A SPLITTING CONDITION USING BLOCK THEORY

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Let A be a normal abelian subgroup of the finite group G (only finite groups are considered here). There are various conditions which imply that G is split over A. The most celebrated, Schur's splitting theorem, is the case when the index of A in G is relatively prime to the order of A. We present here a block theoretic condition for the existence of a complement. The proof follows closely the elegant non-cohomological proof of Schur's splitting theorem given by Wielandt. In fact, we obtain Schur's splitting theorem as a corollary.

1. MODULES IN THE PRINCIPAL BLOCK

Let F be a splitting field of characteristic p for the finite group N, and let e denote the centrally primitive idempotent of the group algebra F[N] corresponding to the principal p-block of N. Write $e = \sum_{x \in N} c_x x$. We list some properties of e:

(a) Each c_x lies in the prime subfield GF(p) of F;

(b) The function $x \to c_x$ is a class function on N;

(c) $\sum_{x \in N} c_x = 1$.

Statement (a) follows from explicit formulas for the coefficients c_x given by M. Osima [2] and the fact that all the algebraic conjugates of a fixed ordinary irreducible character belonging to the principal p-block also belong to the principal p-block. The second statement is clear, as $e \in Z(F[N])$. Finally, if $s = \sum_{x \in N} x$, then s spans the unique one-dimensional space of invariants of F[N]. Since e acts as the identity on this subspace, se = s. However, sx = s for all $x \in N$, so

$$s\left(\sum_{x \in N} c_x x\right) = cs, \qquad \text{where } c = \sum_{x \in N} c_x .$$

Hence c = 1 and (c) follows.

The three statements about e above will be essentially the only facts needed from representation theory.

If V is an F[N]-module, then by definition V belongs to the principal p-block of N, provided Ve = V. Notice that if U is any GF(p)[N]-module, then Ue is defined because of (a) above, and $U \otimes_{GF(p)} F$ lies in the principal p-block of N precisely when Ue = U. Thus, we may define a GF(p)[N]-module U to belong to the principal p-block whenever Ue = U. As the formula for e does not change when F is replaced by any other splitting field, this notion is well defined. Notice that if no composition factor of U belongs to the principal p-block, then Ue = 0.

Received January 30, 1976.

Michigan Math. J. 23 (1976).

2. THE SPLITTING THEOREM

We now state and prove the main result of this paper.

THEOREM. *Let* A *be a normal abelian subgroup of* G. *Every* G-*composition factor* B/C *of* A *is an elementary abelian* p-*group for some prime* p, *and may be regarded as an irreducible* GF(p)[G/A]-*module. Assume that no such module lies in the principal* p-*block of* G/A, *where* p *necessarily depends on* B/C. *Then* A *is complemented in* G *by a subgroup* H, *and all such complements are conjugate.*

Proof. Suppose there exists a normal subgroup B of G with $1 < B < A$. Then induction applies, and A/B is complemented in G/B by a group K/B. The hypotheses of the theorem are satisfied with G and A replaced by K and B so that, by induction again, B is complemented by a subgroup, say H, in K. Clearly, H complements A in G. If H_1 is another complement for A in G, then $H_1 B/B$ is a complement for A/B in G/B. By induction, $H_1 B/B$ is conjugate to K/B, so that $H_1 B$ is conjugate to K. Write $K = (H_1 B)^g = H_1^g B$. Then H_1^g and H are complements for B in K, and again by induction, $H = (H_1^g)^k$ for some $k \in K$. Thus, H and H_1 are conjugate in G.

We may now assume that A is a minimal normal subgroup of G. Regard A as a GF(p)[G/A]-module, where p is the prime dividing $|A|$. Let $N = G/A$ and let e be the block idempotent of GF(p)[N] corresponding to the principal p-block. As before, write $e = \sum_{x \in N} c_x x$. Now, by hypothesis, e annihilates A, which in multiplicative form means

$$\prod_{x \in N} (a^x)^{c_x} = 1 \qquad \text{for all } a \in A .$$

Here, of course, $a^x = a^g$ if $x = Ag \in N$.

By a *transversal for* A *in* G we mean a complete set of coset representatives for A in G. Denote by $\mathscr{T}$ the collection of all transversals for A in G. Notice that if $T \in \mathscr{T}$ and $g \in G$, then $T^g = \{g^{-1} tg : t \in T\}$ belongs to $\mathscr{T}$. In particular, G acts on $\mathscr{T}$ by conjugation. For each $T \in \mathscr{T}$, define the function $f_T : N \to T$ by the condition $f_T(x) \in x$; that is, $f_T(x)$ is the coset representative in T for the coset x. For T and U in $\mathscr{T}$, define $P(T, U) \in A$ by the equation

$$P(T, U) = \prod_{x \in N} (f_T(x)^{-1} f_U(x))^{c_x} .$$

Each term $f_T(x)^{-1} f_U(x)$ in the above product is an element of A, and the product is well defined, as A is abelian. We list some properties of this function.

