

A NOTE ON THE \mathbb{X} -OPERATOR OF AN $[\text{FIA}]^-$ GROUP

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An $[\text{FIA}]^-$ group is a locally compact group G for which the set of inner automorphisms $I(G)$ is relatively compact in the (topologized) group $\text{Aut}(G)$ of all topological automorphisms of G . For such a group, Mosak [4] has constructed a mapping $a \rightarrow a^{\mathbb{X}}$ from the C^* -algebra $C^*(G)$ onto its center Z . This mapping is a center-valued trace, and in this note we show that it is related to an averaging process in $C^*(G)$ similar to Dixmier's process for obtaining the trace in a finite von Neumann algebra [1; Chapitre III, Sections 5, 8].

We begin with some definitions and notation. G will be understood to be an $[\text{FIA}]^-$ group, A will denote its C^* -algebra $C^*(G)$, and \tilde{A} will denote the C^* -algebra obtained by adjoining an identity 1 to A (except in the special situation where A has an identity, in which case \tilde{A} should be taken as equal to A). We will denote closure in the norm topology of A by norm-cl , and convex hull by co . If $\beta \in \text{Aut}(G)$ and f is a complex valued function on G , then $f^\beta(x) = f(\beta^{-1}x)$ ($x \in G$). If f is continuous, then

$$f^{\mathbb{X}}(x) = \int_{I(G)^-} f^\beta(x) d\beta$$

where $d\beta$ is the normalized Haar measure on the compact group $I(G)^-$. For further details, including the definition of a^β and $a^{\mathbb{X}}$ when $a \in A$, we refer the reader to [4].

THEOREM. *For each $a \in A$, $a^{\mathbb{X}} \in \text{norm-cl co} \{u^*au: u \text{ unitary in } \tilde{A}\}$.*

Proof. Since the \mathbb{X} -operator is linear and norm-decreasing [4; 1.5], a simple approximation argument shows that it suffices to consider the case where a is a continuous complex-valued function on G with compact support. By [4; 1.4],

$$(1) \quad a^{\mathbb{X}} \in \text{norm-cl co} \{a^\beta: \beta \in I(G)^-\}.$$

Furthermore, since $I(G)^-$ is a subgroup of $\text{Aut}(G)$, it follows from [4; 1.0] that if $\alpha \in I(G)^-$ then

$$(2) \quad a^\alpha \in \text{norm-cl} \{a^\beta: \beta \in I(G)\}.$$

In view of (1) and (2), it suffices to fix $\beta \in I(G)$ and show that

$$(3) \quad a^\beta \in \text{norm-cl co} \{u^*au: u \text{ unitary in } \tilde{A}\}.$$

Let π be the universal representation of A (extended to \tilde{A} by defining $\pi(1) = 1_\pi$), and let π also denote the associated unitary representation of G . Let $x \in G$ be such that $\beta(g) = x^{-1}gx$ ($g \in G$). From the proof of [4; 1.2] and the discussion of $\Delta(\beta)$ which follows it, we obtain $\pi(a^\beta) = \pi(x)^* \pi(a) \pi(x)$. Since (by the double commutant theorem) $\pi(x)$ lies in the von Neumann algebra generated by $\pi(A)$,

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it follows from [3; Theorem 2] that there is a net (u_λ) of unitaries in \tilde{A} such that $\pi(u_\lambda) \rightarrow \pi(x)$ in the strong operator topology τ_s . Since each $\pi(u_\lambda)$, and also $\pi(x)$, is unitary, it is also true that $\pi(u_\lambda)^* \rightarrow \pi(x)^*$ (in τ_s). Hence

$$\pi(u_\lambda)^* \pi(a) \pi(u_\lambda) \rightarrow \pi(x)^* \pi(a) \pi(x) \quad (\text{in } \tau_s),$$

since multiplication is jointly τ_s -continuous on norm-bounded sets of operators. Thus

$$\begin{aligned} \pi(a^\beta) &\in \pi(A) \cap [\tau_s\text{-cl co } \{\pi(u)^* \pi(a) \pi(u) : u \text{ unitary in } \tilde{A}\}] \\ &= \text{norm-cl co } \{\pi(u)^* \pi(a) \pi(u) : u \text{ unitary in } \tilde{A}\}, \end{aligned}$$

the equality holding by [2; V.2.14] since a linear functional on $\pi(A)$ is τ_s -continuous if and only if it is norm-continuous. Since π is an isometric $*$ -isomorphism, (3) now follows.

Remark. By defining $1^{\otimes} = 1$, the \otimes -operator can be extended to a continuous linear mapping from \tilde{A} to $Z + \mathbb{C}1$ such that $(ab)^{\otimes} = (ba)^{\otimes}$ ($a, b \in \tilde{A}$). Slight modifications of the arguments used in [1; p. 254, Corollaire] then lead to the following facts:

- (i) For each $a \in A$, $Z \cap [\text{norm-cl co } \{u^* a u : u \text{ unitary in } \tilde{A}\}] = \{a^{\otimes}\}$.
- (ii) The \otimes -operator is the only centre-valued trace on A .

REFERENCES

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