ADJOINT REPRESENTATIONS OF FACTOR GROUPS

Arthur Lieberman

Let A be a von Neumann algebra which is a factor, and let G denote either the unitary group of A or the group of invertible members of A. The adjoint representation ϕ of G is defined by $\phi(U)T = UTU^{-1}$ for all $U \in G$, $T \in A$. Below we determine all norm-closed subspaces of A which are invariant under the action of G.

If P, Q are projections in A, we write $P \precsim Q$ if there is a partial isometry $V \in A$ such that $V^*V = P$ and $VV^* \leq Q$. We write $P \sim Q$ if $P \precsim Q$ and $Q \precsim P$; that is, if there is a partial isometry $V \in A$ such that $V^*V = P$ and $VV^* = Q$. We write $P \prec Q$ if $P \precsim Q$ and $P \nsim Q$. If A is a factor, then for any projections $P, Q \in A$, either $P \precsim Q$ or $Q \precsim P$ [2, p. 218].

If $T \in A$, let R(T), the range projection of T, be the least projection $P \in A$ satisfying $PT = T$. Let S(T) be the subspace spanned by $\{UTU^{-1}: U \in G\}$ and let CS(T) be the norm closure of S(T).

We now recall the ideal structure of a factor. Throughout this paper, *ideal* means norm-closed ideal. If A is a finite factor, then A has no proper ideals [2, p. 257]. If A is of infinite type, then the set of ideals of A is well-ordered by set inclusion [7, 8]. If A is type I_∞ or type II_∞, then the *minimal ideal* of A, hereafter denoted K, is the uniform closure of $\{T \in A: R(T) \text{ is a finite projection}\}$; the members of K are called *compact operators*. If A is type I_∞ or type II_∞, then K is the unique minimal proper ideal of A. If A is of infinite type, the unique maximal proper ideal of A is denoted by J; if A is type III and simple, we define J by $J = \{0\}$.

We denote the identity operator by I. If A is semifinite, we denote by tr the trace on A.

THEOREM 1. *Let* A *be a finite von Neumann algebra factor, let* G *be either the unitary group of* A *or the group of invertible members of* A, *and let* ϕ *be the representation of* G *on* A *defined by* $\phi(U)T = UTU^{-1}$ *for all* $U \in G$, $T \in A$. *Then* G *acts irreducibly on* $\{\lambda I: \lambda \in \mathbb{C}\}$ *and on* $\{T \in A: \mathrm{tr}(T) = 0\}$. *Furthermore, these are the only proper invariant subspaces for the action of* G *on* A.

THEOREM 2. *Let* A *be an infinite von Neumann algebra factor, let* G *be either the unitary group of* A *or the group of invertible members of* A, *and let* ϕ *be the representation of* G *on* A *defined by* $\phi(U)T = UTU^{-1}$ *for all* $U \in G$ *and* $T \in A$. *The proper invariant subspaces for the action of* G *on* A *are precisely* $\{\lambda I: \lambda \in \mathbb{C}\}$, *the proper ideals of* A, *and for each proper ideal* $\mathcal{J}$ *of* A, $\{\lambda I + T: \lambda \in \mathbb{C} \text{ and } T \in \mathcal{J}\}$. *The nontrivial irreducible subspaces for the action of* G *on* A *are* $\{\lambda I: \lambda \in \mathbb{C}\}$ *and* (*if* A *is not simple*) *the minimal proper ideal of* A.

If A is a factor of type I or type II, there is associated with A the Hilbert space $L^2(A)$ (see [5]). $L^2(A)$ is the completion of the pre-Hilbert space $\{T \in A: \mathrm{tr}(T^*T) < \infty\}$ with respect to the inner product $\langle S, T \rangle = \mathrm{tr}(T^*S)$. $L^2(A)$ is a $*$-algebra, and A acts in a natural manner on $L^2(A)$.

