AN ANALYTIC CHARACTERIZATION OF GEOMETRICALLY STARLIKE FUNCTIONS

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The class $S^*(\alpha)$ of functions $f(z) = z + a_2 z^2 + \cdots$ which are analytic and univalent in the unit disc $\mathbb{B}$, and which satisfy the condition $|\arg zf'(z)/f(z)| < \alpha\pi/2$, $0 < \alpha \leq 1$, was introduced and studied by Brannan and Kirwan [1]. Since $S^*(1)$ is the usual class of starlike univalent functions, $S^*(1)$ is not studied as such. Recently Leach [3] has extended $S^*(\alpha)$ to multivalent starlike functions. A slight modification of Leach's definition (which yields the same class of functions) follows.

Let p be a positive integer, and $0 \leq \alpha \leq 1$. Then $S^*(\alpha, p)$ denotes those functions f holomorphic in $\mathbb{B}$, with exactly p zeros there (zeros and critical points are counted by their multiplicity), such that

$$\limsup_{r \to 1^-} \max_{|z| = r} |\arg zf'(z)/f(z)| \leq \alpha\pi/2 .$$

This definition clearly has affinities with the analytic definition of weakly starlike functions, $S_w(p)$, as defined by Hummel [2]. $f \in S_w(p)$ if and only if f is holomorphic in $\mathbb{B}$, has exactly p zeros there, and

$$\liminf_{r \to 1^-} \min_{|z| = r} \Re\, zf'(z)/f(z) \geq 0 .$$

For $p > 1$ it is no longer true that, with $\alpha = 1$, $S^*(\alpha, p) = S_w(p)$. It is the purpose of this paper to prove

THEOREM 1. $S^*(1, p) = S_g(p)$.

Here $S_g(p)$ is the class of geometrically starlike functions of order p. That is, $f \in S_g(p)$ if and only if f is holomorphic in $\mathbb{B}$, has exactly p zeros there, and for each point z in $\mathbb{B}$ there is a curve in $\mathbb{B}$, between z and some zero of f, which f maps one-to-one onto the radial line segment between 0 and f(z). An interesting feature of Theorem 1 is that the proof is almost completely combinatorial. Setting the tone, we will need

LEMMA 1. *Let* $f \in S_w(p)$. *Then* $f \in S_g(p)$ *if and only if* f *has* p - 1 *critical points.*

This result may be found in Styer [4, p. 232].

For all $r > 0$ define $\gamma_r: [0, 2\pi] \to \mathbb{C}$ by $\gamma_r(t) = re^{it}$, and let C_r be the image of γ_r. For any closed curve γ which does not pass through 0, let $I(\gamma, 0)$ be the index, or winding number, of γ about 0.

LEMMA 2. *Let* g *be holomorphic in the annulus* $\{z : 0 < \rho < |z| < 1\}$, *and have no zero there. Then the following two statements are equivalent.*

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(i) $I(g \circ \gamma_r, 0) = 0$, $\rho < r < 1$, *and*

$$\liminf_{r \to 1^-} \min_{|z| = r} \Re g(z) \geq 0 .$$

(ii) $\limsup_{r \to 1^-} \max_{|z| = r} |\arg g(z)| \leq \pi/2.$

Note. This means that (i) implies that an appropriate branch of the argument can be chosen in order to satisfy (ii). Also, we may trivially assume that g is not constant.

Proof. Suppose that (i) is true. Choose any s, $\rho < s < 1$, and let $A = \{z : s \leq |z| < 1\}$. Since g omits zero, there is a disc $D = \{w : |w| < \delta\}$, $\delta > 0$, such that $g(C_s) \cap D = \emptyset$. Let $D^- = D \cap \{w : \Re w < 0\}$. It is our objective to prove that $D^- \cap g(A) = \emptyset$. From this it easily follows that a branch of the argument of g may be chosen so that (ii) is true.

It will suffice to show that $D \cap g(A) \cap L = \emptyset$ for each vertical line $L : \Re w = a$, $-\delta < a < 0$. Furthermore, it will suffice to look only at those L such that g has no critical point on $g^{-1}(L)$. In this case $g^{-1}(L)$ is a union of disjoint simple analytic curves. By the hypothesis on the real part of g, $\Re g(z) > a$ for all z with sufficiently large modulus. It thus follows that $g^{-1}(L) \cap A$ is a finite disjoint union of simple curves, each of which must have its first and last point in A on C_s. Also, g maps each of these curves in a one-to-one fashion onto a closed segment S of L. The upper end-point of S is a point where $g \circ \gamma_s$ crosses L from right to left. Similarly, $g \circ \gamma_s$ crosses L from left to right at the lower end-point of S. The proof will be complete when we show that either both end-points of S are in the second quadrant, or both are in the third quadrant, because then $S \cap D = \emptyset$. By hypothesis $I(g \circ \gamma_s, 0) = 0$, so that in the second (third) quadrant $g \circ \gamma_s$ crosses L from right to left and from left to right equally often. But each time $g \circ \gamma_s$ crosses L from left to right in the second quadrant, the point of intersection is the lower end-point of exactly one line segment S as described above. The upper end of this segment must be a point in the second quadrant where $g \circ \gamma_s$ crosses from right to left. Since there is exactly one line segment associated with each crossing of L by $g \circ \gamma_s$ and there are but finitely many crossings, it follows that no segment S with upper end-point in the second quadrant can have lower end-point in the third quadrant. This completes the proof that (i) implies (ii).

Suppose that (ii) is true. In (ii) the condition on the argument makes it clear that for large r, 0 lies in the unbounded component of the complement of $g(C_r)$. Thus $I(g \circ \gamma_r, 0) = 0$ for all r, $\rho < r < 1$.

The rest of the proof is similar to the one given above, but simpler. With the notation used above, it will suffice to show that $g(A) \cap \{w : \Re w < 0\}$ is a bounded set. Choose any ray R of angle θ, $\theta > \pi/2$ (symmetry applies to $\theta < -\pi/2$). One easily sees that (ii) implies $g(A) \cap R$ must be a subset of the segment between 0 and the point of greatest modulus in $g(C_s) \cap R$. This completes the proof.

Proof of Theorem 1. Let $f \in S^*(1, p)$, and let $g(z) = zf'(z)/f(z)$. By the definition of $S^*(1, p)$, there must be an annulus $\{z : 0 < \rho < |z| < 1\}$ in which g is holomorphic and nonzero and satisfies condition (ii) of Lemma 2. There Lemma 2 directly implies that $f \in S_w(p)$. Also $I(g \circ \gamma_r, 0) = 0$, $\rho < r < 1$. But this means that $g(z) = zf'(z)/f(z)$ must have the same number of zeros as poles. Since f has p zeros, f' must have p - 1 zeros. By Lemma 1, $f \in S_g(p)$. Thus $S^*(1, p) \subset S_g(p)$.

On the other hand suppose that $f \in S_g(p)$, and let $g(z) = zf'(z)/f(z)$. Then by the fact that f has exactly p zeros, and p - 1 critical points (Lemma 1), it follows that in some annulus $\{z : 0 < \rho < |z| < 1\}$, g is holomorphic and nonzero and satisfies condition (i) of Lemma 2. By Lemma 2, $f \in S^*(1, p)$, so $S_g(p) \subset S^*(1, p)$.

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