MODELS FOR COMMUTING CONTRACTIONS

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1. INTRODUCTION

Let L denote the unilateral shift on a vector-valued H^2 space $H^2_{\mathscr{H}}$. Let M be a closed invariant subspace for L, let P project $H^2_{\mathscr{H}}$ onto $M^\perp$, and denote by T the restricted shift $T = PL|_{M^\perp}$. G.-C. Rota showed by an amazingly simple argument that T^* is a universal model for a large class of operators; that is, if S is a contraction on $\mathscr{H}$ of norm less than 1 then S is similar to some T^*. Rota's technique was refined by L. de Branges and J. Rovnyak to yield a model up to unitary equivalence: S is unitarily equivalent to some T^* if and only if $\|S\| \leq 1$ and $S^n \to 0$ strongly. (Both these results and basic background material on shifts and vectorial function theory can be found in [3].) Using restricted shifts as models, an extensive structure theory for operators has been developed; see [10].

D. N. Clark [1] extended Rota's theorem to the case of N commuting contractions by using the maps $T_k = PL_k|_{M^\perp}$ as a model, where

$$L_k f(z_1, \cdots, z_N) = z_k f(z_1, \cdots, z_N)$$

in the polydisc space $H^2(U^N)$. (See [6] for a basic reference.) Clark also characterized the commutant of $\{T_1, \cdots, T_N\}$, and hence, up to similarity, the commutant of N commuting contractions; this extended the one variable results of D. Sarason [7] and B. Sz.-Nagy and C. Foiaş [10]. In this paper, we modify the de Branges-Rovnyak technique to construct a unitary equivalence model for N commuting contractions by using a weighted shift analog of the maps T_k, and we extend Clark's description of the commutant to this case. We explain our notation below; for the basic theory of one variable weighted shifts the reader can consult [8].

2. NOTATION AND MODELS

For a fixed positive integer N, we use the notation $z = (z_1, \cdots, z_N)$, $e^{i\phi} = (e^{i\phi_1}, \cdots, e^{i\phi_N})$, and $J = (j_1, \cdots, j_N)$ a multi-index of either nonnegative integers, which we indicate by $J \geq 0$, or arbitrary integers. We let e_k denote the multi-index J with $j_k = 1$ and $j_n = 0$ otherwise; $J \pm e_k$ has the obvious meaning except that by using $J - e_k$ we imply that $j_k \geq 1$. We use $J \cdot \phi = j_1\phi_1 + \cdots + j_N\phi_N$; for $J \geq 0$, $|J| = j_1 + \cdots + j_N$, $J! = j_1! \cdots j_N!$, $z^J = z_1^{j_1} \cdots z_N^{j_N}$; and given commuting operators $S_1, \cdots, S_N$, $S^J = S_1^{j_1} \cdots S_N^{j_N}$. We let U^N and T^N denote the N-dimensional unit polydisc and torus respectively.

For a separable Hilbert space $\mathscr{H}$, $L^2 = L^2_{\mathscr{H}}(T^N)$ and $H^2 = H^2_{\mathscr{H}}(U^n)$ denote the standard Lebesgue and Hardy spaces of square summable vector-valued functions from T^N into $\mathscr{H}$: $f \in L^2$ has Fourier expansion

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$$\sum f_J e^{iJ\cdot\phi} \text{ with } f_J \in \mathscr{H}, \quad \sum \|f_J\|^2_{\mathscr{H}} < \infty,$$

and $f \in L^2$ is in H^2 if and only if $f_J = 0$ unless $J \geq 0$. For $f \in H^2$, we freely identify $f(e^{i\phi})$ with $f(z)$, its analytic extension to U^n. We define $\tilde{L}_k$ on H^2 to be the unique bounded linear map such that

$$\tilde{L}_k(xz^J) = w_{J,k} xz^{J+e_k}, \qquad w_{J,k} = ((j_k + 1)/(|J| + 1))^{1/2}$$

for all $x \in \mathscr{H}$, $J \geq 0$, $k = 1, \cdots, N$.

A simple computation shows that for $x \in \mathscr{H}$, $J \geq 0$, $k = 1, \cdots, N$,

$$\tilde{L}_k^*(xz^J) = \begin{cases} w_{J-e_k,k} xz^{J-e_k} & \text{if } j_k \geq 1 \\ 0 & \text{if } j_k = 0. \end{cases}$$

Clearly, $\tilde{L}_j \tilde{L}_k = \tilde{L}_k \tilde{L}_j$, so $\{\tilde{L}_k\}$ is a family of commuting weighted shifts in N variables.