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THEOREM 3. *Let* A *be a factor of type* I *or type* II, *let* G *be either the unitary group of* A *or the group of invertible members of* A, *and let* ψ *be the representation of* G *on* $L^2(A)$ *defined by* $\psi(U)T = UTU^{-1}$ *for all* $U \in G$, $T \in L^2(A)$. *If* A *is finite, then* $L^2(A) = \{\lambda I: \lambda \in \mathbb{C}\} \oplus \{T \in L^2(A): \mathrm{tr}(T) = 0\}$, *and each of these two direct summands is invariant under the action of* G *and is acted on irreducibly by* G. *If* A *is infinite, then* G *acts irreducibly on* $L^2(A)$.

Before giving the proofs of the theorems, we make some observations.

Remark 1. The only nontrivial fact to be proved is that the subspaces listed above are the only subspaces which are invariant under the action of G. Since the unitary group of A is a subset of the group of invertible members of A, it suffices to prove the theorems when G is the unitary group of A. Therefore, for the remainder of this paper, we assume that G is the unitary group of A.

Remark 2. Let $T = B + iC$ be the cartesian decomposition of T. Let D be a self-adjoint operator which commutes with B. Then

$$e^{itD}Te^{-itD} - T = i(e^{itD}Ce^{-itD} - C) = i[(e^{itD} - I)Ce^{-itD} - C(I - e^{-itD})].$$

Dividing by t and taking the norm limit as $t \to 0$ yields $(DC - CD) \in CS(T)$. If T is not normal, D can be chosen such that $DC - CD \neq 0$, so CS(T) contains a nonzero self-adjoint operator.

LEMMA 1. *Let* A *be a finite factor and let* P *be a projection in* A *with* $P \neq 0, I$. *Then* $CS(P) = A$.

Proof. By the cartesian decomposition and the spectral theorem, it suffices to prove that CS(P) contains all projections in A. By the comparability of projections, it suffices to prove that if $0 < \lambda < 1/2$, then CS(P) contains a projection Q with $\mathrm{tr}(Q) = \lambda$. Since $I \in CS(P)$ [2, p. 254, Corollary], we may assume (replacing P by $I - P$ if necessary) $\mathrm{tr}(P) < 1/2$. Let $0 < \lambda < 1/2$, and let Q be a projection in A such that $\mathrm{tr}(Q) = \lambda$ and $PQ = 0$. Apply [2, p. 254, Corollary] to P considered as a member of the von Neumann algebra $(P + Q)A(P + Q) \oplus (I - P - Q)A(I - P - Q)$ to obtain $(P + Q) \in CS(P)$.

Proof of Theorem 1. It suffices to show that if $T \in A$, $T \notin \{\lambda I: \lambda \in \mathbb{C}\}$, and $\mathrm{tr}(T) \neq 0$, then $CS(T) = A$. Let T satisfy the above conditions.

Assume first that T is normal. Since $I \in CS(T)$ [2, p. 254, Corollary], we can add a scalar to T, if necessary, to assure that the spectral projection Q of T for the open right half-plane satisfies $Q \neq 0, I$. Apply [2, p. 254, Corollary] to T as a member of the von Neumann algebra $QAQ \oplus (I - Q)A(I - Q)$ to obtain

$$(\mathrm{tr}(QT)/\mathrm{tr}(Q))Q + (\mathrm{tr}((I - Q)T)/\mathrm{tr}(I - Q))(I - Q) \in CS(T).$$

Since $\mathrm{tr}(QT)$ has positive real part and $\mathrm{tr}((I - Q)T)$ has nonpositive real part, forming a suitable linear combination with I yields $Q \in CS(T)$.

If T is not normal, by Remark 2, CS(T) contains a nonzero normal operator T_0 with $\mathrm{tr}(T_0) = 0$. Apply the previous part of the proof to $T_0 + I$.

