EXTRINSIC SPHERES IN COMPACT SYMMETRIC SPACES ARE INTRINSIC SPHERES

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1. INTRODUCTION

An n-dimensional submanifold, $n \geq 2$, of an arbitrary Riemannian manifold is called an *extrinsic sphere* if it is umbilical and has nonzero parallel mean curvature vector. An n-dimensional Riemannian manifold is called an *intrinsic sphere* if it is locally isometric to a standard sphere in Euclidean space. Since extrinsic spheres are natural analogues of ordinary spheres in Euclidean spaces from the extrinsic point of view, it is natural to ask when an extrinsic sphere is an intrinsic sphere. In [2], we have proved that a complete, simply connected, extrinsic 2n-sphere in any Kähler manifold is an intrinsic sphere if its normal connection is flat. The main purpose of this paper is to prove the following.

THEOREM 1. *An extrinsic sphere in a compact symmetric space is an intrinsic sphere.*

THEOREM 2. *Let* M *be an extrinsic sphere in a compact symmetric space* $\tilde{M}$. *Then* $\dim M \leq d_{\tilde{M}}$, *where* $d_{\tilde{M}}$ *is the maximal dimension of all totally geodesic submanifolds of constant sectional curvature in* $\tilde{M}$.

Theorem 2 improves the results in [3] for the compact case.

2. PRELIMINARIES

Let M be an n-dimensional submanifold of a symmetric space $\tilde{M}$ with metric g, and let ∇ and $\tilde{\nabla}$ be the covariant differentiations on M and $\tilde{M}$, respectively. Then the second fundamental form σ is defined by $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X and Y are vector fields tangent to M and σ is a normal-bundle-valued symmetric 2-form on M. For a vector field ξ normal to M, we write $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$, where $-A_\xi X$ (respectively, $D_X \xi$) denotes the tangential component (respectively, the normal component) of $\tilde{\nabla}_X \xi$. A normal vector field ξ is said to be *parallel* if $D\xi = 0$. The submanifold is said to be *umbilical* if $\sigma(X, Y) = g(X, Y) H$, where $H = (\text{trace}\ \sigma)/n$ is the *mean curvature vector* of M in $\tilde{M}$. If $\sigma = 0$, M is said to be *totally geodesic* in $\tilde{M}$.

Let R^N, R, and $\tilde{R}$ be the curvature tensors associated with D, ∇, and $\tilde{\nabla}$, respectively. For example, $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$. The submanifold M is locally symmetric if $\nabla R = 0$, and the normal connection of M in $\tilde{M}$ is flat if $R^N = 0$.

For the second fundamental form σ, we define the covariant derivative, denoted by $\overline{\nabla}_X \sigma$, to be

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$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.1}$$

Then, for all vector fields X, Y, Z, W tangent to M, the equations of Gauss and Codazzi take the forms

$$R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)); \tag{2.2}$$

$$(\tilde{R}(X, Y)Z)^{\perp} = (\bar{\nabla}_X \sigma)(Y, Z) - (\bar{\nabla}_Y \sigma)(X, Z), \tag{2.3}$$

where $R(X, Y; Z, W) = g(R(X, Y)Z, W)$ and $^{\perp}$ in (2.3) denotes the normal component.

Let X and Y be two orthonormal vectors which span a plane section $\pi = \pi(X, Y)$ in M. The sectional curvature $K(\pi)$ of π is given by $K(\pi) = R(X, Y; Y, X)$. We denote by $\tilde{K}$ the sectional curvature for $\tilde{M}$.

3. EXTRINSIC SPHERES IN SYMMETRIC SPACES

Let M be an extrinsic n-sphere, $n \geq 2$, in a symmetric space $\tilde{M}$. Then $\sigma(X, Y) = g(X, Y)H$ and $DH = 0$. Thus (2.1) implies $\bar{\nabla}_X \sigma = 0$. From (2.2) and (2.3) we find

$$R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \alpha^2 \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}; \tag{3.1}$$

$$\tilde{R}(X, Y; Z, \xi) = 0 \tag{3.2}$$

for all vector fields X, Y, Z, W tangent to M and every vector field ξ normal to M, where $\alpha^2 = g(H, H)$. From (3.1) and (3.2), we get

$$R(X, Y)Z = \tilde{R}(X, Y)Z + \alpha^2 \{g(Z, Y)X - g(X, Z)Y\}. \tag{3.3}$$

Since $DH = 0$ and $H \neq 0$, α^2 is a positive constant. By the assumption, we have $\tilde{\nabla}\tilde{R} = 0$. Thus we find (see, for instance, p. 6 of [1])

$$\begin{aligned} \tilde{\nabla}_U(\tilde{R}(X, Y)Z) &= \tilde{R}(\tilde{\nabla}_U X, Y)Z + \tilde{R}(X, \tilde{\nabla}_U Y)Z + \tilde{R}(X, Y)\tilde{\nabla}_U Z \\ &= \tilde{R}(\nabla_U X, Y)Z + \tilde{R}(X, \nabla_U Y)Z + \tilde{R}(X, Y)\nabla_U Z \\ &\quad + \tilde{R}(h(X, U), Y)Z + \tilde{R}(X, h(U, Y))Z + \tilde{R}(X, Y)(h(U, Z)). \end{aligned} \tag{3.4}$$

