GROWTH CONDITIONS AND UNIQUENESS FOR WALSH SERIES

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Let Ψ_0, Ψ_1, $\cdots$ denote the Walsh functions defined on the group 2^{ω} (see [2]). Let F be a finite-valued function belonging to $L^1(2^{\omega})$, and let E^* be a countable subset of the group 2^{ω}. Skvortsov [5] has shown that if $S = \sum a_k \Psi_k$ is a Walsh series satisfying

$$\liminf_{n \to \infty} S_{2^n}(x) \leq F(x) \leq \limsup_{n \to \infty} S_{2^n}(x), \quad x \notin E^*, \tag{1}$$

and if a_k converges to zero as $k \to \infty$, then S is the Walsh-Fourier series of F. Crittenden and Shapiro [1] have studied Walsh series which satisfy the weaker growth condition $\lim_{n \to \infty} 2^{-n} S_{2^n}(x) = 0$, $x \in 2^{\omega}$. Under this condition, they showed that if the right-hand side of (1) holds, and if

$$\limsup_{n \to \infty} |S_{2^n}(x)| < \infty, \quad x \notin E^*, \tag{2}$$

then S is a Walsh-Fourier series.

We shall obtain the following result concerning uniqueness under an even weaker growth condition.

THEOREM. *Let* $S = \sum a_k \Psi_k$ *be a Walsh series and suppose that* (1) *holds for a finite-valued integrable function* F *and a countable subset* E^* *of the group* 2^{ω}. *Suppose further that*

$$\liminf_{n \to \infty} 2^{-n} S_{2^n}(x) \leq 0 \leq \limsup_{n \to \infty} 2^{-n} S_{2^n}(x), \quad x \in E^* \cup D^*, \tag{3}$$

where D^* *is the set of points in the group* 2^{ω} *which terminate in* 0*'s or terminate in* 1*'s. Then* S *is the Walsh-Fourier series of* F.

Our result generalizes Skvortsov's theorem, but it does not generalize Crittenden and Shapiro's theorem. Instead, it restores the left-hand side of (1) in order to get rid of hypothesis (2). In connection with this, it is interesting to note that Šaginjan [4] has shown that condition (1) holds off a set of measure zero whenever (2) is satisfied. Hence, our theorem shows that if we insist that this null-set be countable, we can discard (2) altogether. For other related results, see [6] and [7].

To prove the theorem, for each $x \in [0, 1)$ set $f(x) = F(\mu(x))$, $\psi(x) = \Psi(\mu(x))$, and $s(x) = S(\mu(x))$, where μ is Fine's map from $[0, 1)$ to the group 2^{ω} (see [2]). Extend f and s to $(-\infty, \infty)$ as periodic functions of period 1. Let $E = \mu^{-1} E^*$, and let D be the set of dyadic rationals.

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As in [1] and [3], set $H(x) = \int_0^x f(t)\,dt$ and $L(x) = \sum_{k=0}^{\infty} a_k \int_0^x \psi_k(t)\,dt$, and observe that L exists for every dyadic rational x. Use the Vitali-Carathéodory theorem to choose absolutely continuous functions g_k and h_k, $k = 1, 2, \cdots$, such that

$$H(x) = \lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} h_k(x), \quad x \in [0, 1),$$

and such that the derivatives of g_k (respectively, h_k) are less than (respectively, greater than) f at each point in [0, 1).

Fix an integer k and set $G(x) = g_k(x) - L(x)$, $x \in D$. What properties does G inherit from hypotheses (1) and (3)? Recall [2] that $s_{2^n}(x) = 2^n[L(\beta_n(x)) - L(\alpha_n(x))]$, where $\alpha_n(x) = p/2^n$, $\beta_n(x) = (p+1)/2^n$, and p is an integer satisfying $p \le 2^n x < p + 1$. In particular, since all the derivatives of g_k are less than f,

$$\liminf_{n\to\infty} 2^n[G(\beta_n(x)) - G(\alpha_n(x))] \le f(x) - \limsup_{n\to\infty} s_{2^n}(x).$$

Hence, by hypothesis (1), we conclude that

$$\liminf_{n\to\infty} 2^n[G(\beta_n(x)) - G(\alpha_n(x))] \le 0, \quad x \notin E. \tag{4}$$

Similarly, since g_k is continuous,

$$\liminf_{n\to\infty} [G(\beta_n(x)) - G(\alpha_n(x))] = -\limsup_{n\to\infty} 2^{-n} s_{2^n}(x).$$

Hence, by hypothesis (3),

$$\liminf_{n\to\infty} [G(\beta_n(x)) - G(\alpha_n(x))] \le 0, \quad x \in E. \tag{5}$$

Finally, if we set $\alpha_n'(x) = \alpha_n(x) - 2^{-n}$ for $x \in D$, then

$$G(\alpha_n'(x)) - G(x) = 2^{-n} S_{2^n}(\mu'(x))$$

[1, p. 553]. Since $\mu'(x) \in D^*$ when $x \in D$, we conclude from (3) that

$$\limsup_{n\to\infty} G(\alpha_n'(x)) \ge G(x), \quad x \in D. \tag{6}$$

Lindahl [3] has shown that conditions (4), (5), and (6) are sufficient to conclude that G is increasing on D. And by considering -G, we can show that G is decreasing on D. Hence, G is constant on D, and upon letting k tend to ∞, we conclude that

$$L(x) = \int_0^x f(t)\,dt, \quad x \in D. \tag{7}$$

It suffices, therefore, to prove that s is the Walsh-Fourier series of f when (7) holds. Crittenden and Shapiro [1] established this in the special case that $f \equiv 0$.

Their proof was by induction, and if we use the fact that certain linear combinations of Walsh functions are Haar functions, we can obtain the general result in much the same way.

However, N. R. Ladhawala has suggested a much simpler proof which uses the fact that each Walsh function is a step function with jumps at dyadic rationals. Indeed, fix k and choose dyadic rationals α_i, β_i so that $\psi_k(x) = \psi_k(\alpha_i)$, $x \in [\alpha_i, \beta_i)$, and so that $[0, 1)$ is partitioned by the intervals $[\alpha_i, \beta_i)$. Then by (7),

$$\int_0^1 f(x)\,\psi_k(x)\,dx = \sum_i \int_{\alpha_i}^{\beta_i} f(x)\,\psi_k(x)\,dx = \sum_i \psi_k(\alpha_i)\,[L(\beta_i) - L(\alpha_i)]$$

$$\equiv \sum_i \psi_k(\alpha_i) \left\{ \lim_{n\to\infty} \int_{\alpha_i}^{\beta_i} \sum_{\ell=1}^{n} a_\ell \psi_\ell(t)\,dt \right\}.$$

But $\int_{\alpha_i}^{\beta_i} \psi_\ell(t)\,dt \equiv 0$ for ℓ large, so there exists an $n_0 > k$ such that

$$\int_0^1 f(x)\,\psi_k(x)\,dx = \sum_i \psi_k(\alpha_i) \int_{\alpha_i}^{\beta_i} \sum_{\ell=1}^{n_0} a_\ell \psi_\ell(t)\,dt .$$

Hence, $\int_0^1 f(x)\,\psi_k(x)\,dx = \int_0^1 \psi_k(t) \sum_{\ell=1}^{n_0} a_\ell \psi_\ell(t)\,dt \equiv a_k$, by orthogonality of the Walsh functions.

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