SUBORDINATION AND INSUPERABLE ELEMENTS

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1. INTRODUCTION, NOTATION, AND OUTLINE OF RESULTS

Given a family of holomorphic functions on the unit disk, we wish to investigate the subset of functions that are maximal with respect to the relation of subordination. A little more explicitly, we are interested in those functions f which are subordinate to no other function in the family except for any "rotations" of f that may be present. We shall refer to these maximal functions as the *insuperable elements* of the family. In Section 2 of the paper we describe the basic domination properties of the set of insuperable elements in a compact family. The most fundamental of these properties (Theorem 1) is the expected result that every function in the family is subordinate to some insuperable element. The other properties, which will be stated in detail in Section 2, are roughly described by the statement that certain extremal problems are solved by the insuperable elements. Two very different types of problems are discussed.

General results aside, we would like to identify the insuperable elements in certain specific families. On this project, however, we can only record a few obvious conclusions, list the outstanding problems, and partially solve two of the latter. Section 3, the longest and most difficult part of the paper, is devoted to these partial solutions. The families that interest us are obtained from the most familiar classes of univalent functions either by differentiation or by division by z. An example of the first possibility is the family of derivatives of (normalized) starlike functions. The problem of determining the insuperable elements of this family is a natural one as a result of the disproof of the Marx conjecture by J. A. Hummel [6]. We shall not investigate this problem, but we partially solve the corresponding problem for the family C' of derivatives of close-to-convex functions. We are concerned here also with an example of the second mentioned possibility, the family C/z of close-to-convex functions divided by z. Perspective for our study of this family comes from two well known subordination facts: If f and g belong to the simpler families K/z and S^*/z respectively (classes of convex and starlike functions divided by z), then $f(z) \prec 1/(1 - z)$ and $g(z) \prec 1/(1 - z)^2$. With obvious notation for sets of insuperable elements, we note that these results imply the equations

$$I(K/z) = \{1/(1 - yz): |y| = 1\} \quad \text{and} \quad I(S^*/z) = \{1/(1 - yz)^2: |y| = 1\},$$

while the converse holds by our Theorem 1. These old theorems together with the information in [2] on extreme points have led W. E. Kirwan and one of the present authors to ask whether every function in C/z is subordinate to some function in $E(\overline{co}\ C/z)$, the set of extreme points of the closed convex hull of C/z. (Since $E(\overline{co}(C/z)) = (E(\overline{co}\ C))/z = E((\overline{co}\ C)/z)$, we need not be careful with parentheses.) In Section 3, we answer this question negatively by means of a specific example. It then follows from Theorem 1 that I(C/z) is not contained in $E(\overline{co}\ C/z)$. We prove, however, that the reverse containment is correct, that is, that $E(\overline{co}\ C/z) \subset I(C/z)$. Similarly, we show that $E(\overline{co}\ C') \subset I(C')$. Actually, we obtain general results that

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imply $E(G) \subset I(G)$, where G denotes either $\overline{co}\, C/z$ or $\overline{co}\, C'$. Incidentally, we obtain strong evidence for the conjecture that the set of support points of C coincides with $E(\overline{co}\, C)$. Here, the paper [5] is of major importance.

Our notation and terminology is as follows. We shall let $\mathbb{C}$ denote the set of complex numbers, $\mathbb{R}$ the set of real numbers, Δ the unit disk $\{z \in \mathbb{C}: |z| < 1\}$, and Γ the unit circle $\{z \in \mathbb{C}: |z| = 1\}$. We denote by $H(\Delta)$ the linear space of complex-valued functions holomorphic in Δ. We shall let Ω denote the collection of functions ω in $H(\Delta)$ satisfying $|\omega(z)| \leq |z|$ for each z in Δ. As usual, S denotes the class of univalent functions f in $H(\Delta)$ satisfying $f(0) = 0$ and $f'(0) = 1$. The subclasses of convex, starlike, and close-to-convex functions are denoted by K, S^*, and C respectively.