Consequently, from (3.2) and (3.4), we find

$$\begin{aligned} U(\tilde{R}(X, Y; Z, W)) &= \tilde{R}(\nabla_U X, Y; Z, W) + \tilde{R}(X, \nabla_U Y; Z, W) \\ &\quad + \tilde{R}(X, Y; \nabla_U Z, W) + \tilde{R}(X, Y; Z, \nabla_U W) \end{aligned} \tag{3.5}$$

for all vector fields X, Y, Z, W, U tangent to M. By using (3.3), (3.5), and $\nabla g = 0$, we have

$$\begin{aligned} U(\tilde{R}(X, Y; Z, W)) &= U(R(X, Y; Z, W)) - g((\nabla_U R)(X, Y)Z, W) \\ &\quad + \alpha^2 U\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\}. \end{aligned} \tag{3.6}$$

Comparing (3.1) and (3.6), we get $\nabla R = 0$; *i.e.*, M is locally symmetric. Consequently, we have proved the following.

PROPOSITION 1. *Every extrinsic sphere in any symmetric space is locally symmetric.*

Let π be any plane section in M. Equation (3.3) implies that the sectional curvatures on M and $\tilde{M}$ satisfy

$$K(\pi) = \tilde{K}(\pi) + \alpha^2 . \tag{3.4}$$

Thus, if $\tilde{M}$ is nonnegatively curved, then M is positively curved. Hence, M is irreducible and the rank of M is one. In summary, we have the following.

LEMMA 2. *Every extrinsic sphere in a nonnegatively curved symmetric space is a positively curved locally symmetric space of rank one.*

Since $\tilde{M}$ is a symmetric space, there is a triple (G, H, τ) consisting of a connected Lie group G, a closed subgroup H of G, and an involutive automorphism τ of G such that $\tilde{M} = G/H$ and H lies between G_τ and the identity component of G_τ, where G_τ is the closed subgroup of G consisting of all elements left fixed by τ. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively, and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ associated with σ. Then $\mathfrak{m}$ may be identified with the tangent space of $\tilde{M}$ at a point $0 \in \tilde{M}$. It is well known that the curvature tensor $\tilde{R}$ of $\tilde{M}$ at 0 satisfies

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = -[[\tilde{X}, \tilde{Y}], \tilde{Z}] \quad \text{for } \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{m} . \tag{3.5}$$

Without loss of generality, we may assume that $0 \in M$. Let $\mathfrak{m}' = T_0(M)$. Then, by (3.2), we have

$$[\mathfrak{m}', [\mathfrak{m}', \mathfrak{m}']] \subset \mathfrak{m}' ; \tag{3.6}$$

i.e., $\mathfrak{m}'$ forms a Lie triple system. Thus, by a result of É. Cartan, we have the following.

LEMMA 3. *Let* M *be an extrinsic sphere in a symmetric space* $\tilde{M}$. *Then, for any point* $0 \in M$, *there exists a totally geodesic submanifold* M' *in* $\tilde{M}$ *such that* $T_0(M) = T_0(M')$. *Moreover,* M *is of constant sectional curvature if and only if* M' *is of constant sectional curvature.*

As a corollary to Lemma 3, we have the following.

LEMMA 4. *If an extrinsic sphere* M *in a symmetric space* $\tilde{M}$ *is an intrinsic sphere, then* $\dim M \leq d_{\tilde{M}}$, *where* $d_{\tilde{M}}$ *is the maximal dimension of all totally geodesic submanifolds of constant sectional curvature in* $\tilde{M}$.

Theorem 2 follows from Theorem 1 and Lemma 4.

