ON QUASI-AFFINE TRANSFORMS OF SPECTRAL OPERATORS

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Throughout this paper, "an operator" means a bounded linear transformation defined on a fixed separable Hilbert space H.

It is known [6, Lemma 7] that a spectral subnormal operator is necessarily normal. Here we show, among the other things, that if a quasi-affine transform of a hyponormal (subspectral) operator T is spectral, then T is normal (spectral) (see below for definitions). This, in particular, answers a question raised by J. G. Stampfli in [7, Remark to Theorem 4].

Definitions. (1) An operator T is called *spectral* if $T = S + Q$, where S (called the scalar part) is similar to a normal operator, Q is quasi-nilpotent, and $SQ = QS$. Every spectral operator has a resolution of the identity which is the same as that of its scalar part. The decomposition $T = S + Q$ is called the canonical reduction of T [2, page 1939].

(2) The restriction of a normal (spectral) operator to an invariant subspace is called a *subnormal* (*subspectral*) operator; a *cosubnormal* (*cosubspectral*) operator is the adjoint of a subnormal (subspectral) operator.

(3) An operator T is called *hyponormal* (*cohyponormal*) if $T^*T - TT^* \geq 0$ ($T^*T - TT^* \leq 0$).

(4) For an operator T and a closed subset F of the complex plane $\mathbb{C}$, we define

$$X_T(F) = \{x \in H : \text{there exists an analytic function}$$
$$f_x: \mathbb{C} \setminus F \to H \text{ such that } (\lambda - T) f_x(\lambda) \equiv x\}.$$

(5) An operator T is said to be a *quasi-affine transform* of an operator S if there exists a one-to-one operator W such that $WT = SW$ and WH is dense in H.

We need the following two lemmas.

LEMMA 1. *Let* A, B, *and* C *be three operators such that* $AC = CB$. *Let* g *be an* H-*valued function* (*not necessarily analytic*) *defined on a subset* G *of* $\mathbb{C}$ *such that* $(\lambda - B) g(\lambda) \equiv x$ *for some* $x \in H$. *Then* $(\lambda - A) C g(\lambda) \equiv Cx$.

The proof is trivial.

The next lemma plays an important role in this paper; our main results are easy applications of this lemma and some results due to C. R. Putnam [4] and Radjabalipour [5].

LEMMA 2. *Let* T *be a spectral operator with the resolution of the identity* E. *Let* F *be a closed subset of the plane. Let* $x \in H$, *and assume there exists a bounded function* $g: \dot{\mathbb{C}} \setminus F \to H$ *such that* $(\lambda - T) g(\lambda) \equiv x$. *Then* $E(F)x = x$.

Proof. We assume without loss of generality that the scalar part of T is normal. Let $T = S + Q$ be the canonical reduction of T. By [1, Theorem 1 (page 208)],

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there exist a family H_z ($z \in \sigma(S)$) of Hilbert spaces and a positive measure μ on $\sigma(S)$ such that H, N, and Q are unitarily equivalent to

$$\int^{\oplus} H_z\, d\mu(z), \quad \int^{\oplus} zI_z\, d\mu(z), \text{ and } \int^{\oplus} Q_z\, d\mu(z),$$

respectively, where I_z denotes the identity on H_z and Q_z is a quasi-nilpotent operator on H_z for $d\mu$-almost all z. Moreover, $T = \int^{\oplus} T_z\, d\mu(z)$, where $T_z = zI_z + Q_z$ for $d\mu$-almost all z.

Choose a fixed sequence $\{\lambda_n\}$ dense in $\mathbb{C} \setminus F$. Let $x = \int^{\oplus} x_z\, d\mu(z)$ and $g(\lambda_n) = \int^{\oplus} g_z(\lambda_n)\, d\mu(z)$ ($n = 1, 2, \cdots$). It is easy to see that

$$(1) \qquad (\lambda_n - T_z)\, g_z(\lambda_n) = x_z \text{ and } \|g_z(\lambda_n)\| \leq K \ (n = 1, 2, \cdots),$$

for $d\mu$-almost all z, where $K = \sup_\lambda \|g(\lambda)\|$.

