GLEASON PARTS AND BOUNDARY VALUE INTEGRALS FOR CERTAIN FRÉCHET-NUCLEAR ALGEBRAS

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INTRODUCTION

Let $\mathscr{B}$ be a uniform Banach algebra. Call two points ϕ, ψ of its spectrum *equivalent* if with respect to the norm topology of the dual space $\mathscr{B}'$ we have $\|\phi - \psi\| < 2$. This is indeed an equivalence relation, and the corresponding equivalence classes are called the Gleason parts for $\mathscr{B}$. The importance of Gleason parts is derived from the fact that the elements of $\mathscr{B}$ often behave like analytic functions when restricted to a Gleason part (*cf.* Stout [11]).

Therefore, it is of interest to give conditions which assure the existence of non-trivial Gleason parts. Most of the known results are concerned with the one-dimensional case. In this paper, we present a natural condition to yield Gleason parts "as big as possible". For this purpose we view the given Banach algebra as a local piece of a greater system; *i.e.*, $\mathscr{B}$ is to be understood as a member of a projective limit defining a uniform Fréchet algebra $\mathscr{A}$. Under the condition that $\mathscr{A}$ enjoy a strengthened nuclearity property (and that its spectrum have a "good topology"—see (1.1.1)), we shall be able to show that every connected component of the spectrum of $\mathscr{B}$, except for a nowhere dense set, is contained in a Gleason part.

As an example, consider a domain $D \subset \mathbb{C}^n$ and the algebra $\mathscr{A} = \mathscr{O}(D)$ of all holomorphic functions on D. For every compact subset $K \subset D$, we have a uniform Banach algebra $(\mathscr{O}(D))_K$ which is the closure of $\mathscr{O}(D)$ in the uniform Banach algebra $\mathscr{C}(K)$ of all continuous complex-valued functions on K. If K and L are compact subsets of D such that $K \subset \overset{\circ}{L}$, then it is a well known fact that the restriction map $(\mathscr{O}(D))_L \to (\mathscr{O}(D))_K$ is a nuclear operator (*cf.* Pietsch [8]). It is this property which constitutes the strengthened nuclearity condition (see Section 4). The example given above illustrates our Gleason parts theorem. Each connected component of $\overset{\circ}{K}$ is contained in a Gleason part for $(\mathscr{O}(D))_K$. (Here $\overset{\circ}{K}$ denotes the interior of K.)

In Section 2, we extend the notion of Gleason parts to the class of uniform Fréchet algebras. Let us suppose a uniform Fréchet algebra satisfies the modified nuclearity condition and its spectrum has a "good topology"; if its spectrum is connected, then we obtain the surprising fact that the whole spectrum is a single Gleason part! (See Theorem (7.2).)

As a byproduct of the proof we get the result that the spectra of such algebras are locally connected.

We introduce a concept of $\mathscr{A}$-morphic maps which is modelled on the concept of holomorphic maps with values in locally convex spaces. Using the strengthened nuclearity condition and applying the maximum modulus principle obtained by the Gleason parts theorem, we deduce the existence of $\mathscr{A}$-morphically dependent integral formulas over the (Shilov) boundary. In complex analysis, formulas of this type are known as Cauchy-Weil integral formulas. The proof uses some ideas of L. Bungart [1].

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1. PRELIMINARIES

(1.1) A *Fréchet algebra* is a commutative, locally convex, complete algebra with unit over the complex field $\mathbb{C}$, whose topology is generated by a countable number of seminorms. Now let $\mathscr{A}$ be a Fréchet algebra. By $\sigma\mathscr{A}$ we denote the *spectrum of* $\mathscr{A}$, the set of all continuous $\mathbb{C}$-algebra homomorphisms $\phi: \mathscr{A} \to \mathbb{C}$ with $\phi \neq 0$; as usual, it is endowed with the Gelfand topology (= weak-* topology $\sigma(\mathscr{A}', \mathscr{A})$). Let $\mathscr{C}(\sigma\mathscr{A})$ denote the algebra of all continuous functions on $\sigma\mathscr{A}$ endowed with the compact-open topology. Then the standard *Gelfand representation*

$$\Gamma: \mathscr{A} \to \mathscr{C}(\sigma\mathscr{A}), \quad a \mapsto \hat{a},$$

given by setting $\hat{a}(\phi) := \phi(a)$ for $a \in \mathscr{A}$, $\phi \in \sigma\mathscr{A}$, is a continuous $\mathbb{C}$-algebra homomorphism.

Call $\mathscr{A}$ a *uniform Fréchet algebra* if the Gelfand representation Γ induces a topological isomorphism of $\mathscr{A}$ onto a closed subalgebra $\Gamma(\mathscr{A}) \subset \mathscr{C}(\sigma\mathscr{A})$. From now on we consider only uniform Fréchet algebras. Then we may identify $\mathscr{A}$ and $\Gamma(\mathscr{A})$; we also identify the elements $a \in \mathscr{A}$ of the algebra and their Gelfand transforms $\hat{a} \in \Gamma(\mathscr{A})$. We often require the condition

(1.1.1) *the spectrum is locally compact, and all open subsets are hemicompact.*

The latter property assures that for all open subsets $U \subset \sigma\mathscr{A}$ the derived algebras $\mathscr{A}_U$ (see below (1.2)) are uniform Fréchet algebras, too. For example, if $\sigma\mathscr{A}$ is locally compact and has a countable basis for its topology, then all open subsets are hemicompact.