For $J \geq 0$, let $\beta_J = (J!/|J|!)^{1/2}$ and define the weighted H^2 space $H^2_\beta = H^2_{\beta,\mathscr{H}}(U^n)$ to be the set of all $\mathscr{H}$-valued power series $\sum_{J\geq 0} f_J z^J$ such that $\sum \|f_J\|^2 \beta_J^2 < \infty$. We define L'_k on H^2_β by $L'_k f = z_k f$, $k = 1, \cdots, N$. Let W mapping H^2 to H^2_β be the unique linear map such that $W(xz^J) = (|J|!/J!)^{1/2} xz^J$. It is easy to see that W is unitary and $\tilde{L}_k = W^* L'_k W$, $k = 1, \cdots, N$. Hence, the maps $\tilde{L}_k$ and L'_k are easily identified and can be used interchangeably. This is directly analogous to the one variable result in [8]. It can be shown in general that any family of commuting weighted shifts in N variables is unitarily equivalent to the family of unilateral shifts on a weighted H^2 space in N variables; details concerning this will appear elsewhere [4]. The maps $\{\tilde{L}_k^*\}$, or equivalently $\{L'^*_k\}$, will serve as our model; we will denote the compression of $\tilde{L}_k$ or L'_k to a space of the form $M^\perp$ by $\tilde{T}_k$ or T'_k respectively, where M is invariant under $\tilde{L}_k$ or L'_k, $k = 1, \cdots, N$ respectively.

We note that each $\tilde{L}_k$ is a direct sum of one variable weighted shifts. For the case $k = 1$, let J^* be an arbitrary multi-index of $(N - 1)$ nonnegative integers $(j_2, j_3, \cdots, j_N) = J^*$, and let

$$H_{J^*} = \left\{ \sum_{n=0}^{\infty} f_n z_1^n z_*^{J^*} \in H^2 \right\}, \qquad \text{where } z_* = (z_2, \cdots, z_N).$$

Each H_{J^*} clearly reduces $\tilde{L}_1$, $H^2 = \oplus \sum_{J^*} H_{J^*}$ and $L_1 |_{H_{J^*}}$ is a weighted shift having weight sequence $\{((n+1)/(n + |J^*| + 1))^{1/2}\}$. Hence, since $\tilde{L}_1$ is the unweighted unilateral shift on $H_0 = \left\{\sum f_n z_1^n \colon \sum \|f_n\|^2 < \infty\right\}$, we see from [8] that the spectrum of $\tilde{L}_1$ is $\{z\colon |z| \leq 1\}$. It is also easy to see that L_1 is not similar to an unweighted shift operator. By [9], it follows that each $\tilde{L}_1 |_{H_{J^*}}$, and hence $\tilde{L}_1$, is subnormal. By symmetry the same results hold for $\tilde{L}_k$, $k = 2, \cdots, N$. However, the maps $\tilde{L}_k$ do not have commuting normal extensions; we also note that $\sum_{k=1}^N \tilde{L}_k \tilde{L}_k^* = I$. This last condition characterizes $\{\tilde{L}_k\}$ among the commuting weighted shifts. Details will appear elsewhere [5].

3. THE REFINED ROTA-CLARK THEOREM

THEOREM 1. *Let* $S_1, \cdots, S_N$ *be commuting contractions such that*

$$\sum_{k=1}^{N} S_k^* S_k < I, \quad \textit{and} \quad \sum_{J \geq 0} b_J S^J x$$

converges for all $x \in \mathscr{H}$. *Then there exist a closed subspace* $M \subseteq H^2$ *invariant under* $\{\tilde{L}_k : k = 1, \cdots, N\}$ *and a unitary* $W: \mathscr{H} \to M^\perp \subseteq H^2$ *such that* $WS_kW^* = \tilde{L}_k^* |_M = \tilde{T}_k^*$.

Proof. Let $R = [I - (S_1^* S_1 + \cdots + S_N^* S_N)]^{1/2}$ be the unique positive root of the positive operator. We define W on $\mathscr{H}$ by

$$Wx = \sum_{J \geq 0} b_J (RS^J x) z^J, \qquad \text{where } b_J = \beta_J^{-1} = (|J|!/J!)^{1/2}.$$

We note that

$$\|RS^J x\|^2 = \|S^J x\|^2 - \sum_{k=1}^{N} \|S^{J+e_k} x\|^2,$$

and

$$\sum_{|J|=M} b_J^2 \sum_{k=1}^{N} \|S^{J+e_k} x\|^2 = \sum_{|J|=M+1} \left(\sum_{k=1}^{N} \frac{(|J| - 1)!}{(J - e_k)!} \right) \|S^J x\|^2$$

$$= \sum_{|J|=M+1} (|J| - 1)! \left(\sum_{k=1}^{N} \frac{j_k}{J!} \right) \|S^J x\|^2$$

$$= \sum_{|J|=M+1} \frac{|J|!}{J!} \|S^J x\|^2 = \sum_{|J|=M+1} b_J^2 \|S^J x\|^2.$$

