable. Thus by Theorem 1, (10) is satisfied, and we have given a new proof of Theorem A for this particular type of measure μ.

THEOREM 2. *If the measure* μ *defined by* (11) *is a Carleson measure, then the sequence* $\{z_n\}$ *is uniformly square-summable.*

Proof. In order to prove the theorem, we must show that for the kernel functions k_n defined by (6), there exists an M such that

$$\int |k_n(z)|^2 d\mu \leq M, \qquad n = 1, 2, \cdots .$$

We shall use the identity $(1 - |z_m|^2)(1 - |z_n|^2) + |z_m - z_n|^2 = |1 - \bar{z}_n z_m|^2$, so that

$$\int |k_n(z)|^2 d\mu = \int \frac{(1 - |z_m|^2)}{|z - z_m|^2 + (1 - |z|^2)(1 - |z_m|^2)} d\mu(z). \tag{12}$$

We let $z_m = r_m e^{i\theta_m}$, and define the following sets in Δ. Let $h = 2(1 - r_m)$ and let $A_n = \{z \in \Delta: z = re^{i\theta},\ 1 - nh \leq r < 1,\ \theta_m - nh \leq \theta \leq \theta_m + nh\}$. There exists N such that if $z \notin A_n$, for all $n \leq N$, then $|z - z_n| \geq \sqrt{1 - |z_m|^2}$. We subdivide Δ into the set $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$, $2 \leq i \leq N$, and $B_{N+1} = \Delta \setminus A_{N+1}$. We estimate the integrand in (12) as follows:

$$\frac{(1 - |z_m|^2)}{|z - z_m|^2 + (1 - |z|^2)(1 - |z_m|^2)} \leq \begin{cases} \dfrac{1}{1 - |z_m|^2}, & z \in B_1; \\[2ex] \dfrac{1}{(N - 1)(1 - |z_m|^2)}, & z \in B_i,\ 2 \leq i \leq N; \\[2ex] 1, & z \in B_{N+1}. \end{cases}$$

We assume the measure achieves its largest possible concentration where the integrand is largest, so that $\mu(B_i)$ is estimated as follows:

$$\mu(B_1) = 2Ah; \qquad \mu(B_2) = \mu(A_2) - \mu(A_1) = 2Ah;$$

$$\mu(B_N) = \mu(A_N) - \mu(A_{N-1}) = 2Ah; \qquad \mu(B_{N+1}) \leq C_2,$$

the first three estimates coming from (9) and the last from the finiteness of the measure μ. Now since

$$\int_\Delta |k_m(z)|^2 d\mu(z) = \sum_{i=1}^{N+1} \int_{B_i} |k_m(z)|^2 d\mu(z)$$

$$\leq \frac{1}{1 - |z_m|^2} C(1 - |z_m|) + \sum_{i=2}^{N+1} \frac{1}{(i - 1)^2 (1 - |z_m|^2)} \mu(B_i) + \mu(B_{N+1})$$

$$\leq C + 2C \sum_{i=2}^{N} \frac{1}{(i - 1)^2} + \mu(\Delta),$$

we have proved $\int_\Delta |k_m(z)|^2 d\mu(z) \leq M = C + 2C \sum_{n=1}^{\infty} n^{-2} + \mu(\Delta)$, where M is independent of m, and thus the sequence $\{z_m\}$ is uniformly square-summable. Note that when Theorem 2 is combined with Lemma 2, we obtain the following result.

COROLLARY 1. *If the discrete measure* μ *defined by* $\mu(z_n) = 1 - |z_n|^2$ *is a Carleson measure, then* $\{z_n\}$ *is a finite union of uniformly separated subsequences.*

4. SPACES CONTAINING H^2

We recall some definitions and theorems from [7]. A bounded sequence of elements $\{y_n\}$ in a Hilbert space H gives rise to a mapping $T: H \to \ell^\infty$ given by $Tx = \{(x, y_n)\}$. If to each sequence $\{c_n\} \in \ell^2$ there corresponds at least one $x \in H$ for which $(x, y_n) = C_n$ for all n, and $\|x\| \leq m\|C\|_2$, then $\{y_n\}$ is called a *Riesz-Fischer sequence* with bound m. If $\|Tx\|_2 \leq M\|x\|$ for all $x \in H$, then $\{y_n\}$ is called a *Bessel sequence* with bound M. (Note that the condition (1) says that the sequence $\{k_m\}$ defined by (6) is a Bessel sequence in H^2.)

The following theorem of Bari is stated in [7, p. 524].

