INDUCED COBORDISM THEORIES--AN EXAMPLE

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1. INTRODUCTION

The object of this note is to describe a way to construct new cobordism theories, basically by means of one example. First, recall that if M is a manifold, then the total space of the tangent bundle of M is an almost complex manifold, for $\tau(E(\tau_M)) \cong \pi^* \tau_M \oplus \pi^* \tau_M \cong \pi^* \tau_M \otimes_R \mathbb{C}$. If M is also a complex manifold, then τ_M is a complex bundle, so $\tau(E(\tau_M))$ is the complexification of a complex bundle and thus is a quaternionic (or symplectic) vector bundle.

One then introduces the notion of a *weakly weakly almost complex manifold* as a manifold together with a quaternionic structure on the complexification of the normal bundle. In bundle-theoretic terms, there is a fibering $BSp \to BU$ obtained by considering a quaternionic bundle as just a complex bundle, and a map $\otimes\mathbb{C}$: $BO \to BU$ obtained by classifying the complexification of the universal bundle. One may then form the induced fibering

$$\begin{array}{ccc} B & \longrightarrow & BSp \\ {\scriptstyle f}\downarrow & & \downarrow \\ BO & \xrightarrow{\otimes\mathbb{C}} & BU \end{array} ,$$

and a weakly weakly almost complex manifold is a manifold M together with a chosen equivalence class of liftings of the normal map ν: $M \to BO$ to B. (See Lashof [2] for the precise formalism of manifold with (B, f)-structure.)

Noting that the complexification of a complex bundle is quaternionic shows that the composite $BU \xrightarrow{\pi} BO \xrightarrow{\otimes\mathbb{C}} BU$ lifts to BSp, and hence every weakly almost complex manifold (for which ν lifts to BU) is weakly weakly almost complex.

Following Lashof, one may introduce the cobordism group $\Omega_*^{(B,f)}$ of weakly weakly almost complex manifolds. The main result of this paper is then:

THEOREM. *The forgetful homomorphism* F: $\Omega_*^{(B,f)} \to \mathfrak{N}_*$ *into unoriented cobordism is monic, and one may choose generators* x_i *of* $\mathfrak{N}_* = Z_2[x_i\colon i \neq 2^s - 1]$ *so that the image of* F *is the polynomial subalgebra on the* x_i (i *odd*) *and* x_i^2 (i *even*).

Note. The image of the complex cobordism ring Ω_*^U in $\mathfrak{N}_*$ is the polynomial subalgebra consisting of the squares (Milnor [3]). The odd-dimensional generators needed may be taken to be U/O manifolds in the sense of Smith-Stong [4]; *i.e.*, manifolds for which the complexification of the normal bundle is trivial.

The results will include a general structure theorem (Remark following Lemma 3.1) showing that many theories are 2-torsion, and an analysis of Wall's cobordism theory W_* (section 4) in a form similar to weakly weakly almost complex cobordism.

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2. COHOMOLOGY OF B

Following the procedure established by Thom, one first computes the cohomology of B.

First, one has the fibering π: BSp $\to$ BU which has fiber U/Sp. The cohomology of BU with integer coefficients is $Z[c_i]$, where $c_i \in H^{2i}(BU; Z)$ is the universal Chern class $(i > 0)$, and that of BSp is $Z[\mathscr{P}_i^s]$, where $\mathscr{P}_i^s$ is the symplectic Pontrjagin class in $H^{4i}(BSp; Z)$. Then $\pi^*(c_{2i+1}) = 0$ and $\pi^*(c_{2i}) = \pm\mathscr{P}_i^s$ (the sign depending on your choice of definition). Since BU is simply connected, the Serre spectral sequence for π has a trivial coefficient system, and $H^*(U/Sp; Z)$ is the exterior algebra over Z on classes z_{4i+1}, $i \geq 0$, with z_{4i+1} transgressing to c_{2i+1}.

Note. One may also consider $Sp \xrightarrow{i} U \to U/Sp$, where i^* sends the exterior algebra on classes a_{2i+1} onto the exterior algebra on classes b_{4j+3} $(i, j \geq 0)$.

Now considering the induced fibering $U/Sp \to B \xrightarrow{f} BO$, the coefficient system is again trivial. In mod 2 cohomology, $H^*(BO; Z_2)$ is the Z_2 polynomial ring on the Stiefel-Whitney classes $w_i \in H^i(BO; Z_2)$. The mod 2 reduction of the Chern class c_i is w_{2i} and $w(\gamma \otimes \mathbb{C}) = w(\gamma \oplus \gamma) = w(\gamma)^2$, so $(\otimes\mathbb{C})^*(w_{2i}) = w_i^2$. Thus, in the fibration f, z_{4i+1} transgresses to w_{2i+1}^2 and one has

LEMMA 2.1. *The map* f: B $\to$ BO *induces an epimorphism in* mod 2 *cohomology, identifying* $H^*(B; Z_2)$ *with* $Z_2[w_i]/(w_{2i+1}^2 = 0)$.