For $f \in H(\Delta)$ and $|x| \leq 1$, f_x is the function (in $H(\Delta)$) defined by $f_x(z) = f(xz)$. If $|x| = 1$, we call f_x a *rotation of* f. The notation $f \prec g$ means that $f, g \in H(\Delta)$ and f is subordinate to g in Δ; that is, there exists $\omega \in \Omega$ such that $f = g \circ \omega$. If $\mathscr{F} \subset H(\Delta)$, $\mathscr{F}'$ denotes the set of derivatives of functions in $\mathscr{F}$; and $\mathscr{F}/z$ is the set of functions in $\mathscr{F}$, each divided by z. All topological notions (for example, "compact" or "$f_n \to f$") refer to the usual topology of uniform convergence on compact subsets of Δ. For $\mathscr{F} \subset H(\Delta)$, $\overline{co}\, \mathscr{F}$ denotes the closure of the convex hull of $\mathscr{F}$, and $E(\overline{co}\, \mathscr{F})$ is the set of extreme points of $\overline{co}\, \mathscr{F}$. A *support point* of a family $\mathscr{F}$ is a function $g \in \mathscr{F}$ for which there exists a continuous linear functional J on $H(\Delta)$ such that $\Re J$ is nonconstant on $\mathscr{F}$, and $\Re J(g) \geq \Re J(f)$ for every $f \in \mathscr{F}$.

2. GENERAL RESULTS

The principal definition of the paper is the following.

Definition 1. Let $\mathscr{F} \subset H(\Delta)$, and let $f \in \mathscr{F}$. Then f is *insuperable* in $\mathscr{F}$ if $h \in \mathscr{F}$ and $f \prec h$ imply that h is a rotation of f. The set of insuperable elements of $\mathscr{F}$ is denoted by $I(\mathscr{F})$.

Examples. $I(S) = S$, $I(K/z) = \{1/(1 - yz): |y| = 1\}$, and

$$I(K') = I(S^*/z) = \{1/(1 - yz)^2: |y| = 1\}.$$

For classes of spirallike functions divided by z, the insuperable elements are analogous to those of S^*/z. (See, for example, [7], p. 506.) To the best of our knowledge, the following are unknown: $I(S/z)$, $I(S')$, $I((S^*)')$, $I(C')$, and $I(C/z)$. We investigate the last two of these in Section 3. Regarding $(S^*)'$, the Marx conjecture can be stated as the equation

$$I((S^*)') = \left\{ \frac{d}{dz} \frac{z}{(1 - yz)^2}: |y| = 1 \right\}.$$

As mentioned earlier, this was disproved in [6], so $I((S^*)')$ is larger than the indicated set of derivatives. (If $g(z) = \frac{d}{dz}[z/(1 - yz)^2]$, then $g \in I((S^*)')$, because $|g'(0)|$ is maximal.)

We shall establish the fundamental dominating property of $I(\mathscr{F})$, for a compact family $\mathscr{F}$, immediately after the following simple lemma. The lemma asserts that subordination is preserved upon passage to the limit.

LEMMA 1. *Let* $f_n \prec h_n$ $(n = 1, 2, \cdots)$, *let* $f_n \to f$, *and let* $h_n \to h$. *Then* $f \prec h$.

Proof. Let $f_n = h_n \circ \omega_n$, with $\omega_n \in \Omega$. After passing to a subsequence of $\{\omega_n\}$, we can assume $\omega_n \to \omega \in \Omega$. Then the required result that $h_n \circ \omega_n \to h \circ \omega$ follows easily from the inequality

$$|h_n \circ \omega_n - h \circ \omega| \leq |h_n \circ \omega_n - h \circ \omega_n| + |h \circ \omega_n - h \circ \omega|.$$

THEOREM 1. *Let* $\mathscr{F} \subset H(\Delta)$, *where* $\mathscr{F}$ *is compact. Then for each* $f \in \mathscr{F}$, *there exists* $g \in I(\mathscr{F})$ *such that* $f \prec g$.