THEOREM B. *Let* $\{y_n\}$ *be a sequence of elements of a Hilbert space* H, *and let* A *denote the inner product matrix* $\{(y_i, y_j)\}$. *Then*

(i) $\{y_n\}$ *is a Bessel sequence with bound* M *if and only if the matrix* A *is bounded on* ℓ^2 *with bound* M;

(ii) $\{y_n\}$ *is a Riesz-Fischer sequence with bound* m *if and only if the matrix* A *is bounded below on* ℓ^2 *with bound* m.

A matrix is bounded below by m if $\|a\| \leq m\|A_n a\|$ for all n-tuples $a = (a_1, a_2, \cdots, a_n)$ and all n. Here the norms are ℓ_2 norms, and A_n is the $n \times n$ matrix $\{(y_i, y_j)\}$, $i, j = 1, 2, \cdots, n$.

Note that Theorem 1 proves that the matrix

$$\{(k_i, k_j)\} = \left\{ \frac{(1 - |z_i|^2)^{1/2}(1 - |z_j|^2)^{1/2}}{(1 - \bar{z}_i z_j)} \right\}$$

is bounded if and only if the columns are bounded in ℓ^2.

We now consider some weighted interpolation problems in other Hilbert spaces of analytic functions on $\triangle$. As in [7], the space A_2 shall be the Bergman space of functions $f(z) = \sum_{n=0}^\infty a_n z^n$ satisfying $\|f\|^2 = \sum |a_n|^2 (n+1)^{-1} < \infty$. The kernel functions in A_2 are $K_w(z) = 1/(1 - \bar{w}z)^2$, and the orthonormalized kernel functions are

$$k_w^1(z) = (1 - |w|^2)(1 - \bar{w}z)^{-2}. \tag{13}$$

The space H_2' is the space of functions f satisfying

$$\|f\|^2 = \frac{1}{\pi} \iint |f(z)|^2 (1 - |z|^2)\, dx\, dy < \infty. \tag{14}$$

Here the normalized kernel function is given by

$$k_w^2(z) = (1 - |w|^2)^{3/2}/(1 - \bar{w}z)^3. \tag{15}$$

As was shown in [7, p. 529], if the sequence $\{z_i\}$ is uniformly separated, then the associated sequence of orthonormalized kernel functions in A_2 is both a Riesz-Fischer and a Bessel sequence. By Lemma 1, we remark that if the sequence of orthonormalized kernel functions in H^2 is a Bessel sequence, then $\{z_n\}$ is a finite union of uniformly separated sequences. Since a finite union of Bessel sequences is again a Bessel sequence, we have proved:

THEOREM 3. *If the normalized kernel functions in* H^2 (*given by* (6)) *associated with* $\{z_i\}$ *form a Bessel sequence, then the normalized kernel functions in* A_2 *associated with* $\{z_n\}$ *form a Bessel sequence.*

Also in [7, p. 529] the following was proved.

THEOREM C. *If the sequence* $\{z_n\}$ *is weakly separated, then the sequence*

$$\{k_n^1\} = \{(1 - |z_n|^2)/(1 - \bar{z}_n z)^2\} \tag{14}$$

is a Bessel sequence in A_2, *and the sequence*

$$\{k_n^2\} = \{(1 - |z_n|^2)^{3/2}/(1 - \bar{z}_n z_m)^3\} \tag{15}$$

is a Bessel sequence in H_2'.

We prove a partial converse to both of these statements.

THEOREM 3. (i) *The sequence* $\{k_n^1\}$ *given by* (14) *is a Bessel sequence in* A_2 *if and only if* $\{z_n\}$ *is the finite union of subsequences, each of which is weakly separated.*

(ii) *Similarly, the sequence* $\{k_n^2\}$ *is a Bessel sequence in* H_2' *if and only if the sequence* $\{z_n\}$ *is a finite union of subsequences, each of which is weakly separated.*

Proof. (i) By Theorem C, and again since a finite union of Bessel sequences is again a Bessel sequence, the sufficiency is obvious. We now prove necessity.

If the sequence $\{k_n^1\}$ is a Bessel sequence in A_2, then by Bari's theorem, the matrix $\{(k_n^1, k_m^1)\}$ is bounded on ℓ^2. Consequently, the columns must be uniformly bounded in ℓ^2. Thus

$$\sum_{n=1}^{\infty} \frac{(1 - |z_n|^2)^2(1 - |z_m|^2)^2}{|1 - \bar{z}_n z_m|^4} \leq M, \qquad \text{for some } M,\ m = 1, 2, 3, \cdots. \tag{16}$$

Choose an integer N so that $N > M/\alpha^2$ for some α, $0 < \alpha < 1$. By the method of Lemma 2, we may subdivide the sequence $\{z_n\}$ into at most N subsequences, in each of which $(1 - |z_n|^2)(1 - |z_m|^2)|1 - \bar{z}_m z_n|^{-2} < \alpha < 1$. This proves (i).