(1.2) Let X be a topological space. Then we call a countable *exhaustion* $\cdots \subset K_n \subset K_{n+1} \subset \cdots$ of X by compact subsets *admissible* if for every compact subset $K \subset X$ there exists an index $n \in \mathbb{N}$ such that $K \subset K_n$.

Now let X be the spectrum of a uniform Fréchet algebra $\mathscr{A}$. Then every admissible exhaustion $\cdots K_n \subset K_{n+1} \subset \cdots$ of X describes the topology of $\mathscr{A}$ by means of the corresponding seminorms $\|\cdot\|_{K_n}$, $n \in \mathbb{N}$. Here for $a \in \mathscr{A}$ and a compact set $K \subset X$, the seminorm $\|\cdot\|_K$ is defined as usual by $\|a\|_K := \sup_{\phi \in K} |\hat{a}(\phi)|$.

Let $M \subset X$ be an arbitrary subset. By $\mathscr{A}_M$ we denote the separated completion of the restriction algebra $\{\hat{f}|_M : f \in \mathscr{A}\}$ under the topology of uniform convergence on compact subsets of M. Obviously we have $\sigma\mathscr{A}_M = \hat{M}$, where $\hat{M}$ is the $\mathscr{A}$-convex hull of M in $\sigma\mathscr{A}$; more precisely, $\hat{M}$ is the union of all sets $\hat{K} = \{\phi \in X: |f(\phi)| \leq \|f\|_K \text{ for all } f \in \mathscr{A}\}$ with $K \subset M$ compact.

If M is compact, then $\mathscr{A}_M$ is even a uniform Banach algebra with norm $\|\cdot\|_M$.

Let $\mathscr{B}$ be a uniform Banach algebra. By $\gamma\mathscr{B}$ we denote its Shilov boundary, whereas the symbol "∂" indicates the topological boundary.

2. GLEASON PARTS FOR UNIFORM FRÉCHET ALGEBRAS

Let $\mathscr{B}$ be a uniform Banach algebra. Consider the spectrum under the canonical norm topology. Recall that for $\phi, \psi \in \sigma\mathscr{B}$, the following defines an equivalence relation:

$$\phi \sim \psi :\Longleftrightarrow \|\phi - \psi\| < 2;$$

the corresponding equivalence classes are called the *Gleason parts* (or *parts*, for short) for $\mathscr{B}$. For the theory of Gleason parts, see the books of Stout [11] and Gamelin [3].

The notion of parts has been introduced for the study of analytic structure in spectra. The following example illustrates this as well as motivates the subsequent definition and Theorem (5.1).

Example. Let V be an analytic variety in a domain $D \subset \mathbb{C}^n$. Consider the algebra $\mathscr{O}(V)$ of all holomorphic functions on V; $\mathscr{O}(V)$ becomes a uniform Fréchet algebra when endowed with the topology of uniform convergence on compact sets. Its spectrum $\sigma\,\mathscr{O}(V)$ contains V, and equals V if D is holomorphically convex. Let K be a compact subset of V. It is well known that any connected component of $\mathring{K}$ is contained in a part for the algebra $(\mathscr{O}(V))_K$ (*cf.* [11], p. 162). Thus for algebras of this type there always exist parts "as big as possible". Further parts consist of the strong boundary points. Little is known about the remaining boundary points of ∂K. For a special case, *cf.* Wilken [12]. If we choose a compact set L containing K, then the "big parts" for $(\mathscr{O}(V))_K$ increase to "big parts" for $(\mathscr{O}(V))_L$, because we have

$$(+)\qquad \sup_{\|f\|_K \leq 1} |f(\phi) - f(\psi)| \geq \sup_{\|g\|_L \leq 1} |g(\phi) - g(\psi)|$$

whenever $\phi, \psi \in K$ and $f, g \in \mathscr{O}(V)$.

This suggests the

Definition. Let $\mathscr{A}$ be a uniform Fréchet algebra. Then $\phi, \psi \in \sigma\mathscr{A}$ are said to belong to the same Gleason part for $\mathscr{A}$ if there exists a compact set $K \subset \sigma\mathscr{A}$ with $\phi, \psi \in K$ such that ϕ and ψ belong to the same Gleason part for $\mathscr{A}_K$.

The transitivity of this relation is shown by applying (+) for $\mathscr{A}$ instead of $\mathscr{O}(V)$.

3. $\mathscr{A}$-MORPHIC MAPPINGS

(3.1) The notion of $\mathscr{A}$-morphic maps is modelled on the notion of holomorphic functions with values in locally convex spaces. We need this concept for Sections 5 and 8.