Proof. Let $f \in \mathscr{F}$ and let $\mathscr{F}_f = \{h \in \mathscr{F}: f \prec h\}$. We shall show that $\mathscr{F}_f$ contains an insuperable element of $\mathscr{F}$. In the (trivial) case where $\mathscr{F}_f$ contains only constant functions, f must be constant, and $\mathscr{F}_f = \{f\}$. Therefore $f \in I(\mathscr{F})$, and we can choose $g = f$. If $\mathscr{F}_f$ contains nonconstant functions, we define m to be the smallest positive integer n satisfying $h^{(n)}(0) \neq 0$ for some $h \in \mathscr{F}_f$. Now, by Lemma 1 and the compactness of $\mathscr{F}$, $\mathscr{F}_f$ is compact. Therefore we can choose $g \in \mathscr{F}_f$ with $|g^{(m)}(0)| \geq |h^{(m)}(0)|$ for all $h \in \mathscr{F}_f$. We claim that $g \in I(\mathscr{F})$ as required. To prove this, we assume $g = h \circ \omega$, where $h \in \mathscr{F}$ and $\omega \in \Omega$, and we show $\omega(z) = \lambda z$, with $|\lambda| = 1$. Since $f \prec g$ and $g \prec h$, it follows that $h \in \mathscr{F}_f$. Thus the equation $g = h \circ \omega$ implies that $g^{(m)}(0) = h^{(m)}(0)\, \omega'(0)$. By our choice of g, we conclude that $|\omega'(0)| \geq 1$. But then Schwarz' lemma yields the desired result that $\omega(z) = \lambda z$ with $|\lambda| = 1$.

The second dominating property of $I(\mathscr{F})$ that we shall describe requires that $\mathscr{F}$, besides being compact, be rotation invariant in the following sense.

Definition 2. A family $\mathscr{F} \subset H(\Delta)$ is *rotation invariant* if $f \in \mathscr{F}$ and $|x| = 1$ imply that $f_x \in \mathscr{F}$.

Examples. Let G equal S, S^*, K, C, or any other subset of $H(\Delta)$ with the property that $g \in G$ and $|x| = 1$ imply that $x^{-1} g_x \in G$. Then both G' and G/z are rotation invariant. More generally, the operator $z^{n-1}(d/dz)^n$ $(n = 0, 1, 2, \cdots)$ takes G into a rotation invariant family.

THEOREM 2. *Let* $\mathscr{F}$ *be a compact and rotation invariant subset of* $H(\Delta)$. *Let* $z_0 \in \Delta \setminus \{0\}$, *and let* F *be entire and nonconstant. Then each of the functionals* $f \to \Re F(f(z_0))$ *and* $f \to |F(f(z_0))|$ *achieves its maximum over* $\mathscr{F}$ *only on* $I(\mathscr{F})$.

Proof. We generalize an idea of T. H. MacGregor [7, Theorem 3]. Suppose $f \in \mathscr{F} \setminus I(\mathscr{F})$. Then there exists $g \in \mathscr{F}$ such that $f \prec g$ and g is not a rotation of f. It follows that g is nonconstant, and therefore so is $F \circ g$. Let $f = g \circ \omega$ with $\omega \in \Omega$. Then $|\omega(z_0)| < |z_0|$. Hence the maximum principle gives

$$\Re\,(F \circ g)(\omega(z_0)) < \max_{|x|=1} \Re\,(F \circ g)(xz_0) = \Re\,(F \circ g)(yz_0)$$

for some y with $|y| = 1$. Rewriting this inequality, we obtain $\Re F(f(z_0)) < \Re F(g_y(z_0))$ with $g_y \in \mathscr{F}$. This completes the proof for the first of the two described functionals, and the proof for the second is the same.

Remark. All the rotation invariant families mentioned earlier have the following stronger property: $f \in \mathscr{F}$ and $|x| \leq 1$ imply that $f_x \in \mathscr{F}$. It follows that for any $f \in \mathscr{F} \setminus I(\mathscr{F})$, $F(f(z_0))$ is an interior point of $\{F(h(z_0)): h \in \mathscr{F}\}$.