The proof of the necessity of (ii) is similar, except that by Bari's theorem we have

$$\sum_{n=1}^{\infty} \frac{(1 - |z_n|^2)^3(1 - |z_m|^2)^3}{|1 - \bar{z}_n z_m|^6} \leq M.$$

COROLLARY 2. *The sequence* $\{k_n^1\}$ *given by* (14) *is a Bessel sequence in* A_2 *if and only if for some* $M > 0$,

$$\sum_{j=1}^{\infty} (1 - |z_i|^2)^2(1 - |z_j|^2)^2/|1 - \bar{z}_i z_j|^4 \leq M, \qquad i = 1, 2, \cdots.$$

We conclude this section with a surprising arithmetic fact about sequences of the type $S_i = \{(1 - |z_i|^2)^{1/2}(1 - |z_j|^2)^{1/2}/|1 - \bar{z}_i z_j|\}_{j=1}^{\infty}$.

THEOREM 4. *If there exists* $M_1 > 0$ *so that for some* p, $4 < p < \infty$, $\|S_i\| \leq M_1$ *in* ℓ^p *for all* i, *then there exists* M_2 *so that* $\|S_i\| \leq M_2$ *in* ℓ^4 *for all* i.

Proof. The proof is again similar to the proof of Lemma 2. By the hypotheses of the theorem.

$$\sum_{j=1}^{\infty} ((1 - |z_i|^2)^{1/2}(1 - |z_j|^2)^{1/2}|1 - \bar{z}_i z_n|^{-1})^p \leq M, \qquad i = 1, 2, \cdots .$$

Choose an integer N so that $N > M/\alpha^p$. Then we may subdivide the sequence $\{z_n\}$ into subsequences $\{z_{n,k}\}_{k=1}^{\infty}$ so that each subsequence is weakly separated. By Theorem C, this implies that each subsequence, and consequently the union $\{z_n\}$, gives rise to a Bessel sequence of orthonormalized kernel functions in A_2. Thus, by Corollary 2, we have $\|S_i\| \leq M_2$ in ℓ^4 for some constant M_2.

5. THE DIRICHLET SPACE

We now turn our attention to the Dirichlet space D, the Hilbert space of analytic functions on Δ which vanish at the origin, with $\|f\|^2 = \iint_{\Delta} |f'(z)|^2\, dx\, dy < \infty$. The normalized kernel function in D associated with the point z_i is

$$k^*_{z_i}(z) = \log \frac{1}{1 - \bar{z}_i z}\left(\log \frac{1}{1 - |z_i|^2}\right)^{-1/2}.$$

THEOREM 5. *If* $\{k^*_{z_i}\}$ *is a Riesz-Fischer sequence in* D, *then* $\{z_i\}$ *is uniformly separated.*

Proof. We are given that for each $\{C_n\} \in \ell^2$, there exists $f \in D$ such that $C_n = f(z_n)(\log(1/(1 - |z_n|^2)))^{-1/2}$ and $\sum |C_n|^2 \geq \delta \|f\|_D^2$. By the hypothesis, there must exist functions f_i in D satisfying

(i) $$|f_i(z_i)|^2 = \log \frac{1}{1 - |z_i|^2};$$

(ii) $$f_i(z_j) = 0, \qquad j \neq i;$$

(iii) $$\|f_i\|_D \leq M.$$

We shall also use the fact [3, p. 291] that if $f = B\phi$, where B is a Blaschke product and $f \in D$, then $\phi \in D$ and $\|\phi\|_D \leq \|f\|_D$. By (i) and (ii), $f_i(z) = B_i(z)\phi_i(z)$, where $|B_i(z)| = \prod_{j \neq i} |(z - z_j)/(1 - \bar{z}_j z)|$. However,

$$|\phi(z_i)| = |(\phi, -\log(1 - \bar{z}_i z))| \leq \|\phi\| \left(\log \frac{1}{1 - |z_i|^2}\right)^{1/2}.$$

Therefore,

$$\left(\log \frac{1}{1-|z_i|^2}\right)^{1/2} = |f_i(z_i)| = |B_i(z_i)|\,|\phi_i(z_i)|$$
$$\leq |B_i(z_i)|\,\|\phi_i\|_D\left(\log \frac{1}{1-|z_i|^2}\right)^{1/2}.$$

Therefore, $|B_i(z_i)| \geq (1/(\|\phi_i\|_D)) \geq 1/M$ for all i. Thus the sequence $\{z_i\}$ is uniformly separated. For our next theorem, we shall need the following lemma.

