SYMMETRIC POWERS AND LEFSCHETZ NUMBERS

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0. INTRODUCTION

In this note we point out several ramifications of a result of Dold's [1] concerning the Lefschetz numbers of the symmetric powers of a map. In [2], many consequences were deduced from a connection between the Lefschetz numbers of iterates of a map and a certain characteristic rational function. In [1], Dold provides a similar connection between the Lefschetz numbers of symmetric powers of a map and this same characteristic function. Consequently, a portion of [2] can be carried over to symmetric powers. Theorem 3.1 is an answer to a question raised by Dold in [1]. The author is indebted to A. Dold for stimulating conversations and for shortening some of the proofs.

1. NOTATION AND CONVENTIONS

We denote the rationals by $\mathscr{R}$. Homology is denoted by H and coefficients are taken in $\mathscr{R}$. If f is a linear self-mapping of a finite dimensional vector space over $\mathscr{R}$, then $X(f) = X(f;\, t)$ denotes its characteristic polynomial. Throughout this note Y is a compact CW-space and $g: Y \to Y$ is a continuous map.

$$\Lambda(g) = \mathrm{trace}((Hg)_{\mathrm{even}}) - \mathrm{trace}((Hg)_{\mathrm{odd}})$$

is the Lefschetz number of g, and $X(g) = X((Hg)_{\mathrm{even}})/X((Hg)_{\mathrm{odd}})$ is its characteristic rational function. $X(g)$ is an element of $\mathscr{R}(t)^*$, the multiplicative group of the field $\mathscr{R}(t)$ of rational functions over $\mathscr{R}$ in one indeterminate t. The Euler characteristic of Y is denoted by eY.

The nth symmetric power of Y is $P^{S(n)}Y = Y^n/\sim$, where two elements a and b of Y^n are equivalent under $\sim$ provided some permutation of the coordinates takes a to b. The nth symmetric power of g is the map $P^{S(n)}(g)\colon P^{S(n)}Y \to P^{S(n)}Y$, induced from $\bar{g}\colon Y^n \to Y^n$, where $\bar{g}(y_1, \cdots, y_n) = (g(y_1), \cdots, g(y_n))$.

We denote the nth iterate of g by $g^n = g \circ g \circ \cdots \circ g$, n times.

2. PRELIMINARY RESULTS

First we state Theorem 5.9 of Dold [1].

THEOREM 2.1 (Dold). $$\left[t^{eY} X\left(g; \frac{1}{t}\right)\right]^{-1} = \sum_{n=0}^{\infty} \Lambda(P^{S(n)}(g))\, t^n.$$

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Next we state the result from Kelley-Spanier [3], on which [2] is based. This result essentially goes back to Newton.

THEOREM 2.2. $$-\frac{t\frac{d}{dt}\left(t^{eY}X\left(g;\frac{1}{t}\right)\right)}{t^{eY}X\left(g;\frac{1}{t}\right)} = \sum_{n=1}^{\infty} \Lambda(g^n)\, t^n .$$

We also note that if we write

$$t^{eY}X\left(g;\frac{1}{t}\right) = \sum_{n=0}^{\infty} a_n t^n ,$$

then $a_0 = 1$ and for $n \geq 1$, the a_n are the canonical coefficients as defined in Halpern [2].

3. RESULTS

Our first two results are concerned with how certain Lefschetz numbers and characteristic functions can be explicitly calculated from other such information. The theoretical determination of these numbers and functions is established in Dold [1].

THEOREM 3.1. *There is an explicit way (described below) to calculate* $X(P^{S(n)}(g))$ *in terms of* $X(g)$. *(We consider* eY *as known.)*

Proof. The equation of Theorem 2.2 means

$$-n\,a_n = \ell_n + a_1\,\ell_{n-1} + a_2\,\ell_{n-2} + \cdots + a_{n-1}\,\ell_1 \qquad \text{for } n \geq 1,$$

where $\ell_i = \Lambda(g^i)$. These equations may be solved recursively for the a_n's in terms of the ℓ_n's. Hence, Theorem 2.2 gives an effective means of calculating the $\Lambda(g^n)$'s in terms of $X(g)$, and vice versa.