Hence,

$$\|Wx\|^2 = \lim_{M \to \infty} \sum_{|J|=0}^{M} b_J^2 \|RS^J x\|^2$$

$$= \lim_{M \to \infty} \left[\sum_{|J|=0}^{M} b_J^2 \|S^J x\|^2 - \sum_{|J|=0}^{M} b_J^2 \sum_{k=1}^{N} \|S^{J+e_k} x\|^2 \right]$$

$$= \lim_{M \to \infty} \left[\sum_{|J|=0}^{M} b_J^2 \|S^J x\|^2 - \sum_{|J|=1}^{M+1} b_J^2 \|S^J x\|^2 \right]$$

$$= \|x\|^2 - \lim_{M \to \infty} \sum_{|J|=M+1} b_J^2 \|S^J x\|^2 = \|x\|^2,$$

since the series $\sum_{J\geq 0} b_J S^J x$ converges. For $x \in \mathscr{H}$,

$$\tilde{L}_k^* Wx = \sum_{J\geq 0} b_J \tilde{L}_k^*(RS^J xz^J) = \sum_{J\geq 0} \left(\frac{|J|!}{J!}\cdot\frac{j_k}{|J|}\right)^{1/2} RS^J xz^{J-e_k}$$

$$= \sum_{J\geq 0} b_{J-e_k} RS^{(J-e_k)}(S_k x) z^{J-e_k} = \sum_{J\geq 0} b_J RS^J (S_k x) z^J = WS_k x.$$

(Recall that by our convention, the second and third summations are taken over $J \geq 0$, $j_k > 0$.) Hence, $\tilde{L}_k^* W = WS_k$, so the range of W, which is closed since W is isometric, is invariant under $\tilde{L}_k^*$, $k = 1, \cdots, N$. Hence, W maps $\mathscr{H}$ unitarily onto $M^\perp$ for some M invariant under $\{\tilde{L}_k\}$ and the theorem follows.

Note that equivalently we have S_k unitarily equivalent to $T_k'^*$ in H^2_β. Also, if $\sum_{k=1}^N \|S_k\|^2 < 1$, then $\{S_k\}$ satisfies the hypothesis of the theorem. Given any commuting contractions $S_1, \cdots, S_N$, then for C sufficiently large (any $C > N$ will suffice) Theorem 1 applies to $C^{-1}S_1, \cdots, C^{-1}S_N$. Multiplying by a constant does not affect the invariant subspaces of the operators or their commutant, which we consider below.

4. THE COMMUTANT

Clark's description of the commutant [1] extends immediately to the case of $\{T_k'\}$ on H^2_β. We give the terminology of [1] necessary to state the theorem below; the proof follows verbatim from [1].

For $z \in U^N$, let u(z) be a bounded operator on $\mathscr{H}$ depending analytically on z and suppose that $u(z)x \in H^2_\beta$ for all $x \in \mathscr{H}$. We call u(z) an operator-valued function and note that $(uf)(z) = u(z)f(z) \in H^2_\beta$ for all polynomials $f \in H^2_\beta$; that is, $f = \sum f_J z^J$ where $f_J = 0$ except for finitely many J. We say u is compatible with a subspace M if and only if $uf \in M$ for all $f \in M$ such that $uf \in H^2_\beta$. For any subspace $K \subseteq H^2_\beta$, let K_B denote the projections onto K of all polynomials in H^2_β. For any u compatible with M, define T_u on $(M^\perp)_B$ by

$$T_u f = Pug,$$

where P projects onto $M^\perp$, and g is a polynomial such that $f = Pg$. By the compatibility of u, T_u is a well-defined (possibly unbounded) operator defined on a dense subspace of M. We say $u \in \mathscr{B} = \mathscr{B}(M)$ whenever T_u has a (necessarily unique) bounded extension to all of M.

THEOREM 2. *Any operator* T' *on* $M_\beta^\perp \subseteq H^2_\beta$ *that commutes with* $T_1', \cdots, T_N'$ *has the form* $T' = T_u$ *where* u *is an operator-valued function compatible with* M *and* $u \in \mathscr{B}$. *Conversely, every such* T_u *commutes with* $T_1', \cdots, T_N'$.

As we remarked earlier, this describes the commutant of any N commuting contractions up to unitary equivalence. Clark showed in [1] that u cannot necessarily be chosen such that $\|T_u\| = \sup_{z\in U^n} \|u(z)\|$; see [2] for an example where $\sup \|u(z)\| = +\infty$.

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