If M and M' are both locally symmetric spaces of rank one, then from Lemma 3, either they are both of constant sectional curvature, or they are both 1/4-pinched. If M' is 1/4-pinched, then we have either $c/4 \leq \tilde{K} \leq c$ or $-c \leq K \leq -c/4$, $c > 0$, for all the plane sections in $T_0(M)$. Therefore, we have either

$$\alpha^2 + c/4 \leq K \leq \alpha^2 + c \quad \text{or} \quad \alpha^2 - c \leq K \leq \alpha^2 - c/4$$

for all plane sections in $T_0(M)$. Since M is also 1/4-pinched, we get $\alpha^2 = 0$. This contradicts $\alpha^2 > 0$. Consequently, we have

LEMMA 5. *Let* M *be an extrinsic sphere in a symmetric space* $\tilde{M}$. *If* M *and* M' *are both of rank one, then* M *is of constant sectional curvature.*

In particular, if $\tilde{M}$ is of rank one, then M' is of rank one. Thus, Lemma 5 implies the following.

PROPOSITION 6. *Let* M *be an extrinsic sphere of rank one in a rank-one symmetric space* $\tilde{M}$. *Then* M *is of constant sectional curvature. In particular, if* $\tilde{M}$ *is compact, then* M *is an intrinsic sphere.*

4. EXTRINSIC SPHERE OF RANK ONE

Assume that M is an extrinsic n-sphere in a nonnegatively curved symmetric space $\tilde{M}$, and M is not an intrinsic sphere. Then, from Lemma 2, M is of rank one and it is 1/4-pinched.

From Lemma 3, there exists a complete totally geodesic submanifold M' such that $T_0(M) = T_0(M')$ at some point 0 in M. Since M is 1/4-pinched and of rank one, Lemma 5 implies that the rank of M' is greater than one. Thus, there exists a constant $c \geq 0$ such that

$$0 \leq \tilde{K}(\pi) \leq c \tag{4.1}$$

for all plane sections π in $T_0(M')$. Moreover, $\tilde{K}$ takes both values 0 and c for some plane sections in $T_0(M')$. From (4.1) we have

$$\alpha^2 \leq K(\pi) \leq \alpha^2 + c \tag{4.2}$$

for all plane sections in $T_0(M)$. Moreover, K takes both values α^2 and $\alpha^2 + c$ for some plane sections.

Let V be a linear subspace of $T_0(M)$. Then $\tilde{K}(\pi) = 0$ for all plane sections π in V if and only if $K(\pi) = \alpha^2$ for all plane sections π in V. Since M is not an intrinsic sphere, $c > 0$. Moreover, by the definition of rank for symmetric spaces, we have the following.

LEMMA 7. *Let* M *be an extrinsic sphere in a nonnegatively curved symmetric space* $\tilde{M}$. *Then we have*

$$\text{rank}\,(M') = \max\{\dim V\colon V \text{ a linear subspace of } T_0(M) \text{ with } K(\pi) = \alpha^2 \text{ for all plane sections } \pi \text{ in } V\}. \tag{4.3}$$

Since M is of rank one and it is not an intrinsic sphere, M is an open submanifold of either a complex projective space $P^k(C)$ with $k = n/2 > 1$, or a quaternion projective space $P^k(Q)$ with $k = n/4 > 1$, or a Cayley projective plane P^2 (Cayley). Since $P^k(C)$ (respectively, $P^2(C)$) can be isometrically imbedded in $P^k(Q)$ (respectively, P^2 (Cayley)) as a totally geodesic submanifold, and every totally geodesic submanifold of M is an extrinsic sphere in $\tilde{M}$, we may always assume that there exists an extrinsic M in $\tilde{M}$ such that M is an open submanifold of $P^k(C)$ for some k. Because α^2 is the least sectional curvature on M and

$$\max\{\dim V\colon V \subset T_0(M) = P^k(C) \text{ with } K(\pi) = \alpha^2 \text{ for all plane sections } \pi \subset V\}$$

is equal to k, we have by Lemma 7

$$\text{rank } M' = \frac{1}{2} \dim M'. \tag{4.4}$$

On the other hand, let J be the complex structure of $M = P^k(C)$. For any unit vector X in $T_0(M)$, the sectional curvature $K(\pi(X, JX))$ of the holomorphic section $\pi(X, JX)$ is equal to $c + \alpha^2$. Without loss of generality, we may assume that M' is simply connected. Let $M' = M_0 \times M_1 \times \cdots \times M_r$ be the de Rham decomposition of the symmetric space M' into a flat space M_0 and irreducible compact symmetric spaces $M_1, \cdots, M_r$. Then for any unit vector X in $T_0(M') = T_0(M_0 \times \cdots \times M_r)$ tangent to M_0, the sectional curvature $\tilde{K}$ on M' for any plane section containing X vanishes. Thus, the sectional curvature K on M for any plane section containing X is equal to α^2. This is a contradiction because the holomorphic sectional curvatures on M are equal to $c + \alpha^2$. Therefore, M_0 is just a point and

$$M' = M_1 \times \cdots \times M_r,$$

where each M_i, $i = 1, \cdots, r$, is an irreducible compact symmetric space. From (4.4) and Cartan's classification theorem of irreducible symmetric spaces (see, for instance, [5]), we have

$$\dim M_i = 2, \quad \text{rank } M_i = 1, \quad i = 1, 2, \cdots, r. \tag{4.5}$$