LEMMA 2. *Let* A *be a type* I_∞ *or type* II_∞ *factor, and let* T *be a compact operator. Then* CS(T) *contains a nonzero finite rank projection.*

Proof. By Remark 2, CS(T) contains a nonzero normal operator, so we may assume that T is normal. Let P be a spectral projection of T such that $\mathrm{tr}(P) < \infty$ and $\mathrm{tr}(TP) \neq 0$. Apply [2, p. 254, Corollary], and [6, Theorem 10] to the operator

$TP \oplus T(I - P)$ considered as a member of the von Neumann algebra $PAP \oplus (I - P)A(I - P)$ to obtain $P \in CS(T)$.

LEMMA 3. *Let* A *be a type* I_∞ *or type* II_∞ *factor, and let* T *be a compact operator. Then* $CS(T) = K$.

Proof. By Lemma 2, CS(T) contains a nonzero finite rank projection P. Let S be any finite rank operator. Choose a finite rank projection Q such that $P \preceq Q$ and $R(S) \preceq Q$. By Theorem 1, applied to P and Q considered as members of the von Neumann algebra QAQ, $S \in CS(P) \subseteq CS(T)$. However, the finite rank operators are norm-dense in K.

LEMMA 4. *Let* A *be an infinite factor, and let* T *generate the proper ideal* $\mathscr{I}$ *of* A. *Assume* $\mathscr{I} \neq K$. *Then* $CS(T) = \mathscr{I}$.

Proof. Let $P_1 = \text{lub}\{R(T), R(T^*)\}$. Assume $P_1 \in \mathscr{I}$. Let P_2 be any projection in A such that $P_1 P_2 = 0$ and $P_1 \sim P_2$. Let $P = P_1 + P_2$. Then $P \in \mathscr{I}$, and PTP is not a member of any ideal in the infinite factor PAP. By [6, Theorem 10], $P \in CS(T)$.

If Q is any projection in A such that $QP = 0$ and $Q \prec P$, then $(P + Q) \sim P$, so that $P + Q \in CS(T)$ and therefore $Q \in CS(T)$. Thus

$$\{Q \in A\colon Q \text{ is a projection and } Q \preceq R(T)\} \subseteq CS(T);$$

this set of projections spans a dense subset of $\mathscr{I}$.

If $P_1 \notin \mathscr{I}$, let $P(\lambda)$ denote the spectral projection of T^*T for the interval (λ, ∞). Then $R(T^*) = \text{lub}\{P(\lambda)\colon \lambda > 0\}$. Thus if $Q \prec P_1$, then there exists $\lambda < 0$ such that $Q \preceq P(\lambda)$. Therefore, it suffices to show that $P(\lambda) \in CS(T)$ for all $\lambda > 0$. Since $\mathscr{I} \neq K$, $P(\lambda)$ is an infinite projection for λ sufficiently small; thus we may assume $P(\lambda)$ is an infinite projection.

Let F denote the semigroup with identity generated by T and T^*; F is countable. Let $P_3 = \text{lub}\{R(SP(\lambda)\colon S \in F\}$. Then $P(\lambda) \sim P_3$. Then P_3TP_3 is not a member of any ideal in the infinite factor P_3AP_3. Apply [6, Theorem 10] to the operator $T = P_3TP_3 + (I - P_3)T(I - P_3)$ considered as a member of the von Neumann algebra $P_3AP_3 \oplus (I - P_3)A(I - P_3)$ to obtain $P_3 \in CS(T)$. Since $P_3 \sim P(\lambda)$, $P(\lambda) \in CS(T)$.

LEMMA 5. *Let* A *be an infinite factor which is not countably decomposable, let* $T \in A$, *and assume* $T + \lambda I \notin J$ *for any* $\lambda \in \mathbb{C}$. *Then* $CS(T) = A$.

Proof. Let F denote the semigroup with identity generated by T and T^*; note that F is countable. If Q is any infinite projection in A with $Q \prec I$, then $\text{lub}\{R(SQ)\colon S \in F\} \sim Q$. Thus T reduces many subspaces, and by a routine argument, there is a projection $P \in A$ such that $P \sim I$, $(I - P) \sim I$, and $PT = TP$.