Let $\mathscr{E}$ be a locally convex vector space over $\mathbb{C}$, and let $\mathscr{A}$ be a uniform Fréchet algebra with spectrum X.

Definition. A mapping $\lambda: X \to \mathscr{E}$ is called $\mathscr{A}$-morphic (strongly $\mathscr{A}$-morphic, respectively) if it satisfies the following conditions:

(i) for all $\Psi \in \mathscr{E}'$, we have $\Psi \circ \lambda \in \mathscr{A}$;

(ii) λ is continuous when $\mathscr{E}$ is endowed with the weak topology $\sigma(\mathscr{E}, \mathscr{E}')$ (λ is continuous, respectively).

By $\mathscr{H}_{\mathscr{A}}(X, \mathscr{E})$ ($\mathscr{H}^s_{\mathscr{A}}(X, \mathscr{E})$, respectively) we denote the vector spaces of all $\mathscr{A}$-morphic mappings (strongly $\mathscr{A}$-morphic mappings, respectively). Notice that for $\mathscr{E} = \mathbb{C}$ we have $\mathscr{H}_{\mathscr{A}}(X, \mathbb{C}) = \mathscr{H}^s_{\mathscr{A}}(X, \mathbb{C}) = \mathscr{A}$.

Recall that *the Schwartzian* ε*-product for locally convex spaces* is defined by $\mathscr{A}\,\varepsilon\,\mathscr{E} := \mathscr{L}(\mathscr{E}'_\tau, \mathscr{A})$ (see *e.g.*, [1]). Here τ stands for the Mackey topology $\tau(\mathscr{E}', \mathscr{E})$; that is, the topology of uniform convergence on all absolutely convex compact subsets of $\mathscr{E}$.

(3.2) By means of the ε-product we are able to describe the $\mathscr{A}$-morphic mappings.

LEMMA. *Let* $\mathscr{E}$, $\mathscr{A}$, X *be as above. Then there is an isomorphism* $\mathscr{H}_{\mathscr{A}}(X, \mathscr{E}) \xrightarrow{\sim} \mathscr{A} \,\varepsilon\, \mathscr{E}$ *of vector spaces.*

The lemma is proven by displaying the isomorphism and its inverse mapping. An $\mathscr{A}$-morphic mapping $\lambda: X \to \mathscr{E}$ induces the linear operator $T: \mathscr{E}'_T \to \mathscr{A}$, $\Psi \mapsto T_\Psi := \Psi \circ \lambda$. T clearly is injective. The image $\lambda(K)$ of a compact set $K \subset X$ is weakly compact in $\mathscr{E}$, and hence the continuity of T follows from Krein's theorem (*cf.* [9]), which asserts that the absolutely convex hull of a weakly compact set remains weakly compact.

The inverse mapping $\mathscr{A} \,\varepsilon\, \mathscr{E} \to \mathscr{H}_{\mathscr{A}}(X, \mathscr{E})$ is given by

$$T \mapsto \lambda = (\phi \mapsto i^{-1}(\phi \circ T)).$$

Here i denotes the canonical isomorphism $i: \mathscr{E} \to (\mathscr{E}'_T)'$, which exists according to the theorem of Mackey-Arens. The desired properties of the inverse mapping follow from the equality $\Psi(i^{-1}(\phi \circ T)) = T_\Psi(\phi)$.

4. UNIFORM FRÉCHET-NUCLEAR-* ALGEBRAS

(4.1) In this section, we introduce a notion more restrictive than ordinary nuclearity. An analogous notion can be obtained for the larger class of Schwartz spaces if one replaces "nuclear operator" by "compact operator". We do not perform the latter—although we need it for Section 6—because the following (except (4.4)) may be carried over for the case of Schwartz spaces word by word. For the theory of nuclear locally convex spaces, see Pietsch [8], and for the theory of Schwartz spaces, see Horváth [6, p. 271 *ff.*].

Recall that a locally convex space $\mathscr{A}$ is nuclear if for all Banach spaces $\mathscr{E}$, all continuous linear operators $\mathscr{A} \to \mathscr{E}$ are nuclear operators (see below). For uniform Fréchet algebras, this condition can be reformulated more conveniently:

LEMMA. *Let* $\mathscr{A}$ *be a uniform Fréchet algebra with spectrum* X. *Then* $\mathscr{A}$ *is a nuclear space if and only if for every compact subset* $K \subset X$ *there exists a* (*larger*) *compact subset* $L \subset X$ *such that the restriction map* $\mathscr{A}_L \to \mathscr{A}_K$ *is a nuclear operator.*

We omit the simple proof.

(4.2) The condition in the above lemma often fails to be sufficiently strong, because one does not know to what extent L is larger than K. For certain local questions, one wishes to choose $K \subset L \subset X$ such that L is only "a little bit larger" than K. This leads to the

Definition. Let $\mathscr{A}$ be a uniform Fréchet algebra whose spectrum X satisfies (1.1.1). Then $\mathscr{A}$ is called a nuclear-* algebra (or (N*)-algebra, for short) if the following holds: for all compact subsets K, $L \subset X$ such that $K \subset \mathring{L}$, the restriction map $\mathscr{A}_L \to \mathscr{A}_K$ is a nuclear operator.

