ON A UNIQUENESS THEOREM IN CONFORMAL MAPPING

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Let $f(z) = z + b_0 + b_1 z^{-1} + \cdots + b_n z^{-n} + \cdots$ be univalent and holomorphic in $D^* = \{z: |z| > 1\}$, up to the simple pole at ∞. The set $B = \mathbb{C} \setminus f(D^*)$ is a compact continuum whose diameter we denote by d. According to [3, Abschnitt IV, Aufgabe 141, S.25 und S.199], d satisfies the inequalities $2 \leq d \leq 4$. Moreover, d equals 4 if and only if B is a straight segment of length 4, and d equals 2 if and only if B is a disc; that is, if $f(z) = z + b_0$. But a proof that d equals 2 only if $f(z) = z + b_0$ was missing. In 1969, J. A. Jenkins [2] closed this gap with a proof based on the following facts: if $d = 2$, then B is bounded by a rectifiable Jordan curve of length at most 2π; hence, f' belongs to the Hardy class H_1, the function $f(e^{i\theta})$ is absolutely continuous, and $df(e^{i\theta})/d\theta = ie^{i\theta} f'(e^{i\theta})$ almost everywhere. It is the purpose of this note to give an *elementary* proof.

THEOREM. *The diameter* d *of* B *satisfies the inequality* $d \geq 2$, *and equality occurs if and only if* $f(z) = z + b_0$.

Let C_r denote the circle $\{z: |z| = r\}$ oriented in the positive sense. For $r > 1$, let $\Gamma_r = f(C_r)$. Let d_r denote the diameter of Γ_r and L_r the length of Γ_r.

The proof of the inequality $d_r \geq 2$ is elementary (*cf.* [3]). From the relation

$$f(z) - f(-z) = 2z + 2b_1 z^{-1} + 2b_3 z^{-3} + \cdots, \qquad |z| > 1,$$

it follows that

$$2 = \left| \frac{1}{2\pi i} \int_{C_r} (f(z) - f(-z)) \frac{dz}{z^2} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(-re^{i\theta})| \frac{d\theta}{r} \leq \frac{1}{r} d_r, \quad r > 1;$$

and this implies $d \geq 2$.

We now give three lemmas that we need for the proof of the uniqueness part of the theorem.

LEMMA 1. *For* $r \geq 1$, $d_r \leq rd$.

Proof. Let z_1 and z_2 be two points on C_r such that $|f(z_1) - f(z_2)| = d_r$, and let $\Phi(z) = f(z_1 z/z_2) - f(z)$. Then, for $\rho > 1$, it follows from the relations $\max_{|z|=\rho} |\Phi(z)| \leq d_\rho$ and $\lim_{\rho \to 1} d_\rho = d$ that

$$\limsup_{|z| \to 1} |\Phi(z)/z| \leq d.$$

By the maximum modulus principle, $|\Phi(z)| \leq |z| d$. Because $\Phi(z_2) = f(z_1) - f(z_2)$, it follows further that $d_r \leq rd$.

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It will be shown later that equality holds in Lemma 1 for some $r > 1$ if and only if $d = 2$ and $f(z) = z + b_0$.

LEMMA 2. *If* B *is convex, then* Γ_r, $r > 1$, *is a convex curve.*

Proof. The set B is convex if and only if $\Re\{1 + z f''(z)/f'(z)\} > 0$, $|z| > 1$; see [4, p. 47]. Thus the lemma follows from the maximum principle.

LEMMA 3. *If the curve* Γ *is convex and of diameter* d, *then its length* L *satisfies the inequality* $L \leq \pi d$.

Proof (*cf.* [1, p. 65]). If $h(\phi)$ is the supporting function of Γ, then the length L is given by $L = \int_0^{2\pi} h(\phi)\, d\phi$. Because $h(\phi) + h(\phi + \pi)$ is the width of Γ in the direction ϕ, it follows that $h(\phi) + h(\phi + \pi) \leq d$, and finally

$$L \leq \int_0^{\pi} h(\phi)\, d\phi + \int_0^{\pi} (d - h(\phi))\, d\phi = \pi d.$$

Observe that equality holds if and only if Γ is of constant width d.

We now turn to the uniqueness part of the proof of the theorem: *If* $d = 2$, *then* f *is a translation.* From the formula $f'(z) = 1 - b_1 z^{-2} - 2b_2 z^{-3} - \cdots$ for $|z| > 1$, it follows that

$$2\pi = \left| \int_{C_r} f'(z) \frac{dz}{z} \right| \leq \frac{1}{r} \int_{C_r} |f'(z)|\, |dz| = \frac{1}{r} L_r. \tag{1}$$

Assume now that the continuum B is not convex and has diameter d. Then its convex hull B_c still has diameter d, and there is a mapping

$$f_1(z) = az + a_0 + a_1 z^{-1} + \cdots, \qquad a > 1,$$

taking D^* onto $\overline{\mathbb{C}} \setminus B_c$. This shows that the continuum $B_c = B(f_1)$ has diameter d and transfinite diameter $a > 1$, and we conclude that the univalent function $a^{-1} f_1(z) = z + b_0 + b_1 z^{-1} + \cdots$ takes D^* onto a domain $\overline{\mathbb{C}} \setminus B_0$, where B_0 has diameter $d/a < d$. Hence, if $d = 2$, the continuum B must be convex. Lemmas 1, 2, and 3 show that equality occurs in (1). This implies $f'(z) = |f'(z)|$ for $z = re^{i\theta}$, and this holds only if f' is a constant; that is, if $f'(z) = 1$ and $f(z) = z + b_0$. This completes the proof of the theorem.

Remark 1. Equality holds in Lemma 1 for some $r > 1$ if and only if $f(z) = z + b_0$. This follows from the proof of Lemma 1 and from the theorem. If $d_r = dr$ for some $r > 1$, then $|z_2| d = |f(z_2) - f(z_1)| = |\Phi(z_2)|$ for suitably chosen z_1 and z_2. Hence, for some $\alpha \in \mathbb{R}$, $\Phi(z) = (e^{i\alpha} - 1)z + \cdots \equiv zd$. This implies $d = 2$ and $f(z) = z + b_0$.

Remark 2. If B is convex and of diameter d, then $L_r \leq \pi r d$ for $r > 1$. This follows immediately from the Lemmas 2, 3, and 1. Equality occurs for some $r > 1$ only if $d_r = rd$, hence only if $f(z) = z + b_0$.

Remark 3. Let $2 < d < 4$. Then the preceding remarks show that for $r > 1$ we have $d_r < rd$ if f is any univalent function $f(z) = z + b_0 + b_1 z^{-1} + \cdots$ mapping D^* onto $\overline{\mathbb{C}} \setminus B$, where B has diameter d. Moreover, $L_r < \pi r d$ if the continuum B with diameter d is convex. Which are the mappings maximizing d_r and L_r, respectively, for a fixed $r > 1$?

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