COROLLARY 1. *Let* $\mathscr{F}$ *be a compact, rotation invariant subset of* $H(\Delta)$ *containing some nonconstant functions. Let* $z_0 \in \Delta \setminus \{0\}$ *and* $c \in \mathbb{C} \setminus \{0\}$. *Then the functions in* $\mathscr{F}$ *which maximize the functional* $f \to \Re\, cf(z_0)$ *over* $\mathscr{F}$ *are both insuperable elements and support points of* $\mathscr{F}$.

Proof. The first assertion follows from Theorem 2 with $F(w) = cw$. The second assertion is valid by definition, provided that the linear map $f \to cf(z_0)$ has nonconstant real part on $\mathcal{F}$. But if this is not true, the rotation invariance of $\mathcal{F}$ implies that for each $f \in \mathcal{F}$, $\Re\, cf(xz_0)$ is independent of x, $|x| = 1$. It follows by routine reasoning that f is constant, contradicting the hypotheses.

Theorem 3 below, expressing the final dominating property to be mentioned, is an immediate consequence of Theorem 1 and Littlewood's subordination theorem [4, pp. 10-11].

THEOREM 3. *Let* $\mathcal{F}$ *be compact in* $H(\Delta)$, *and let* $f \in \mathcal{F}$. *Then there exists* $g \in I(\mathcal{F})$ *such that*

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \le \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta$$

for all r *between* 0 *and* 1, *and for all* $p > 0$.

3. THE INSUPERABLE ELEMENTS OF C/z AND C'

We begin this section with some joint examples of extreme points and insuperable elements. By [2], the extreme points of $\overline{co}\, K$, $\overline{co}\, S^*$, and $\overline{co}\, C$ are given by $\{z/(1 - yz): |y| = 1\}$, $\{z/(1 - yz)^2: |y| = 1\}$, and

$$\{[z - (1 + x)yz^2/2]/(1 - yz)^2: |x| = |y| = 1,\ x \ne 1\}$$

respectively. As mentioned in the introduction,

$$I(K/z) = \{1/(1 - yz): |y| = 1\} \quad \text{and} \quad I(S^*/z) = \{1/(1 - yz)^2: |y| = 1\}.$$

These examples make the following question a natural one. Is it true that for each $f \in C$, there exists x with $|x| = 1$, $x \ne 1$, and $f(z)/z \prec [1 - (1 + x)z/2]/(1 - z)^2$? It is not difficult to show that the last written function maps the unit disk Δ univalently onto the "outside" of a parabola, that for different values of x the parabolas have different directions, and therefore that none of these functions is subordinate to any other. As a result of this and of Theorem 1, an affirmative answer to our question is equivalent to the equation $I(C/z) = E(\overline{co}\, C/z)$. In this section, however, we exhibit a function $f \in C$ for which the subordination described does not hold for any x. Therefore $I(C/z) \not\subset E(\overline{co}\, C/z)$. On the other hand, we subsequently prove that $E(\overline{co}\, C/z) \subset I(C/z)$. Similarly we show that $E(\overline{co}\, C') \subset I(C')$. (Here we have not attempted to disprove the possibility of equality, but we consider this possibility unlikely.) The exact determination of $I(C/z)$ and $I(C')$ appears to be a difficult problem.

Before presenting our counterexample (Theorem 4), we require a preliminary definition and lemma. These will be needed later also.

Definition 3. Let $g \in H(\Delta)$. Then g is a *Brannan-Clunie-Kirwan* (*or* BCK) *function* if $h \prec g$ implies that $h(z) = \int_\Gamma g(yz)\, d\mu(y)$ for some probability measure μ *on the unit circle* Γ.