(4.3) LEMMA. *Let* $\mathscr{A}$, X *be as above. Then the following statements are equivalent:*

(i) $\mathscr{A}$ *is a nuclear-* algebra;*

(ii) *for all* $\mathscr{A}$*-convex compact subsets* $K, L \subset X$ *such that* $K \subset \mathring{L}$, *the restriction map* $\mathscr{A}_L \to \mathscr{A}_K$ *is a nuclear operator;*

(iii) $\mathscr{A}_U$ *is a nuclear-* algebra, for all open subsets* $U \subset X$;

(iv) $\mathscr{A}_U$ *is a nuclear algebra, for all open subsets* $U \subset X$.

Proof. The implications (i) ⇒ (ii) and (iii) ⇒ (iv) are trivial. The implication (ii) ⇒ (iii) follows from the fact that $\mathscr{A}_K = (\mathscr{A}_U)_K$ for all $K \subset \hat{U}$.

Now we shall prove the implication (iv) ⇒ (i). Let $K, L \subset X$ be compact subsets such that $K \subset \mathring{L}$. We have to show that the restriction map $\mathscr{A}_L \to \mathscr{A}_K$ is a nuclear operator. By hypothesis, $\mathscr{A}_{\mathring{L}}$ is a nuclear algebra. By Lemma (4.1), there exists a compact set K_1 with $K \subset K_1 \subset \hat{\mathring{L}}$ such that the restriction map $\mathscr{A}_{K_1} \to \mathscr{A}_K$ is a nuclear operator. Since nuclear operators enjoy the two-sided ideal property (*cf.* [8], p. 47), we conclude that the following composed restriction mappings constitute a nuclear operator:

$$\mathscr{A}_L \to \mathscr{A}_{\mathring{L}} = \mathscr{A}_{\hat{\mathring{L}}} \to \mathscr{A}_{K_1} \to \mathscr{A}_K .$$

Thus $\mathscr{A}$ is a nuclear-* algebra.

(4.4) For the reader's convenience we insert the definition of nuclear operators between locally convex spaces, as needed for Section 5. This definition is more general than the one in Pietsch's book (*cf.* [8], p. 44), but it is of course consistent with the theory of nuclearity.

Definition. Let $\mathscr{E}$, $\mathscr{F}$ be locally convex spaces and $T: \mathscr{E} \to \mathscr{F}$ a continuous linear operator. Then T is called a nuclear operator if there exist a bounded sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{F}$, an equicontinuous family $(\phi_n)_{n\in\mathbb{N}} \subset \mathscr{E}'$, and a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ satisfying $\sum_{n\in\mathbb{N}} |\lambda_n| < \infty$, such that $T(e) = \sum_{n\in\mathbb{N}} \lambda_n \phi_n(e) f_n$, for all $e \in \mathscr{E}$.

(4.5) The following theorem provides examples of uniform Fréchet-nuclear-* algebras. For the terminology, see the book of Gunning and Rossi [5].

THEOREM. *Let* $(X, \mathscr{O}_X)$ *be a holomorphically convex (reduced) complex space, having a countable basis for its topology. Then the algebra* $\mathscr{O}(X)$ *of all holomorphic functions on* X *is a uniform Fréchet-nuclear-* algebra.*

Proof. In the special case of X being an analytic variety in an open polycylinder $P \subset \mathbb{C}^n$, it is well known that $\mathscr{O}(P)$ is a uniform Fréchet-nuclear algebra and $\mathscr{O}(X) = \mathscr{O}(P)/J$ is a semisimple Fréchet-nuclear algebra under the canonical quotient topology (*cf.* [1], p. 322).

In the general case, X can be covered by a countable number of analytic varieties X_n of the above type. Hence, we can construct $\mathscr{O}(X)$ as a countable projective limit built by the Stein algebras $\mathscr{O}(X_n)$, algebraically and topologically. Therefore, $\mathscr{O}(X)$ is a semisimple Fréchet-nuclear algebra. From a theorem of Grauert-Remmert [4], it follows that $\mathscr{O}(X)$ is complete under uniform convergence on compact subsets of X. Now Forster's theorem on the uniqueness of the topology of a finite dimensional Stein algebra [2] and a sheaf-theoretic consideration yield the fact that the above semisimple Fréchet-nuclear topology for $\mathscr{O}(X)$ is equivalent to the uniform topology.

In order to show that $\mathscr{O}(X)$ is a nuclear-* algebra, we wish to apply Lemma (2.3) (iv). Let $U \subset \sigma\mathscr{O}(X)$ be an open set. It has been assumed that $(X, \mathscr{O}_X)$ is holomorphically convex. Thus the canonical map $g\colon X \to \sigma\mathscr{O}(X)$ is surjective and, since g is continuous, the set $V := g^{-1}(U)$ is open in X. The closure of $\mathscr{O}(X)$ with respect to the topology of uniform convergence on compact subsets of V equals the algebra $(\mathscr{O}(X))_U$. By the above considerations, we know that $\mathscr{O}(V)$ also is a uniform Fréchet-nuclear algebra. Now, $(\mathscr{O}(X))_U$ is a closed subalgebra of $\mathscr{O}(V)$ and hence uniform Fréchet-nuclear, too. This establishes our theorem.