By "determines effectively" we will mean "gives an effective procedure for calculating".

Suppose now we are given $X(g)$. Theorem 2.2 determines effectively the numbers $\Lambda(g^n)$ for $n \geq 0$, and hence the numbers $\Lambda((g^m)^n)$ for $n, m \geq 0$. Theorem 2.2 now determines effectively $X(g^m)$ for $m \geq 0$. Next, Theorem 2.1 determines effectively $\Lambda(P^{S(n)}(g^m))$ for all $n, m \geq 0$. Since $P^{S(n)}$ is a functor,

$$P^{S(n)}(g^m) = (P^{S(n)}(g))^m .$$

Finally, Theorem 2.2 determines effectively $X(P^{S(n)}(g))$ for all $n \geq 0$. (By Theorem 5.7 of Dold [1], $eP^{S(n)}(Y) = \binom{eY + n - 1}{n}$.)

THEOREM 3.2. *Given bounds on* $\dim H(Y)_{\text{even}}$ *and* $\dim H(Y)_{\text{odd}}$, *that is,* $\dim H(Y)_{\text{even}} \leq p$ *and* $\dim H(Y)_{\text{odd}} \leq q$, *there is an explicit way (described below) to calculate* $X(g)$ *and hence all* $\Lambda(P^{S(n)}(g))$, *for* $n \geq 0$, *in terms of the numbers* $\Lambda(P^{S(n)}(g))$ *for* $0 \leq n \leq p + q$. *(We consider* eY *as known.)*

Proof. First we will show that there is exactly one rational function $R(t)$ of the form $R(t) = \frac{P(t)}{Q(t)}$, where P and Q are monic polynomials with $\deg P \leq p$ and

$\deg Q \leq q$, such that if

$$(*) \qquad \frac{P(t)}{Q(t)} = \sum_{n=0}^{\infty} b_n t^n,$$

then $b_n = \Lambda(P^{S(n)}(g))$ for $0 \leq n \leq p+q$.

It is easy to see from Theorem 2.1 that $\overline{X} = \left[t^{eY} X\left(g; \frac{1}{t}\right) \right]^{-1}$ is one such rational function. Suppose $R = \frac{P}{Q}$ and $\overline{R} = \frac{\overline{P}}{\overline{Q}}$ are two such rational functions. Then

$$\frac{P(t)}{Q(t)} = \frac{\overline{P}(t)}{\overline{Q}(t)} + t^{p+q+1} Z(t)$$

for some rational function $Z(t) = \sum_{n=0}^{\infty} z_n t^n$. It follows that

$$P(t)\overline{Q}(t) - \overline{P}(t)Q(t) = t^{p+q+1} Z(t)Q(t)\overline{Q}(t).$$

The left-hand side is a polynomial whose degree is less than or equal to $p+q$. This implies $Z(t)Q(t)\overline{Q}(t) = 0$. Since Q and $\overline{Q}$ are monic, we can conclude that $Z(t) = 0$. Hence $R = \frac{P}{Q} = \frac{\overline{P}}{\overline{Q}} = \overline{R}$, and uniqueness is established.