Since M' is not of constant sectional curvature, (4.5) implies that $k = r \geq 2$ and $n \geq 4$. This leads to a contradiction because we have the following identities:

$$\dim\{X \in T_0(M)\colon K(\pi(X, Y)) = c + \alpha^2 \text{ for some } Y \in T_0(M)\} = n;$$

$$\dim\{X \in T_0(M')\colon K(\pi(X, Y)) = c \text{ for some } Y \in T_0(M')\} = 2; \text{ and}$$

$$T_0(M) = T_0(M') = T_0(M_1 \times \cdots \times M_k).$$

Consequently, we obtain the following.

PROPOSITION 8. *If* M *is an extrinsic sphere in a nonnegatively curved symmetric space* $\tilde{M}$, *then* M *is an intrinsic sphere.*

If particular, if $\tilde{M}$ is a compact symmetric space, then $\tilde{M}$ is positively curved. Thus, by Proposition 8, M is an intrinsic sphere. This proves Theorem 1.

5. REMARKS

Remark 1. Let M be an n-dimensional Riemannian manifold isometrically immersed in a Euclidean space E^{n+N}, and let ξ be any unit normal section on M. Then by a suitable change of the metric on E^{n+N}, without changing the metric on M, M may become an extrinsic sphere in the ambient space with ξ as its mean curvature vector [8]. Therefore, locally, a Riemannian manifold is an extrinsic sphere in some other Riemannian manifold.

Remark 2. Since every Riemannian manifold of constant sectional curvature admits extrinsic hyperspheres, every symmetric space $\tilde{M}$ admits extrinsic spheres of any dimension less than $d_{\tilde{M}}$.

Remark 3. Theorem 2 and Proposition 8 can be used to obtain the maximal dimension of all extrinsic spheres in the compact symmetric space $\tilde{M}$ if we know the maximal dimension of the totally geodesic submanifolds in $\tilde{M}$ which are intrinsic spheres. For example, from [4], we see that if $\tilde{M}$ is the complex quadric $SO(2 + m)/SO(2) \times SO(m)$, then $d_{\tilde{M}} = m$. Thus, every extrinsic sphere in

$$SO(2 + m)/SO(2) \times SO(m)$$

has dimension less than or equal to m.

Remark 4. If M is a complete orientable extrinsic sphere in any nonnegatively curved Hermitian symmetric space $\tilde{M}$, then Proposition 8 implies that M is isometric to an ordinary n-sphere S^n. Thus, M is simply connected. If we further assume that the normal connection is flat, then, by Theorem 1 of [2], we see that the curvature tensor $\tilde{R}$ of $\tilde{M}$ satisfies $\tilde{R}(X, Y) = 0$ for all vectors X, Y tangent to M when n is even. If n is odd, then, for any (n - 1)-dimensional linear subspace V of $T_p(M) = T_p(S^n)$, there exists an ordinary (n - 1)-sphere S^{n-1} in $S^n = M$ such that $T_p(S^{n-1}) = V$ and S^{n-1} is a totally geodesic submanifold of S^n. Since $M = S^n$ is an extrinsic sphere in $\tilde{M}$ and S^{n-1} is totally geodesic in S^n, S^{n-1} is an extrinsic sphere with flat normal connection in $\tilde{M}$. By applying Theorem 1 of [2] again, we get $\tilde{R}(X, Y) = 0$ for all vectors X, Y tangent to S^{n-1}. Because V can be any hyperplane in $T_p(M)$ for any $p \in M$, we see that $\tilde{R}(X, Y) = 0$ for all X, Y tangent to M. Let $\mathfrak{m} = T_0(M)$. Then, by the fact that $\tilde{R}(X, Y) = 0$ for all $X, Y \in T_0(M)$, $\mathfrak{m}$ is abelian with respect to the Lie bracket [,] on $\mathfrak{g}$. Therefore $\dim M \leq \operatorname{rank} \tilde{M}$. In particular, we have the following.

THEOREM 3. *If* M *is a complete, orientable extrinsic* n-*sphere*, $n \geq 2$, *with flat normal connection in any compact Hermitian symmetric space* $\tilde{M}$, *then* M *is isometric to an ordinary* n-*sphere and* $n \leq \operatorname{rank} \tilde{M}$.

This proposition generalizes the main theorem of [6] for the compact case.

Remark 5. In [7], extrinsic spheres were studied from the point of view of circles.

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