A theorem of D. A. Brannan, J. G. Clunie, and W. E. Kirwan [1], to which we shall refer as the BCK theorem, asserts that $[(1 + cz)/(1 - z)]^p$ is a BCK function if $|c| \leq 1$ and $p \geq 1$. We observe that if g is a BCK function, $ag + b$ is a BCK function for any $a, b \in \mathbb{C}$.

LEMMA 2. *If* $\Re x \leq 1$, *then* $[1 - (1 + x)z/2]/(1 - z)^2$ *is a* BCK-*function.*

Proof. If $x = 1$, the assertion is a special case of the BCK theorem ($p = 1$, $c = 0$) or an easy consequence of the Herglotz representation for functions with positive real part. For the other values of x under consideration, our conclusion follows from the identity

$$\frac{1 - (1 + x)z/2}{(1 - z)^2} + \frac{(1 + x)^2}{8(1 - x)} = \frac{1}{8(1 - x)} \left[\frac{(x - 3) + (x + 1)z}{1 - z} \right]^2,$$

the observation made above, and the BCK theorem with $p = 2$ and $c = (x + 1)/(x - 3)$.

THEOREM 4. *Let* $|v| = 1$, $v \neq \pm 1$, *and let* $f \in C$ *with*

$$f'(z) = \frac{1}{2}\left[\frac{1 + \bar{v}z}{1 - \bar{v}z} + \frac{1 + vz}{1 - vz} \right] \frac{1}{(1 - vz)^2}.$$

Then for v *sufficiently close to* ± 1, $f(z)/z$ *is not subordinate to any of the functions in* $E(\overline{co}\ C/z)$.

Proof. Suppose v is a number as described above for which there exists x with $|x| = 1$, $x \neq 1$, and $f(z)/z \prec [1 - (1 + x)z/2]/(1 - z)^2$. By Lemma 2, we then obtain a probability measure μ on Γ for which

$$f(z) = \int_\Gamma \{[z - (1 + x)yz^2/2]/(1 - yz)^2\}\, d\mu(y).$$

Hence $f'(z) = \int_\Gamma [(1 - xyz)/(1 - yz)^3]\, d\mu(y)$, and therefore

$$1 = \lim_{z \to \bar{v}} \int_\Gamma [(1 - vz)^3 (1 - xyz)/(1 - yz)^3]\, d\mu(y).$$

If we let $z \to \bar{v}$ radially, the last integrand stays bounded by 2. Consequently Lebesgue's dominated convergence theorem implies that $1 = (1 - x)\, \mu(\{v\})$. It follows that $x = -1$ and $\mu(\{v\}) = 1/2$. Returning to our two formulas for $f'(z)$, we obtain the equations

$$f'(z) = (1/2)(1 + vz)/(1 - vz)^3 + \int_{\Gamma \setminus \{v\}} [(1 + yz)/(1 - yz)^3]\, d\mu(y),$$

$$(1/2)(1 + \bar{v}z)/(1 - \bar{v}z)(1 - vz)^2 = \int_{\Gamma \setminus \{v\}} [(1 + yz)/(1 - yz)^3]\, d\mu(y).$$

We now multiply by 2, observe that $2\mu(\Gamma \setminus \{v\}) = 1$, and conclude that the function $(1 + \bar{v}z)/(1 - \bar{v}z)(1 - vz)^2$ belongs to $\overline{co}\,(S^*)'$. But by [2, Theorem 14], this is false for v sufficiently near ± 1. Our theorem now follows.

We take this opportunity to record a correction of Theorem 14 of [2] concerning the nearness of v to ± 1. A correct requirement is that $20/21 < \cos^2\theta < 1$, where $v = e^{i\theta}$. The statement in [2] containing the inequality $5/6 < \cos^2\theta < 1$ is unjustified; it results from the incorrect factor $1 + 14\cos^2\theta - 15\cos^4\theta$ occurring in the proof. This factor should be replaced by $1 + 2\cos^2\theta - 3\cos^4\theta$. The essential content of the theorem remains unchanged.

As a result of the next lemma, we shall obtain "half" of our ultimate result that $E(\overline{co}\ C/z) \subset I(C/z)$. More importantly, the lemma leads to results on support points of C.