Apply [6, Theorem 10] to the operator $T = PTP + (I - P)T(I - P)$ in the von Neumann algebra $PAP \oplus (I - P)A(I - P)$ to obtain $\lambda_1 P + \lambda_2(I - P) \in CS(T)$ for some λ_1, λ_2 with $\lambda_1 \neq \lambda_2$. Since $I \in CS(T)$ [6, Theorem 10], forming a suitable linear combination yields $P \in CS(T)$. Now proceed as in the second paragraph of the proof of Lemma 4.

LEMMA 6. *Let* A *be an infinite countably decomposable factor, let* $T \in A$, *and suppose* $T + \lambda I \notin J$ *for any* $\lambda \in \mathbb{C}$. *Then* $CS(T) = A$.

Proof. If T is normal, let P be a spectral projection of T such that $P \sim I$ and $(I - P) \sim I$; now apply the last paragraph of the proof of Lemma 5. If T is not normal, let $T = B + iC$ be the cartesian decomposition of T. Without loss of generality, $C + \lambda I \notin J$ for any $\lambda \in \mathbb{C}$. Let P be a spectral projection of C such that $P \sim I$ and $(I - P) \sim I$. By Remark 2, $BP - PB \in CS(T)$. If $BP - PB \notin J$, then proceed as in the case when T is normal; by [3, Theorem 3.8], $BP - PB + \lambda I \notin J$ for any $\lambda \in \mathbb{C}$.

If A is type III, the proof is finished since in this case A is a simple algebra. If A is type I_∞ or type II_∞, note that $J = K$. If $BP - PB \in K$, then by Lemma 3, $K \subseteq CS(T)$. By [4, Lemma 1], there is an operator $D \in K$ such that $BP - PB = DP - PD$. Then $T - D \in CS(T)$, and $(T - D)P - P(T - D) = 0$. Now apply the last paragraph of the proof of Lemma 5 to the operator $T - D$.

Proof of Theorem 2. The proof of Theorem 2 follows from the ideal theory of infinite factors and the lemmas. One need only note that if $T = \lambda I + X$, where $\lambda \neq 0$ and $X \in J$, then by [6, Theorem 10], $I \in CS(T)$, so that $X \in CS(T)$.

Proof of Theorem 3. If A is finite of type I, then $L^2(A) = A$ as a linear space, and both are finite dimensional. Thus the two topologies on this space coincide. Now apply Theorem 1.

Assume now that A is type II_1. For bounded operators, the convergence of a sequence in A implies its convergence in $L^2(A)$. Thus if $T \in A$ and $\mathrm{tr}(T) = 0$, the closed subspace $CS\psi(T)$ spanned by $\psi(G)T$ contains, by Theorem 1, $\{W \in A: \mathrm{tr}(W) = 0\}$; this set is obviously dense in $\{W \in L^2(A): \mathrm{tr}(W) = 0\}$.

If T is unbounded and normal, let P be a spectral projection of T such that PT is bounded and the spectrum of PT contains at least two points. Choose a unitary U in A such that $U(I - P) = I - P$ and $UPT \neq PTU$. Then $UTU^{-1} - T$ is bounded and is in $CS\psi(T)$. Now apply the previous part of this proof. If T is not normal, let T have cartesian decomposition $T = B + iC$. Then $CS\psi(T)$ contains the normal operators $e^{itB}Te^{-itB} - T = i(e^{itB}Ce^{-itB} - C)$; for some t, this operator must be nonzero. Now apply the previous parts of this proof.

Assume now that A is of infinite type. Let $T \in L^2(A)$; it suffices to show that $CS\psi(T)$ contains a nonzero finite rank operator. We may assume that T is normal, since $CS\psi(T)$ must contain a nonzero normal operator. Let P be a spectral projection for T such that $\mathrm{tr}(P) < \infty$, $TP \neq 0$, and $T(I - P)$ is bounded. By "averaging" T using unitary operators U satisfying $UP = P$, one obtains $TP \in CS\psi(T)$.

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Department of Mathematics
The Cleveland State University
Cleveland, Ohio 44115