The author conjectures that the above theorem remains true when the holomorphic convexity condition is dropped. Of course, the present proof does not work to obtain that result.

5. CONTINUITY OF THE MAP δ

Let $\mathscr{A}$ be a uniform Fréchet algebra with spectrum X, $U \subset X$ an open and relatively compact set, and $V \subset X$ also open such that $\overline{U} \subset V$. Consider the canonical inclusion map $\delta_V^U\colon U \to \mathscr{A}_V'$. If $\mathscr{A}_V'$ is endowed with the strong topology, then in general this map fails by far to be continuous, since U carries the weak-* topology. It is the crucial step for our Gleason parts theorem (7.1) that this map becomes continuous under the assumption (N*).

LEMMA. *Let $\mathscr{A}$, $U \subset V \subset X$ be as above and assume that $\mathscr{A}$ is a nuclear-* algebra. Then the canonical map $\delta_V^U\colon U \to \mathscr{A}_V'$ is strongly $\mathscr{A}$-morphic. If only nuclearity is assumed, then at least $\delta_X^U\colon U \to \mathscr{A}'$ remains strongly $\mathscr{A}$-morphic.*

Proof. Note first that the condition (FN*) implies the reflexivities $\mathscr{A}_V \xrightarrow{\sim} \mathscr{A}_V''$, $\mathscr{A}_V' \xrightarrow{\sim} \mathscr{A}_V'''$. From the former isomorphism, one derives easily that δ_V^U is an $\mathscr{A}$-morphic map.

Since $\overline{U}$ is a compact subset of V, the restriction map $\mathscr{A}_V \to \mathscr{A}_U$ is a nuclear operator. Therefore, the canonical map

$$\mathscr{A}_V \,\varepsilon\, \mathscr{A}_V' \longrightarrow \mathscr{A}_U \,\varepsilon\, \mathscr{A}_V'$$

factorizes through a subspace consisting of nuclear operators:

$$\begin{array}{ccc} \mathscr{L}((\mathscr{A}_V')_\tau', \mathscr{A}_V) & \longrightarrow & \mathscr{L}((\mathscr{A}_V')_\tau', \mathscr{A}_U) \\ & \searrow \qquad \nearrow & \\ & \mathscr{N}((\mathscr{A}_V')_\tau', \mathscr{A}_U) & \end{array}$$

Hence, the operator L_V^U corresponding by Lemma (3.2) to δ_V^U is nuclear, and by (4.4), L_V^U can be written as a series

$$L_V^U(\Psi) = \sum_{n=1}^{\infty} \lambda_n \cdot \phi_n(\Psi) \cdot a_n, \qquad \text{for all } \Psi \in \mathscr{A}_V''.$$

Here $(a_n)_{n\in\mathbb{N}} \subset \mathscr{A}_U$ is a bounded sequence, $(\phi_n)_{n\in\mathbb{N}} \subset \mathscr{A}_V''' \xrightarrow{\sim} \mathscr{A}_V'$ an equicontinuous family, and $\lambda_n \in \mathbb{C}$ a sequence such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$. By the equicontinuity of (ϕ_n), there exist a compact set $K \subset V$ and a constant $C' > 0$ such that $|\phi_n(a)| \le C' \|a\|_K$, for all $n \in \mathbb{N}$ and $a \in \mathscr{A}_V$.

Again by Lemma (3.2), $\delta_V^U \in \mathscr{H}_{\mathscr{A}_U}(U, \mathscr{A}'_V)$ can be represented as a series

$$\delta_V^U(\phi) = \left(a \mapsto \sum_{n=1}^{\infty} \lambda_n \phi_n(a) a_n(\phi) \right)$$

which is convergent for all $\phi \in U$, $a \in \mathscr{A}_V$.

We first show that the members of this series belong to $\mathscr{C}(U, \mathscr{A}'_V)$ and then that the series itself is strongly $\mathscr{A}$-morphic. Now recall that the strong topology for $\mathscr{A}'_V$ is the topology of uniform convergence on $\sigma(\mathscr{A}_V, \mathscr{A}'_V)$-bounded sets of $\mathscr{A}_V$. In each topology of a dual pair, the same sets are bounded (*cf.* [9], p. 77). The finest topology of the dual pair is $\tau(\mathscr{A}_V, \mathscr{A}'_V)$, and this topology coincides with the original topology on $\mathscr{A}_V$, since $\mathscr{A}_V$ is a Fréchet space (*cf.* [9], p. 77). Thus the strong topology for $\mathscr{A}'_V$ coincides with the topology of uniform convergence on the sets $A \subset \mathscr{A}_V$ of the form $A = \{a \in \mathscr{A}_V \colon \|a\|_{K_n} \leq C_n\}$; here $\cdots \subset K_n \subset K_{n+1} \subset \cdots$ denotes a fixed admissible exhaustion of V by compact sets and $(C_n)_{n \in \mathbb{N}}$ an arbitrary sequence of nonnegative numbers.