LEMMA 3. *Let* $0 < r < 1$. *Then the functional* $f \to f(-r)$ *attains its maximum real part over* $E(\overline{co}\ C)$ *at the function* $g_{x,y}(z) = \dfrac{z - (1+x)yz^2/2}{(1-yz)^2}$ *if and only if* $\Re x = \max\{-1, -(3 - 2r - r^2)/4r\}$ *and* $y = -(1+rx)/(x+r)$.

Proof. From the equation

$$g_{x,y}(-r) = g(-r) = \frac{-r - yr^2/2}{(1+ry)^2} + \frac{-xyr^2/2}{(1+ry)^2},$$

it follows that for any given y, $\max_x \Re g(-r)$ is obtained for x satisfying

$$\frac{-xyr^2/2}{(1+ry)^2} = \frac{r^2/2}{|1+ry|^2}, \quad \text{or} \quad \frac{-x}{1+ry} = \frac{1}{y+r}.$$

Hence, $\max_{(x,y)} \Re g(-r)$ can be attained only for pairs (x, y) satisfying the last equation. For such pairs, the expression for $g(-r)$ assumes its simplest form when written in terms of x. Thus the last equation gives $y = -(1+rx)/(x+r)$, and if this is substituted into the expression for $g(-r)$,

$$g(-r) = (-2 + r + r^2 + r^3 + r^2x + r^2\bar{x} + r^3\bar{x} - 3r\bar{x} - r^2\bar{x}^2)r/2(1-r^2)^2.$$

Now we need only seek the values of x that maximize the real part of the last expression. With the notation $t = \Re x$, this is equivalent to determining t, $-1 \le t \le 1$, for which $(r^2 + 2r - 3)t - 2rt^2$ is maximal. The rest follows easily.

COROLLARY 2. *If* $0 < r \le 2\sqrt{3} - 3$, *then* $\Re g_{x,y}(-r)$ *is maximal over* $E(\overline{co}\ C)$ *only for* $x = -1$ *and* $y = 1$, *that is, for the function* $z/(1-z)^2$.

Theorem 5 below is not the best possible. The last assertion of the theorem will be superseded by Theorem 8, and the statement about support points is undoubtedly not the final one to be made. We present the theorem because of the explicit information on evaluation functionals contained in the proof.

THEOREM 5. *Let* E^- *be the set of functions* $g_{x,y}(z) = \dfrac{z - (1+x)yz^2/2}{(1-yz)^2}$ *with* $|x| = |y| = 1$ *and* $\Re x < 0$. *Then each point of* E^- *is a support point of* C, *and* $E^-/z \subset I(C/z)$.

Proof. Let $g_{x,y} \in E^-$. We choose r, $0 < r < 1$, so that $-(3 - 2r - r^2)/4r = \Re x$. Then by Lemma 3, for a suitable y_0, g_{x,y_0} maximizes the functional $f \to \Re f(-r)$ over $E(\overline{co}\ C)$, and therefore over C. We let $u = y_0/y$. Then $g_{x,y_0}(z) = g_{x,y}(uz)/u$. Therefore, from the known inequality $\Re g_{x,y_0}(-r) \ge \Re f(-r)$ $(f \in C)$, we deduce that

$\Re\, g_{x,y}(-ur)/u \geq \Re\, f(-r)$ $(f \in C)$ and $\Re\, g_{x,y}(-ur)/u \geq \Re\, f(-ur)/u$ $(f \in C)$. In other words, $g_{x,y}$ maximizes the real part of the functional $f \to f(-ur)/u$. Hence $g_{x,y}$ is a support point of C. Finally, we must show that if $g_1(z) = g_{x,y}(z)/z$, then $g_1 \in I(C/z)$. But the last inequality above can be written $\Re\,[-rg_1(-ur)] \geq \Re\,[-rf_1(-ur)]$, where f_1 is an arbitrary function in C/z. Therefore $g_1 \in I(C/z)$ by Corollary 1 of Section 2.

