BROWDER-LIVESAY INDEX INVARIANT AND EQUIVARIANT KNOTS

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Let T be a differentiable fixed point free involution on a $(4n+3)$-dimensional homotopy sphere Σ^{4n+3}, denoted by (T, Σ^{4n+3}). We know that (T, Σ^{4n+3}) always admits an invariant $(4n+1)$-dimensional sphere S^{4n+1} [6, p. 81]. Furthermore, we may require that $(\Sigma^{4n+3}, S^{4n+1})$ be a simple knot [2].

As in [4] or [7], we may apply equivariant surgery in $X = \overline{\Sigma - (S \times D^2)}$ to obtain two $2n$-connected Seifert submanifolds V_1 and V_2 of dimension $(4n+2)$ such that $TV_1 = V_2$ and $\partial V_1 = S^{4n+1} \times \{0\}$, $\partial V_2 = S^{4n+1} \times \{\pi\}$. The set $V_1 \cup V_2$ divides X into two parts W_1 and W_2 with $TW_1 = W_2$.

Gluing V_1 and V_2 in the boundary of W_1 by the map $T: V_1 \to V_2$, we obtain a $(4n+3)$-manifold Y. Let $\Sigma_1 = Y \cup S^{4n+1} \times D^2$ by some PL-homeomorphism $h: \partial Y \to S^{4n+1} \times S^1$. We see that (Σ_1, S^{4n+1}) is a simple $(4n+3)$-knot. Choosing a basis $\{b_1, \cdots, b_m\}$ for $H_{2n+1}(V_1)$, we have a Seifert matrix A, which is the matrix for the mapping $j_1: H_{2n+1}(V_1) \to H_{2n+1}(W_1)$ with respect to the bases $\{b_i\}$ and $\{c_i\}$ determined by the Alexander duality. Let A^T be the transpose of A. Then $(-1)^{2n+2} A^T$ is the matrix for the mapping $j_2: H_{2n+1}(V_2) \to H_{2n+1}(W_1)$ with respect to the bases $\{T_* b_i\}$ and $\{c_i\}$.

From [3], we know that $A + (-1)^{2n+1} A^T = A - A^T$ is unimodular. But by using the same argument in [4], we can show that $A + A^T$ is also unimodular. For the involution (T, Σ^{4n+3}), Browder and Livesay defined an index invariant $\sigma(T, \Sigma^{4n+3})$. (For its definition, see [1] or [5].) The purpose of this note is to prove the following result.

THEOREM. $\sigma(T, \Sigma^{4n+3}) = \text{index}(A + A^T)$.

Proof. In Σ^{4n+3}, we construct an invariant submanifold M of codimension 1 as follows:

$$M = V_1 \cup S^{4n+1} \times re^0 \cup S^{4n+1} \times re^{i\pi} \cup V_2 .$$

It is easy to see that M is $2n$-connected, and $\{b_1, \cdots, b_m, T_* b_1, \cdots, T_* b_m\}$ forms a basis for $H_{2n+1}(M)$ by the natural inclusion $V_i \to M$, $i = 1$ or 2. M divides Σ^{4n+3} into two parts E_1 and E_2, with $TE_1 = E_2$. Under the inclusion $W_i \to E_i$, $i = 1$ or 2, we have a basis $\{c_i\}$ for $H_{2n+1}(E_1)$ and $\{T_* c_i\}$ for $H_{2n+1}(E_2)$.

Browder and Livesay [1] defined a symmetric bilinear form B on $H_{2n+1}(M)$ by $B(x, y) = x \cdot T_* y$. Since [3, p. 542] the $m \times m$ matrix $(b_i \cdot b_j) = A - A^T$, and $b_i \cdot T_* b_j = 0$, we see that B is represented by the matrix

$$\begin{pmatrix} 0 & A - A^T \\ A^T - A & 0 \end{pmatrix}$$

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with respect to the basis $\{b_1, \cdots, b_m, T_*b_1, \cdots, T_*b_m\}$.

Let $i_j: H_{2n+1}(M) \to H_{2n+1}(E_j)$, j = 1 or 2, denote the map induced by the inclusion. From the Mayer-Vietoris sequence

$$0 \to H_{2n+1}(M) \xrightarrow{(i_1, -i_2)} H_{2n+1}(E_1) \oplus H_{2n+1}(E_2) \to 0,$$

we have $(i_1, -i_2)$ represented by

$$\begin{pmatrix} A & A^T \\ A^T & A \end{pmatrix}$$

with respect to the bases $\{b_i, T_* b_i\}$ and $\{c_i\} \cup \{T_* c_i\}$.

Let D be the $2m \times 2m$ matrix

$$\frac{1}{2}\begin{pmatrix} (A + A^T)^{-1} + (A - A^T)^{-1}, & (A + A^T)^{-1} - (A - A^T)^{-1} \\ (A + A^T)^{-1} - (A - A^T)^{-1}, & (A + A^T)^{-1} + (A - A^T)^{-1} \end{pmatrix}.$$

It is easy to check that $D^{-1} = \begin{pmatrix} A & A^T \\ A^T & A \end{pmatrix}$.

We construct a new basis $\{a_i, T_* a_i\} = F$ for $H_{2n+1}(M)$ from $\{b_i, T_* b_i\} = G$ by $F = D \cdot G$. Then with respect to the bases $\{a_i, T_* a_i\}$ and $\{c_i\} \cup \{T_* c_i\}$, $(i_1, -i_2)$ is represented by the identity matrix. In particular, $\{a_i\}$ is a basis for kernel i_2. With respect to the new basis $\{a_i, T_* a_i\}$, the symmetric bilinear form B is given by the matrix

$$D\begin{pmatrix} 0 & A - A^T \\ A^T - A & 0 \end{pmatrix} D^T = \begin{pmatrix} (A + A^T)^{-1} & 0 \\ 0 & -(A + A^T)^{-1} \end{pmatrix}.$$

From [1, p. 75] or [5], we see that the Browder-Livesay index invariant

$$\sigma(T, \Sigma^{4n+3}) = \text{index}(A + A^T)^{-1} = \text{index}(A + A^T). \qquad \textit{q.e.d.}$$

From [3, p. 544], we see that the Kervaire invariant of V_1 is the Arf invariant of A. Since $A + A^T$ is a symmetric, even, unimodular matrix, Lemma 2 in [6, p. 36] shows that the Arf invariant of A is zero. Hence, S^{4n+1} is standard [6, p. 27]. But the same proof also works [4] for semifree involutions $(T, \Sigma^{4n+3}, S^{4n+1})$. Using the fact that $(T, \Sigma^{4n+3}, S^{4n+1})$ can be made equivariant knot-cobordant to a simple one ([2] or [7]), we have the following corollary.

COROLLARY. *Let* S^{4n+1} *be a homotopy sphere embedded as the fixed point set of some semifree involution* T *on a homotopy sphere* Σ^{4n+3}. *Then* S^{4n+1} *is standard.*

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