The mappings $U \to \mathscr{A}'_V$ and $\phi \mapsto (a \mapsto \phi_n(a) \cdot a_n(\phi))$ are continuous for all $n \in \mathbb{N}$. For, let $(\psi_i)_{i \in \mathbb{N}}$ be a sequence in U converging to $\psi \in U$; then one obtains the estimation

$$\begin{aligned} \sup_{a \in A} |\phi_n(a)\,(a_n(\psi_i) - a_n(\psi))| &\leq \sup_{a \in A} C' \cdot \|a\|_K \cdot |a_n(\psi_i) - a_n(\psi)| \\ &\leq C' \cdot C_{n_0} \cdot |a_n(\psi_i) - a_n(\psi)|, \end{aligned}$$

when $n_0 \in \mathbb{N}$ is chosen large enough such that $K \subset K_{n_0}$. The last term of the estimation tends to zero as $i \to \infty$.

Now endow $\mathscr{C}(U, \mathscr{A}'_V)$ with the compact-open topology. Then this is a complete space, since $\mathscr{A}'_V$ is complete. Thus we are finished if we show that the above series converges in the space $\mathscr{C}(U, \mathscr{A}'_V)$.

Let $L \subset U$ be a compact set and $A \subset \mathscr{A}_V$ a τ-bounded set. Then we have the estimate

$$\begin{aligned} \sup_{\phi \in L} \sup_{a \in A} \left| \sum_{n \geq m} \lambda_n \phi_n(a)\, a_n(\phi) \right| &\leq \sup_{a \in A} C' \|a\|_K \sum_{n \geq m} |\lambda_n| \, \|a_n\|_L \\ &\leq C' C_{n_0} C'' \sum_{n \geq m} |\lambda_n| \end{aligned}$$

with a constant $C'' > 0$, which exists by the boundedness of the sequence (a_n). Again this tends to zero as $m \to \infty$.

The second half of the lemma is proven by setting $V = X$ in the above proof.

6. LOCAL CONNECTIVITY OF THE SPECTRUM

The theorem of this section is needed for the Gleason parts theorem (7.1). In addition, Theorem 6 provides a new and comparatively short proof for the well known fact in complex analysis that complex analytic spaces are locally connected. (For a standard proof, see [5], p. 103.)

One can easily give examples of uniform Fréchet algebras whose spectra are not locally connected. Therefore, it seems a little surprising that the Schwartz-* assumption—and not even nuclearity-*—implies the desired property.

THEOREM. *If the spectrum* X *of a uniform Fréchet-Schwartz-* algebra* $\mathscr{A}$ *is locally compact, then* X *is locally connected.*

Proof. Let $\phi_0 \in X$ be given. We argue indirectly: suppose there does not exist a neighborhood basis of ϕ_0 consisting of connected sets. Then there must exist a neighborhood U of ϕ_0 such that no neighborhood of ϕ_0 contained in U can be connected. Now choose a compact neighborhood $L \subset U$. Consider the family $(L_i)_{i\in I}$ of all L-clopen sets $L_i \subset L$ such that $L_i \cap L_j = \emptyset$ for $i \neq j$, and $\phi_0 \notin L_i$ for all $i \in I$. The index set I cannot be finite, since otherwise $L - \bigcup_{i\in I} L_i$ would be a connected neighborhood of ϕ_0.

Since X is required to be locally compact, there exists a compact neighborhood K of ϕ_0 with $K \subset \overset{\circ}{L}$.

The set $J := \{j \in I : K \cap L_j \neq \emptyset\}$ is an infinite set, too. This can be shown as follows. First we observe that ϕ_0 is an element of the closure of $\bigcup_{i\in I} L_i$. For if not, then its complement $L_0 := L - \bigcup_{i\in I} L_i$ is a neighborhood of ϕ_0 and therefore nonconnected. L_0 is open with respect to L and hence there exist L-open disjoint sets L_0', L_0'' such that $L_0 = L_0' \cup L_0''$. By calculating complements, one also finds that L_0', L_0'', and thus L_0, are closed. Thus we have obtained a contradiction to the maximality of the family $(L_i)_{i\in I}$, to which L_0' or L_0'' must belong. Hence $\phi_0 \in \overline{\bigcup_{i\in I} L_i}$. Since each L_i, $i \in I$, is compact and $\phi_0 \notin L_i$, we have the fact that each neighborhood of ϕ_0 meets infinitely many L_i.