Now that we know that $\overline{X}$ is the only rational function of the form $\frac{P}{Q}$ with P and Q satisfying (*), we can effectively calculate $\overline{X}$, and hence X, by effectively calculating any two polynomials P and Q which satisfy (*). Let $P_0 = Q_0 = 1$, let $P(t) = \sum_{n=0}^{p} P_n t^n$, and let $Q(t) = \sum_{n=0}^{q} Q_n t^n$; then condition (*) reduces to the following set of equations.

$$E_n\colon P_n = \sum_{i=0}^{n} \Lambda(P^{S(n-i)}(g))\, Q_i \qquad \text{for } 1 \leq n \leq p+q,$$

where $P_n = 0$ for $p+1 \leq n \leq p+q$. From basic linear algebra we know that we can effectively calculate a solution $Q_1, Q_2, \cdots, Q_q$ to the equations E_{p+1}, $E_{p+2}, \cdots, E_{p+q}$, because at least one solution exists. Now the equations E_1, $E_2, \cdots, E_p$ effectively determine $P_1, P_2, \cdots, P_p$. This completes the proof.

The proofs of the next three theorems are so closely parallel to the proofs of the analogous results in Halpern [2], that we only indicate the analogous results and leave the proofs to the reader.

A. Dold has observed that if in Theorem 3.2 we do not assume eY as known, then we may still calculate $X(g)$ up to a factor t^j. This factor does not enter into the calculation of $\Lambda(P^{S(n)}(g))$, and hence one can calculate all the $\Lambda(P^{S(n)}(g))$ in terms of $\Lambda(P^{S(i)}(g))$ with $0 \leq i \leq p+q$ without knowing eY.

THEOREM 3.3 (Analogous to Theorem 4 of [2]). *If* $(Hg)_{odd}$ *is nilpotent, then* $\Lambda(P^{S(n)}(g)) \neq 0$ *for some* n, *where* $1 \leq n \leq \dim H(Y)_{even}$.

THEOREM 3.4 (Analogous to Corollary 9 of [2]). *If* Hg *is an isomorphism and* $eY \neq 0$, *then*

$$\Lambda(P^{S(n)}(g)) \neq 0 \text{ for some } n,$$

where $1 \leq n \leq \max(\dim H(Y)_{even}, \dim H(Y)_{odd})$.

THEOREM 3.5 (Analogous to Theorem 30 of [2]). *Suppose* $\bar{Y}$ *is another* CW-*space and* $\bar{g}: \bar{Y} \to \bar{Y}$ *another continuous map. If* Hg *and* $H\bar{g}$ *are isomorphisms and* $\Lambda(P^{S(n)}(g)) = \Lambda(P^{S(n)}(\bar{g}))$ *for all* n, *then* $eY = e\bar{Y}$.

Our last theorem can be geometrically motivated by comparing the numbers of fixed points for g^n and $P^{S(n)}(g)$ under the supposition that $g^1, g^2, \cdots, g^{n-1}$ are fixed point free.

THEOREM 3.6. *The smallest* $n \geq 1$ *such that* $\Lambda(g^n) \neq 0$ *is equal to the smallest* $n \geq 1$ *such that* $\Lambda(P^{S(n)}(g)) \neq 0$. *Let* m *be this common value. Then* $n\Lambda(P^{S(n)}(g)) = \Lambda(g^n)$ *for* $m \leq n \leq 2m - 1$.

Proof. Let $A = A(t) = t^{eY} X\left(g; \frac{1}{t}\right)$, let $B = B(t) = \sum_{n=1}^{\infty} \Lambda(g^n)t^n$, and let $C = C(t) = \sum_{n=0}^{\infty} \Lambda(P^{S(n)}(g))t^n$. Theorems 2.1 and 2.2 may now be written as

$$A^{-1} = C \quad \text{and} \quad -t\frac{A'}{A} = B.$$

It follows that $t\frac{C'}{C} = B$. Since $\Lambda(P^{S(0)}(g)) = 1$, the theorem now follows.

REFERENCES

1. A. Dold, K-*theory of non-additive functors of finite degree.* Math. Ann. 196 (1972), 177-197.

2. B. Halpern, *Fixed points for iterates.* Pacific J. Math. 25 (1968), 255-275.

3. J. L. Kelley and E. H. Spanier, *Euler characteristics.* Pacific J. Math. 26 (1968), 317-339.

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