Fix $z \notin F$ satisfying (1) for which Q_z is quasi-nilpotent. Since $\sigma(T_z) = \{z\}$, the analytic function $h(\lambda) = (\lambda - z - Q_z)^{-1} x_z$ ($\lambda \neq z$) agrees with $g_z(\lambda)$ on a dense subset of $\mathbb{C} \setminus F$ and thus $h(\lambda)$ is bounded in a deleted neighbourhood of z. Hence $h(\lambda)$ is an entire function. Therefore $x_z = 0$, and

$$x = \int_F^{\oplus} x_z\, d\mu(z) \in E(F)H\,.$$

COROLLARY 1. *Lemma* 2 *remains true if* T *is assumed to be a subspectral operator.*

Remark 1. Lemma 2 is true for normal operators with no boundedness condition on g [3, Theorem 1]. This is not true for a spectral operator in general: any nonzero vector in the range of a quasi-nilpotent operator yields a counterexample.

Now we prove our generalizations. For convenience, we state the results in terms of cohyponormal and cosubspectral operators.

THEOREM 1. *Let* T, S, D, *and* W *be operators satisfying the following conditions:*

(i) $(T - \lambda)(T^* - \bar{\lambda}) \geq D \geq 0$ *for all* $\lambda \in \mathbb{C}$;

(ii) S *is a subspectral operator;*

(iii) W *is one-to-one;*

(iv) $WT = SW$.

Then $D = 0$.

Note. Any cohyponormal operator T satisfies condition (i) with $D = TT^* - T^*T$ [4, page 167].

Proof of Theorem 1. Since every eigenvalue of T is also an eigenvalue of S, it follows that the point spectrum of T has no interior [2, page 1958]. Assume, if possible, that $D \neq 0$. By [4, Theorems 1 and 3] there exist a nonzero vector x and a bounded function $g: \mathbb{C} \to H$ such that $(\lambda - T)g(\lambda) \equiv x$. In view of Lemma 1, Wg is a bounded function from $\mathbb{C}$ into H with the property that $(\lambda - S)Wg(\lambda) \equiv Wx$. Now it follows from Corollary 1 that $Wx \in E(\emptyset)H = \{0\}$, a contradiction.

THEOREM 2. *Let* T, S, *and* W *be operators satisfying the following conditions:*

(i) T *is cosubspectral;*

(ii) *there exists a sequence* $\{G_n\}$ *of open sets forming a base for the topology of* $\mathbb{C}$ *such that* $X_S(\partial G_n) = \{0\}$ $(n = 1, 2, \cdots)$;

(iii) W *is one-to-one;*

(iv) $WT = SW$.

Then T *is spectral.*

Note. Any subspectral operator S satisfies condition (ii) of the theorem.

Proof of Theorem 2. In light of Lemma 1, $WX_T(\partial G_n) \subseteq X_S(\partial G_n) = \{0\}$ $(n = 1, 2, \cdots)$. The rest of the proof follows from [5, Theorem 2].

THEOREM 3. (a) *If a cohyponormal operator* T *is a quasi-affine transform of a subspectral operator* S, *then* T *is normal and* S *is similar to* T.

(b) *If a cosubspectral operator* T *is a quasi-affine transform of a subspectral operator* S, *then* T *and* S *are spectral.*

Proof. The normality of T follows from Theorem 1. By Theorem 2, applied to the cosubspectral operator S^*, the operator S is spectral. To finish the proof of (a), we have to show that if A is a normal operator, if B is a spectral operator with a normal scalar part, and if $WA = BW$ for some one-to-one positive operator W, then $A = B$. Let E_A and E_B be the resolutions of the identity for A and B, respectively. Let F be a closed subset of $\mathbb{C}$ such that

$$E_A(\partial F) = E_B(\partial F) = 0. \tag{2}$$

It is easy to see that $X_V(F) = E_V(F)H$ and $X_V(\overline{\mathbb{C} \setminus F}) = E_V(\mathbb{C} \setminus F)H$, where V stands for A and B. By Lemma 1 and the observations above,

$$W^2 X_A(F) \subseteq WX_B(F) = WX_{B^*}(F^*) \subseteq X_{A^*}(F^*) = X_A(F)$$

and, by a similar proof, $W^2 X_A(\overline{\mathbb{C} \setminus F}) \subseteq X_A(\overline{\mathbb{C} \setminus F})$. (Here F^* denotes the set of complex conjugates of the elements of F.) Thus $W^2 E_A(F) = E_A(F) W^2$. Since every closed set in the plane is the intersection of a decreasing sequence of closed sets satisfying (2), it follows that $W^2 E_A = E_A W^2$ and hence $W^2 A = AW^2$. Therefore $WA = AW = BW$, which implies that $A = B$.

(b) Apply Theorem 2 to the cosubspectral operators T and S^*.

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