Now consider the characteristic functions, for $j \in J$,

$$\chi_j(\phi) := \begin{cases} 1, & \text{if } \phi \in L_j \\ 0, & \text{if } \phi \notin L_j. \end{cases}$$

These functions belong to $\mathscr{A}_L$, by Shilov's idempotent theorem (*cf.* [11]). Clearly the family $(\chi_j)_{j\in J}$ is bounded in $\mathscr{A}_L$. Since $K \subset \overset{\circ}{L}$, the condition Schwartz-* implies that the restriction map $\mathscr{A}_L \to \mathscr{A}_K$ is a compact operator. Therefore, there exists a sequence $(j_n)_{n\in\mathbb{N}} \subset J$ such that the corresponding sequence of restricted characteristic functions $(\chi_{j_n}|_K)_{n\in\mathbb{N}}$ is convergent in $\mathscr{A}_K$. This establishes a contradiction to the fact that $\|\chi_{j_k} - \chi_{j_l}\|_K = 1$ for $j_k \neq j_l$.

7. BIG GLEASON PARTS FOR UNIFORM FRÉCHET-NUCLEAR-* ALGEBRAS

(7.1) THEOREM. *Let* $\mathscr{A}$ *be a uniform Fréchet-nuclear-* algebra whose spectrum* X *satisfies* (1.1.1), *and let* $V \subset X$ *be an open relatively compact subset. Then each connected component of* V *lies in a Gleason part for the Banach algebra* $\mathscr{A}_{\overline{V}}$.

Proof. By Lemma (5.1), we know that the mappings $\delta_V^U: U \to \mathscr{A}'_V$ are continuous for all open $U \subset X$ satisfying $\overline{U} \subset V$. Therefore, the weak-* topology on U and the strong topology induced by $\overline{V}$ on U coincide. Hence, for each $\phi \in U$, there exists an open set $U_\phi \subset U$ which belongs to a part for $\mathscr{A}_{\overline{V}}$.

Now consider finite chains $U_{\phi_1}, \cdots, U_{\phi_n}$ of such sets satisfying

$$U_{\phi_i} \cap U_{\phi_{i+1}} \neq \emptyset.$$

Then $\bigcup_{i=1}^n U_{\phi_i}$ belongs to a single part for $\mathscr{A}_{\overline{V}}$, because the notion of Gleason parts causes a partition of V into equivalence classes. On the other hand, it is easily proven that the set of all points in V which can be reached by such chains starting from a fixed point, is a (weak-*) closed set with respect to V. Thus for each open connected component there exists a part for $\mathscr{A}_{\overline{V}}$ containing it. But from Theorem 6, we conclude that all connected components of V are open. This finishes the proof.

(7.2) Using Definition 2, we reformulate Theorem (7.1). The proof is immediate if one takes into consideration the local compactness of X.

THEOREM. *Let* $\mathscr{A}$ *be a uniform Fréchet-nuclear-* algebra whose spectrum* X *satisfies* (1.1.1). *Then the Gleason parts for* $\mathscr{A}$ *are exactly the connected components of* X (*which are all open*).

(7.3) COROLLARY (*Maximum modulus principle*). *Again let* $\mathscr{A}$ *be a uniform Fréchet-nuclear-* algebra whose spectrum satisfies* (1.1.1), *and let* $U \subset X$ *be an open relatively compact and connected subset. Then for all functions* $f \in \mathscr{A}$ *which are nonconstant on* U *we have* $|f(\phi)| < \|f\|_U$ *for all* $\phi \in U$.

Proof. By Theorem (7.1), U is contained in a Gleason part for $\mathscr{A}_{\overline{U}}$. Now apply [11, 16.8].

(7.4) *Remark.* The question arises whether these big Gleason parts carry analytic structure. In a future paper [7], we shall show that this is the case under rather weak conditions. This will be managed by an appropriate dimension theory for uniform Fréchet algebras and a theorem of Basener and Sibony [10].

8. GENERAL CAUCHY-WEIL INTEGRAL FORMULAS

(8.1) In complex analysis, there are various boundary-value integral formulas. For example, one has Cauchy-Weil integral formulas for relatively compact subsets U of any reduced complex space (X, $\mathscr{O}$) (*cf.* [1], Theorem 19.1). These formulas consist of a family holomorphically varying on U, of complex representing measures supported on the Shilov boundary γ of $\overline{\mathscr{O}(\overline{U})}$ (= closure of $\mathscr{O}(\overline{U})$ under the seminorm $\|\cdot\|_{\overline{U}}$).

If one replaces $\overline{\mathcal{O}(\overline{U})}$ by $(\mathcal{O}(X))_{\overline{U}}$, then one can transfer the holomorphic theorem to the $\mathcal{A}$-morphic and nuclear-* setting. By the way, for Stein spaces these two algebras coincide.

The question arises with what topology the space $\mathcal{M}(\gamma)$ of complex Radon measure should be endowed in order to obtain an $\mathcal{A}$-morphic map $U \to \mathcal{M}(\gamma)$ for the theorem below. Then we find that both natural cases are possible: the map below will be $\mathcal{A}$-morphic with respect to the weak topology $\sigma(\mathcal{M}(\gamma), (\mathcal{M}(\gamma))')$ if and only if it is $\mathcal{A}$-morphic with respect to the strong topology $\beta(\mathcal{M}(\gamma), \mathcal{C}(\gamma))$. This follows from the fact that $(\mathcal{M}(\gamma), (\mathcal{M}(\gamma))')$ constitutes a dual pair and that all admissible topologies for a dual pair produce the same dual space.