Now, letting p be an odd prime, one considers the diagram

$$\begin{array}{ccccccc} & & & & B & \longrightarrow & BSp \\ & & & & \downarrow f & & \downarrow \pi \\ BSp & \xrightarrow{\pi} & BU & \xrightarrow{\pi'} & BO & \xrightarrow{\otimes\mathbb{C}} & BU \end{array},$$

in which $(\otimes\mathbb{C}) \circ \pi'$ and $(\otimes\mathbb{C}) \circ \pi' \circ \pi$ both lift to BSp. Since π is a principal fibration (U/Sp may be realized as a group), one has an induced fibering

$$\begin{array}{ccc} BSp \times U/Sp & \longrightarrow & B \\ \pi_1 \downarrow & & \downarrow f \\ BSp & \xrightarrow{\pi' \circ \pi} & BO \end{array}$$

Again the coefficient systems are trivial and $\pi' \circ \pi$ induces an isomorphism in Z_p cohomology, so $H^*(B; Z_p) \cong H^*(BO; Z_p) \otimes_{Z_p} H^*(U/Sp; Z_p)$.

3. CALCULATION OF THE COBORDISM

Beginning with the fibration f: B $\to$ BO, one forms the induced fibering $f_n: B_n \to BO_n$, and if γ_n is the universal n-plane bundle over BO_n, one lets TB_n

be the Thom space of $f_n^*(\gamma_n)$. From the Pontrjagin-Thom theorem (Lashof [2]), the cobordism group $\Omega_*^{(B,f)}$ may be identified with the homotopy of the spectrum $\{TB_n\}$. Since π: BSp $\to$ BU is an H-map, B is an H-space and f is an H-map (*i.e.*, if $\xi \otimes \mathbb{C}$ and $\eta \otimes \mathbb{C}$ have quaternionic structures, so does $(\xi \oplus \eta) \otimes \mathbb{C}$), so that $\{TB_n\}$ is a ring spectrum, giving $\Omega_*^{(B,f)}$ the structure of a graded ring. The map on Thom spaces T_f: $TB_n \to T_n$ is a map of ring spectra inducing the forgetful homomorphism F: $\Omega_*^{(B,f)} \to \mathfrak{N}_*$, a ring homomorphism.

Now considering the inclusion i: $S(\gamma_n) \to D(\gamma_n)$ of the sphere bundle in the disc bundle, the projection identifies $D(\gamma_n)$ with BO_n, and $S(\gamma_n)$ may be identified with BO_{n-1}, so that the inclusion i may be identified with the usual map $BO_{n-1} \to BO_n$ classifying the Whitney sum with a trivial bundle (see [5], page 72). In particular, taking induced bundles, one has a cofibration

$$\begin{array}{ccccc} B_{n-1} & \xrightarrow{j_{n-1}} & B_n & & \\ \wr\Vert & & \wr\Vert & & \\ S(f_n^*\gamma_n) & \longrightarrow & D(f_n^*\gamma_n) & \longrightarrow & TB_n \,. \end{array}$$

Letting p be an odd prime and n an odd integer, one has the fiberings

$$\begin{array}{ccc} B_{n-1} & \xrightarrow{j_{n-1}} & B_n \\ {\scriptstyle f_{n-1}}\downarrow & & \downarrow{\scriptstyle f_n} \\ BO_{n-1} & \xrightarrow{i_{n-1}} & BO_n \end{array}$$

with fiber U/Sp, with i_{n-1} being an isomorphism on Z_p cohomology, and with trivial coefficient system. Thus j_{n-1} induces an isomorphism on Z_p cohomology and $\tilde{H}^*(TB_n; Z_p) = 0$ for n odd. Thus the Z_p cohomology of the Thom spectrum $\{TB_n\}$ is zero, giving

LEMMA 3.1. *The groups* $\Omega_*^{(B,f)}$ *are entirely* 2-*torsion.*

Note. Any fibration f: B $\to$ BO with trivial coefficient system gives a 2-primary cobordism theory using this argument. In particular, this holds whenever f is induced from a fibering over a simply connected space.

To study the 2-primary structure, one considers Z_2 cohomology for which there is a Thom isomorphism. Thus $\tilde{H}^*(TB; Z_2) \cong Z_2[w_i]/(w_{2i+1}^2 = 0) \cdot U$ is a free $H^*(B; Z_2)$ module of rank 1 on the Thom class U. Since TB = $\{TB_n\}$ is a ring spectrum, $\tilde{H}^*(TB; Z_2)$ is a connected coalgebra over the Hopf algebra $\mathscr{A}$ (the mod 2 Steenrod algebra). Let ν: $\mathscr{A} \to \tilde{H}^*(TB; Z_2) : \alpha \to \alpha(U)$ denote the action on the Thom class.