The following considerations provide evidence, in addition to that of Theorem 5 and the paper [5], that the set of support points of C coincides with E. (We temporarily abbreviate $E(\overline{co}\, C)$ to E.) If we let σ denote this set of support points, the result of [5] is that $\sigma \subset E$. Therefore our conjecture is equivalent to the inclusion $E \subset \sigma$. Although we cannot prove this, the abstract argument below shows that $E \subset \overline{\sigma}$. Thus we have the equality $\overline{\sigma} = \overline{E} = \{g_{x,y}: |x| = |y| = 1\}$. This conclusion of course allows the possibility that σ is dense in E without equalling E, but we regard this as highly unlikely, especially in view of Theorem 5.

To prove $E \subset \overline{\sigma}$, we assume only that C and $\overline{co}\, C$ are compact subsets of a locally convex linear topological space. First we note that $C \subset \overline{co}\, \sigma$, for otherwise some point of C could be separated from $\overline{co}\, \sigma$ by a hyperplane [3, p. 417, Theorem 10], and a contradiction quickly follows. (It is easy to obtain the stronger conclusion $C \subset \overline{co}(\sigma \cap E)$ by the same type of argument.) From this result, we easily deduce that $\overline{co}\, \overline{\sigma} = \overline{co}\, C$. Finally, since both $\overline{\sigma}$ and $\overline{co}\, \overline{\sigma}$ are compact, another general theorem for locally convex spaces [3, p. 440, Lemma 5] asserts that each extreme point of $\overline{co}\, \overline{\sigma}$ belongs to $\overline{\sigma}$. Thus $E \subset \overline{\sigma}$.

To achieve our results that $E(\overline{co}\, C/z) \subset I(C/z)$ and $E(\overline{co}\, C') \subset I(C')$, we require one further definition.

Definition 4. Let $G \subset H(\Delta)$. Then G is *closed under subordination* if the conditions $g \in G$ and $f \prec g$ imply that $f \in G$.

THEOREM 6. *Both* $\overline{co}(C/z)$ *and* $\overline{co}(C')$ *are closed under subordination.*

Proof. We first suppose $f \prec g$, where

$$g(z) = \int_{\Gamma^2} \frac{1 - (1+x)yz/2}{(1 - yz)^2}\, d\mu(x, y)\,,$$

and where μ is a probability measure on the torus Γ^2. Then

$$f(z) = \int_{\Gamma^2} \frac{1 - (1+x)y\,\omega(z)/2}{[1 - y\,\omega(z)]^2}\, d\mu(x, y) \qquad \text{for some } \omega \in \Omega.$$

It follows from Lemma 2 that the integrand belongs to $\overline{co}(C/z)$. Consequently the same is true for f.

Next we assume that, for suitable μ and ω,

$$f(z) = \int_{\Gamma^2} \frac{1 - xy\,\omega(z)}{[1 - y\,\omega(z)]^3}\, d\mu(x, y) = \int_{\Gamma^2} \frac{1 - xy\,\omega(z)}{1 - y\,\omega(z)}\, \frac{1}{[1 - y\,\omega(z)]^2}\, d\mu(x, y)\,.$$

Since $1/(1 - z)^2$ is a BCK function, while $\Re\, e^{i\alpha}\,(1 - xy\,\omega(z))/(1 - y\,\omega(z)) > 0$ for some $\alpha \in \mathbb{R}$, we conclude that the integrand belongs to $\overline{co}(C')$. Therefore $f \in \overline{co}(C')$ as required.

Remark. The proof of Theorem 6 suggests that if G is compact and convex, and if $f \in G$ whenever $h \in E(G)$ and $f \prec h$, then G is closed under subordination. This statement is correct, and the Krein-Milman theorem [3, p. 440] provides an easy proof.

THEOREM 7. *Let* G *be closed, convex, and closed under subordination. Then an extreme point of* G *cannot be subordinate to a nonextreme point. In particular,* $I(E(G)) \subset I(G)$.