(8.2) THEOREM. *Let* $\mathcal{A}$ *be a uniform Fréchet-nuclear-* algebra whose spectrum* X *satisfies* (1.1.1), *and let* $U \subset X$ *be an open, relatively compact subset. Then the Shilov boundary* $\gamma := \gamma\, \mathcal{A}_{\overline{U}}$ *is contained in* ∂U *and there exists an* $\mathcal{A}_U$-*morphic mapping* $\mu: U \to \mathcal{M}(\gamma)$ *such that*

$$f(\phi) = \int_\gamma f(\psi)\, d\mu_\phi(\psi) \qquad \textit{for all } f \in \mathcal{A}_{\overline{U}} \textit{ and } \phi \in U.$$

Proof. By Corollary (7.3), we have for all $f \in \mathcal{A}$ — and then even for all $f \in \mathcal{A}_{\overline{U}}$ — the equation $\|f\|_{\overline{U}} = \|f\|_{\partial U}$. Therefore, the Shilov boundary γ for $\mathcal{A}_{\overline{U}}$ is contained in ∂U.

The canonical map $\mathcal{A}_{\overline{U}} \to \mathcal{C}(\gamma)$ is injective and has closed range. By the theorem of Hahn and Banach, we conclude that the adjoint map $\mathcal{C}(\gamma)' = \mathcal{M}(\gamma) \to \mathcal{A}'_{\overline{U}}$ is surjective. In the sequel, let these dual spaces be endowed with their strong topologies. $\mathcal{A}_U$ is a nuclear space since $\mathcal{A}$ is assumed to be nuclear-*. Therefore, we can apply Bungart's theorem ([1], Theorem 5.3) and obtain the surjectivity of the map

$$\mathcal{A}_U \,\varepsilon\, \mathcal{C}(\gamma)' \to \mathcal{A}_U \,\varepsilon\, \mathcal{A}'_{\overline{U}}.$$

Hence by Lemma (3.2) the map

$$\Phi: \mathcal{H}_{\mathcal{A}_U}(U, \mathcal{M}(\gamma)) \to \mathcal{H}_{\mathcal{A}_U}(U, \mathcal{A}'_{\overline{U}})$$

is surjective.

Now consider the canonical inclusion map $\delta := \delta^U_{\overline{U}}: U \to \mathcal{A}'_{\overline{U}}$. We must prove the continuity of δ with respect to the topology $\sigma(\mathcal{A}'_{\overline{U}}, \mathcal{A}''_{\overline{U}})$ of $\mathcal{A}'_{\overline{U}}$. The canonical map $\mathcal{A}'_U \to \mathcal{A}'_{\overline{U}}$ is continuous under the strong topologies. By a well known theorem ([9], p. 48), it is weakly continuous; *i.e.*, continuous under the topologies $\sigma(\mathcal{A}'_U, \mathcal{A}''_U)$ and $\sigma(\mathcal{A}'_{\overline{U}}, \mathcal{A}''_{\overline{U}})$. Since Fréchet-nuclear spaces are reflexive [8], we have $\sigma(\mathcal{A}'_U, \mathcal{A}''_U) = \sigma(\mathcal{A}'_U, \mathcal{A}_U)$. Hence δ factorizes through the continuous maps

$$U \to (\mathcal{A}'_U, \sigma(\mathcal{A}'_U, \mathcal{A}_U)) \to (\mathcal{A}'_{\overline{U}}, \sigma(\mathcal{A}'_{\overline{U}}, \mathcal{A}''_{\overline{U}}))$$

and thus is continuous, too.

In order to show that δ is an $\mathcal{A}$-morphic map, we consider the following diagram:

$$\begin{array}{ccccc} U & & \xrightarrow{\quad\delta\quad} & & \mathscr{A}'_{\overline{U}} \\ & \searrow & & \nearrow{\scriptstyle \pi} & \downarrow{\scriptstyle \Psi} \\ & & \mathscr{A}'_{U} & \overset{\tilde{\Psi}}{\dashrightarrow} & \mathbb{C} \end{array}$$

For every $\Psi \in \mathscr{A}''_{\overline{U}}$, we have $\tilde{\Psi} = \Psi \circ \pi \in \mathscr{A}''_{U}$; since $\mathscr{A}_U$ is reflexive we obtain $\tilde{\Psi} \circ \delta \in \mathscr{A}_U$.

Thus δ lies in $\mathscr{H}_{\mathscr{A}_U}(U, \mathscr{A}'_{\overline{U}})$. By the surjectivity of Φ, we can lift δ to an $\mathscr{A}_U$-morphic map $\mu \in \mathscr{H}_{\mathscr{A}_U}(U, \mathscr{M}(\gamma))$. Clearly $\mu_\phi := \mu(\phi)$ is a complex representing measure for $\phi \in U$ with respect to $\mathscr{A}_{\overline{U}}$, and thus we gain the desired integral formula.

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