ON INERTIA GROUPS AND BORDISM

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In this paper a manifold will be a compact oriented differential manifold of dimension greater than 4, and all diffeomorphisms will be orientation preserving. θ_m denotes the group of homotopy m-spheres, or equally the group of diffeomorphism classes of manifolds homeomorphic to the m-sphere S^m. For a closed m-manifold M the inertia group I(M) is the subgroup of θ_m consisting of those homotopy m-spheres Σ for which M and the connected sum M # Σ are diffeomorphic. $\theta_m(k)$ is the subgroup of θ_m consisting of those homotopy m-spheres Σ that occur as boundaries of k-connected manifolds. Brender [1] has shown that if M is (k - 1)-connected then $I(M) \subset \theta_m(k - 1)$. The purpose of this note is to show that this result can be improved if it is assumed that M is k-parallelisable.

THEOREM. *If* M *is a closed* (k - 1)*-connected* k*-parallelisable* m*-manifold, then* $I(M) \subset \theta_m(k)$.

Before proving the theorem we consider some of its consequences. Excluding the case where m = 2r + 1 and k = r + 1,

$$\theta_m(k - 1) = \theta_m(k) \text{ if } k \equiv 3, 5, 6, 7 \pmod 8.$$

This follows since if $k \geq [m/2] + 1$, then $\theta_m(k - 1) \simeq 0 \simeq \theta_m(k)$; while if $k < [m/2] + 1$, then for $\Sigma \in \theta_m(k - 1)$, $\Sigma = \partial V$ where V is (k - 1)-connected, and as $\pi_{k-1}(SO) \simeq 0$ for $k \equiv 3, 5, 6, 7 \pmod 8$, V is also k-parallelisable. Since V is k-parallelisable it can be surgered to give a k-connected manifold with boundary Σ. The required surgeries exist by Wall [4] for m = 2r and k = r, and by Milnor [3] for all other cases.

The identities $\theta_m(k - 1) = \theta_m(k)$ above together with the theorem give necessary conditions for a homotopy sphere Σ to belong to an inertia group I(M), possibly showing $I(M) \simeq 0$. For example, from above $\theta_{2r}(r - 3) = \theta_{2r}(r) \simeq 0$ for $r \equiv 7 \pmod 8$, $\theta_{2r}(r - 2) = \theta_{2r}(r) \simeq 0$ for $r \equiv 6 \pmod 8$, and $\theta_{2r}(r - 1) = \theta_{2r}(r) \simeq 0$ for $r \equiv 3, 5 \pmod 8$. Combining these identities with the theorem, we obtain the following corollary which extends Theorem 3.1 of Kosiński [2] from the stably parallelisable case to the appropriate degree of parallelisability below.

COROLLARY. *If* M *is a closed* 2r*-manifold, then* $I(M) \simeq 0$ *in each of the following cases:*

(a) M *is* (r - 1)*-connected and* r*-parallelisable,*

(b) M *is* (r - 2)*-connected and* (r - 1)*-parallelisable,*

where $r \equiv 3, 5, 6, 7 \pmod 8$,

(c) M *is* (r - 3)*-connected and* (r - 2)*-parallelisable, where* $r \equiv 6, 7 \pmod 8$,

(d) M *is* (r - 4)*-connected and* (r - 3)*-parallelisable, where* $r \equiv 7 \pmod 8$.

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Proof of the theorem. For $k \geq [m/2] + 1$, M is a homotopy sphere and $I(M) \simeq 0 \simeq \theta_m(k)$.

For $m = 2r$ and $k = r$ the obstruction to r-parallelisability of an $(r - 1)$-connected 2r-manifold M is a well defined element $\hat{\alpha} \in H^r(M; \pi_{r-1}(SO))$. By Theorem 10 of Wall [5], $I(M) \simeq 0$ for $\hat{\alpha} \equiv 0 \pmod 2$ and $I(M) = \theta_{2r}(r - 1)$ for $\hat{\alpha} \not\equiv 0 \pmod 2$, and so if M is r-parallelisable, $\hat{\alpha} = 0$ when $I(M) \simeq 0 \simeq \theta_{2r}(r)$.

Excluding the cases above, M is χ-equivalent to a k-connected manifold N, by surgeries of type $(k + 1, m - k)$. Since M is k-parallelisable, the surgeries exist by Wall [4] for $m = 2r + 1$ and $k = r$, and by Milnor [3] for the other cases. Thus N is χ-equivalent to M, by surgeries of type $(m - k, k + 1)$, and a manifold L is defined from $N \times I$ by attaching $(m - k)$-handles to the boundary component $N \times \{1\}$ by the surgery embeddings $S^{m-k-1} \times D^{k+1} \to N$. Adding the handles performs surgery on the boundary, and so the boundary of L consists of the disjoint union of M and N. Since N is k-connected and $m - k - 1 \geq k$, L is also k-connected.

Suppose $\Sigma \in I(M)$ with a diffeomorphism $f: M \to M \# \Sigma$. From the disjoint union of L and $\Sigma \times I$, the boundary connected sum $L + (\Sigma \times I)$ is formed by taking the connected sum $M \# (\Sigma \times \{1\})$ of M and the boundary component $\Sigma \times \{1\}$. $L + (\Sigma \times I)$ is k-connected and its boundary is the disjoint union of N, $M \# \Sigma$, and Σ. A manifold W is now formed by glueing L to $L + (\Sigma \times I)$ by the identity map $N \to N$ and the diffeomorphism $f: M \to M \# \Sigma$. This manifold W has boundary Σ, and it easily follows that $H_1(W) \simeq \pi_1(W) \simeq \mathbb{Z}$ with $H_i(W) \simeq 0$ for $2 \leq i \leq k$. Now killing $\pi_1(W) \simeq \mathbb{Z}$ by surgery, which does not affect $\pi_2(W), \cdots, \pi_k(W)$, gives the required k-connected manifold with boundary Σ. The surgery exists by Theorem 3 of Milnor [3] since W is orientable and so 1-parallelisable.

REFERENCES

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