LEMMA 3.2. ν: $\mathscr{A} \to \tilde{H}^*(TB; Z_2)$ *is monic.*

Proof. Let U/O $\otimes\, \mathbb{C}$ denote the fiber of the map $\otimes\mathbb{C}$: BO $\to$ BU. The inclusion i: U/O $\otimes$ C $\to$ BO then lifts to B, giving maps U/O $\otimes\, \mathbb{C} \xrightarrow{j} B \xrightarrow{f} BO$, and induced maps of spectra. In Smith-Stong [4], it was verified that the $\mathscr{A}$ action on the Thom class in T(U/O $\otimes\, \mathbb{C}$) is monic, and hence ν is monic.

Following the ideas of Browder, Liulevicius, and Peterson [1], $\tilde{H}^*(TB; Z_2)$ is a free $\mathscr{A}$ module and TB has the homotopy type of a product of Eilenberg-MacLane spectra $K(Z_2)$. Since f is epic in mod 2 cohomology, one then has:

PROPOSITION 3.3. *The forgetful homomorphism* $F: \Omega_*^{(B,f)} \to \mathfrak{N}_*$ *is a monomorphism.*

To determine the image of F, one recalls (from Smith-Stong [4]) that generators for $\mathfrak{N}_* = Z_2[x_i: i \neq 2^s - 1]$ may be chosen so that x_{2i+1} is a $U/O \otimes \mathbb{C}$ manifold, and following Milnor [3], the image of complex cobordism in $\mathfrak{N}_*$ consists of the squares. Since $U/O \otimes \mathbb{C}$ manifolds and weakly almost complex manifolds are weakly weakly almost complex, this choice of generators shows that the image of F contains $X = Z_2[x_{2i+1}, x_{2j}^2: 2i + 1 \neq 2^s - 1]$.

Now the Steenrod algebra has the same dimension (degree by degree) as a Z_2 polynomial algebra on classes x_{2^s-1}, so $X \otimes \mathscr{A}$ has the same dimension as $Z_2[x_{2i+1}, x_{2j}^2]$, which is identical with the dimension of

$$\tilde{H}^*(TB; Z_2) \cong Z_2[w_i]/(w_{2i+1}^2 = 0).$$

Thus, the homotopy of TB has the same dimension as X in each degree, and image F = X, completing the proof of

THEOREM 3.4. *The forgetful homomorphism* $F: \Omega_*^{(B,f)} \to \mathfrak{N}_*$ *into unoriented cobordism is monic, and one may choose generators* x_i *of* $\mathfrak{N}_* = Z_2[x_i: i \neq 2^s - 1]$, *so that the image of* F *is the polynomial subalgebra on the* x_i (i *odd*) *and* x_i^2 (i *even*).

4. REMARK ON WALL MANIFOLDS

Surprisingly, the notion of weakly weakly almost complex manifolds is very similar to the manifolds introduced by Wall [6] in computing oriented cobordism. Specifically, Wall considered manifolds together with a reduction of the first Stiefel-Whitney class to an integral class.

To see the analogy, consider a manifold M for which the complexification of the normal bundle has a special unitary structure. In bundle-theoretic terms, one has the induced fibering

$$\begin{array}{ccc} B' & \longrightarrow & BSU \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \pi} \\ BO & \xrightarrow{\otimes \mathbb{C}} & BU \end{array}.$$

The fibering $f': B' \to BO$ is exactly the fibration which gives rise to Wall manifolds.

First one notes that π is induced from the universal bundle $\pi': ES^1 \to BS^1$ by the map $\det: BU \to BS^1$ which classifies the determinant bundle, but

$$\det \circ (\otimes \mathbb{C}) = (\otimes \mathbb{C}) \circ \det,$$

and so f' is induced by

$$\begin{array}{ccccc} B' & \longrightarrow & B'' & \longrightarrow & ES^1 \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f''}\downarrow & & \downarrow \\ BO & \xrightarrow{\det} & BZ_2 & \xrightarrow{\otimes C} & BS^1 \end{array} .$$

If λ denotes the universal line bundle over $BZ_2 = RP(\infty)$, B'' is the sphere bundle of $\lambda \otimes C = \lambda \oplus \lambda$ and f'' may be identified with the projection

$$S^\infty \times S^1/{-1} \times {-1} \to S^\infty/{-1}.$$

The projection $S^\infty \times S^1/{-1} \times {-1} \to S^1/{-1} = S^1$ is a homotopy equivalence, and so f'' is just the usual map $K(Z, 1) \to K(Z_2, 1)$. A lifting to B' is then precisely a reduction of the first Stiefel-Whitney class to an integral class.

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