Proof. Let $f \in E(G)$, $h \in G$, and $f \prec h$. We must prove that $h \in E(G)$. Considering first the case where f is constant, we have the equations

$$f(z) = h(0) = \frac{1}{2\pi} \int_0^{2\pi} h(ze^{i\theta})\, d\theta = \frac{\varepsilon}{2\pi} g_\varepsilon(z) + \frac{2\pi - \varepsilon}{2\pi} k_\varepsilon(z),$$

where $z \in \Delta$, $0 < \varepsilon < 2\pi$, $g_\varepsilon(z) = \frac{1}{\varepsilon} \int_0^{\varepsilon} h(ze^{i\theta})\, d\theta$, and

$$k_\varepsilon(z) = \frac{1}{2\pi - \varepsilon} \int_\varepsilon^{2\pi} h(ze^{i\theta})\, d\theta .$$

Now, the hypotheses concerning G imply that g_ε and k_ε belong to G. Therefore our convex decomposition of f is trivial, so that $f = g_\varepsilon$. Hence

$$f(z) = \lim_{\varepsilon \to 0} g_\varepsilon(z) = h(z),$$

and $h \in E(G)$.

We now suppose that f is nonconstant and that $h = tg_1 + (1 - t)g_2$, where $0 < t < 1$ and $g_1, g_2 \in G$. Then $f = tg_1 \circ \omega + (1 - t)g_2 \circ \omega$ for a suitable function $\omega \in \Omega$. Hence $g_1 \circ \omega = g_2 \circ \omega$. But since f is nonconstant, so is ω. Therefore $g_1 = g_2$ as required.

LEMMA 4. *Let* G *denote either* $\overline{co}(C/z)$ *or* $\overline{co}(C')$. *Then* $I(E(G)) = E(G)$.

Proof. We discuss only the case where $G = \overline{co}(C')$. The proof for $\overline{co}(C/z)$ can be accomplished either by the same method or by the geometric considerations described at the beginning of Section 3. We shall prove that the relation

$$\frac{1 - x_1 y_1 z}{(1 - y_1 z)^3} \prec \frac{1 - x_2 y_2 z}{(1 - y_2 z)^3} \qquad (x_1, x_2, y_1, y_2 \in \Gamma;\ x_1 \neq 1,\ x_2 \neq 1)$$

between two extreme points of $\overline{co}(C')$ implies that $x_1 = x_2$. First we obtain $\omega \in \Omega$ such that

$$\left(\frac{1 - \omega(z)}{1 - z}\right)^3 = \frac{1 - x_2\omega(z)}{1 - x_1 z}.$$

Now, as $z \to 1$ ($z \in \Delta$), the right side of the last equation stays bounded. Therefore, consideration of the left side leads to the conclusion that $\omega(z) \to 1$. Thus the right side converges to $(1 - x_2)/(1 - x_1)$. Hence $\lim_{z \to 1} (1 - \omega(z))/(1 - z) = \alpha$, where $\alpha^3 = (1 - x_2)/(1 - x_1)$. If we can show that $\alpha \in \mathbb{R}$, the desired conclusion that $x_1 = x_2$ will follow. But if we define $\omega(1)$ to equal 1, and

$$h(t) = \Re\, \omega\left(\frac{1}{2} + \frac{1}{2} e^{it}\right) \qquad (t \in \mathbb{R}),$$

then h has a maximum at 0 with $h'(0) = \Re(\alpha i/2)$. Therefore $\Re(\alpha i/2) = 0$ and $\alpha \in \mathbb{R}$.

Our final theorem is an immediate consequence of Theorem 6, Theorem 7, and Lemma 4.

THEOREM 8. *Let* G *denote either* $\overline{co}(C/z)$ *or* $\overline{co}(C')$. *Then* $E(G) \subset I(G)$.

We remark in closing that R. Hornblower and the second-named author have now completed the proof that the set of support points of C coincides with $E(\overline{co}\, C)$.

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