LEMMA 3. *If* x, y *are such that* $0 < |x| < 1$ *and* $0 < |y| < 1$, *and* $1 - |x|\,|y| < 1/e$, *then*

$$(1-y^2)^{1/2}(1-x^2)^{1/2}\log\frac{1}{1-x^2}\log\frac{1}{1-y^2} \leq (1-xy)^2\log^2\frac{1}{1-xy}.$$

Proof. For $0 \leq u < 1/e$, $u(\log u)^2$ is increasing. Since

$$(1-xy)^2 \geq (1-x^2)(1-y^2)$$

and $1 - xy \geq (1-x^2)^{1/2}(1-y^2)^{1/2}$, we have for $1 - xy < 1/e$,

$$(1-xy)\log^2\left(\frac{1}{1-xy}\right) \geq (1-x^2)^{1/2}(1-y^2)^{1/2}\log^2\left(\frac{1}{(1-x^2)^{1/2}}\,\frac{1}{(1-y^2)^{1/2}}\right)$$
$$\geq (1-x^2)^{1/2}(1-y^2)^{1/2}\log\frac{1}{(1-x^2)}\log\frac{1}{(1-y^2)}.$$

This proves the lemma.

THEOREM 6. *If* $\{k^*_{z_i}\}$ *is a Bessel sequence in* D, *then* $\{k_{z_i}\}$ *is a Bessel sequence in* H^2.

Proof. We know by Bari's theorem that if $\{k^*_{z_i}\}$ is a Bessel sequence in D, then the infinite matrix $C = \{(k^*_{z_i}, k^*_{z_j})_D\}$ is a bounded operator on ℓ^2 with bound M. This implies that the rows and columns have uniformly bounded ℓ^2 norms. We shall show that this implies that the matrix $F = \{(k_{z_i}, k_{z_j})_{H^2}\}$ has rows and columns uniformly bounded in ℓ^1. By Schur's theorem, this will imply that the matrix F is bounded.

Assume without loss of generality that $|1 - \bar{z}_i z_j| < \delta < 1/e$ for all i and j. (If not, partition the sequence of points into a finite number of subsequences which obey this condition. Then if each of the subsequences is a Bessel sequence, so is the union.) If $|1 - \bar{z}_i z_j| < 1/e$, then

$$(1-|z_i|^2)^{1/2}(1-|z_j|^2)^{1/2}\log\frac{1}{1-|z_i|^2}\log\frac{1}{1-|z_j|^2}$$
$$\leq (1-|z_i|\,|z_j|)\log^2\left(\frac{1}{1-|z_i|\,|z_j|}\right) \leq |1-\bar{z}_i z_j|\log^2\left|\left(\frac{1}{1-\bar{z}_i z_j}\right)\right|.$$

the second inequality coming from the facts that $u(\log u)^2$ is increasing on $0 < u < 1/e$, and that $1 - |z_i|\,|z_j| \leq |1 - \bar{z}_i z_j|$. Therefore,

$$\frac{(1-|z_i|^2)^{1/2}(1-|z_j|^2)^{1/2}}{|1-\bar{z}_i z_j|} \leq \frac{\left|\log \dfrac{1}{1-\bar{z}_i z_j}\right|^2}{\log \dfrac{1}{1-|z_i|^2} \log \dfrac{1}{1-|z_j|^2}} .$$

By Bari's theorem, we have already remarked that

$$\sum_{i=1}^{\infty} \frac{\left|\log \dfrac{1}{1-\bar{z}_i z_j}\right|^2}{\log \dfrac{1}{1-|z_i|^2} \log \dfrac{1}{1-|z_j|^2}} \leq M \quad \text{for all } j .$$

Therefore,

$$\sum_{i=1}^{\infty} \frac{(1-|z_i|^2)^{1/2}(1-|z_j|^2)^{1/2}}{|1-\bar{z}_i z_j|} \leq M \quad \text{for all } j .$$

Therefore, $\{k_{z_i}\}$ is a Bessel sequence in H^2.

REFERENCES

1. L. Carleson, *An interpolation problem for bounded analytic functions.* Amer. J. Math. 80 (1958), 921-930.
2. ———, *Interpolations by bounded analytic functions and the corona problem.* Ann. of Math. (2) 76 (1962), 547-559.
3. ———, *On the zeros of functions with bounded Dirichlet integrals.* Math. Z. 56 (1952), 289-295.
4. P. L. Duren, *Theory of* H^p *Spaces.* Pure and Applied Mathematics, Vol. 38. Academic Press, New York-London, 1970.
5. K. Hoffman, *Banach Spaces of Analytic Functions.* Prentice-Hall Series in Modern Analysis. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.
6. D. J. Newman, *Interpolation in* H^∞. Trans. Amer. Math. Soc. 92 (1959), 501-507.
7. H. S. Shapiro and A. L. Shields, *On some interpolation problems for analytic functions.* Amer. J. Math. 83 (1961), 513-532.

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