For all $T, U, V \in \mathscr{T}$, for all $a, b \in A$ and for all $g \in G$:

(i) $P(T, T) = 1$;

(ii) $P(T, U) = P(U, T)^{-1}$;

(iii) $P(T, U)\, P(U, V) = P(T, V)$;

(iv) $P(T, T^a) = a$;

(v) $P(T^b, T^a) = b^{-1} a$;

(vi) $P(T^g, U^g) = P(T, U)^g$.

Properties (i) to (iii) are immediate from the definition. Property (iv) follows from the equation $f_{T^a}(x) = f_T(x)^a$, the fact that $e = \sum_{x \in N} c_x x$ annihilates A, and the equation $\sum_{x \in N} c_x = 1$. Property (v) is immediate from (iv) (with T^b in place of T), while (vi) follows from the equation $f_{T^g}(x^g) = f_T(x)^g$ and the fact that the function $x \to c_x$ is a class function.

Define a relation ~ on $\mathscr{T}$ by setting T ~ U if P(T, U) = 1. Because of (i), (ii), and (iii), ~ is an equivalence relation on $\mathscr{T}$. Let Ω be the set of equivalence classes of $\mathscr{T}$ under ~. We already know that G acts on $\mathscr{T}$ by conjugation. Because of (vi), G can be made to act on Ω in the natural way. Specifically, if $\alpha \in \Omega$ and if $T \in \alpha$, define $\alpha^g = \beta$ where β is the equivalence class containing T^g.

Let $T \in \mathscr{T}$ be fixed. For $U \in \mathscr{T}$ and P(T, U) = a, we have

$$P(T^a, U) = P(T^a, T)\, P(T, U) = P(T, T^a)^{-1}\, P(T, U) = a^{-1} a = 1.$$

Hence, $U \sim T^a$. Therefore, every $\alpha \in \Omega$ has a representative of the form T^a for some $a \in A$. Moreover, because of (v), $T^a \sim T^b$ if and only if a = b, and so $\{T^a : a \in A\}$ is a complete collection of representatives for the ~ equivalence classes on $\mathscr{T}$. It follows that A acts regularly on Ω. Let $\alpha \in \Omega$ and, as usual, let G_α denote the stabilizer of α in G. Then $G = AG_\alpha$ and $A \cap G_\alpha = 1$, which proves that A is complemented in G.

Now suppose H and K are two complements for A in G. Notice that both H and K belong to $\mathscr{T}$. Suppose $H \in \alpha$ and $K \in \beta$, where α and β are ~ equivalence classes; that is, $\alpha, \beta \in \Omega$. Now for $g \in H$ we have $H^g = H$, so that g must fix α (as g fixes an element of α). Hence $g \in G_\alpha$, and this implies $H \leq G_\alpha$. Similarly, $K \leq G_\beta$, and by consideration of orders we have $H = G_\alpha$ and $K = G_\beta$. Now A is transitive on Ω, so $\alpha^a = \beta$ for some $a \in A$. Hence, $H^a = (G_\alpha)^a = G_{\alpha^a} = G_\beta = K$, and H is conjugate to K in G.

3. CONCLUDING REMARKS

We obtain the classical Schur splitting theorem as a corollary.

COROLLARY. *Let* A *be a normal abelian subgroup of* G *with* $(|G/A|, |A|) = 1$. *Then* A *is complemented in* G, *and all such complements are conjugate.*

Proof. Arguing as in the first paragraph of the proof of the theorem, we may assume A is a minimal normal subgroup of G. If G/A acts trivially on A, that is, $A \subseteq Z(G)$, then the kernel of the transfer map $G \to A$ is easily seen to be the unique complement for A in G, and we are finished. (For the definition and elementary properties of the transfer homomorphism, see pp. 245-251 of [1].) Suppose, then, G/A is nontrivial on A. If p is the prime dividing $|A|$, then A is not the principal GF(p)[G/A]-module. As $p \nmid |G/A|$, the principal GF(p)[G/A]-module is the only irreducible module in the principal p-block, so A does not belong to the principal p-block. The corollary now follows from the theorem.

We remark that the main result of this paper follows from the stronger assertion that the cohomology groups $H^n(G, U)$ vanish for all $n \geq 0$ and all GF(p)[G]-modules U which are annihilated by the principal block idempotent. This result, although not difficult to prove, was not presented here, as it would have detracted from the elementary noncohomological proof of the theorem of this paper.

REFERENCES

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