COMPACTNESS CONDITIONS IN TOPOLOGICAL GROUPS

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INTRODUCTION

In 1938, H. Fitting proved the following theorem (see [1] or [3, Section 61]).

THEOREM (H. Fitting). *Every finite group is an extension of a soluble group by a semisimple group.*

Thus the study of the theory of finite groups is reduced to the study of soluble groups, semisimple groups, and the theory of group extensions. Semisimple groups were first defined in the context of finite groups, by Fitting. Later, the idea was extended to infinite groups by P. A. Gol'berg [2]. In the paper [6], the idea was further extended to topological groups.

In this paper, we attempt to find a theorem similar to Fitting's for topological groups. For this purpose, we introduce in Section 1 the concept of W-subgroups of a topological group. In fact, each W-subgroup of a topological group G is a member of a well-ordered ascending soluble chain in G, and it is therefore a generalized soluble group. We shall show that every topological group has a unique maximal W-subgroup, called the W-*radical* of the group. A topological group is called W-*soluble* if it coincides with its W-radical. A structure theorem for W-soluble groups will also be given in Section 1.

Now, for an arbitrary topological group G with W-radical Q, one might expect G/Q to be topologically semisimple (see [6] for the definition of a topologically semisimple group). However, this is not true unless G/Q satisfies a certain minimal condition. This is justified by the following characterization of a topologically semisimple group G:

(a) G has no nontrivial, normal, abelian subgroups, and

(b) every nontrivial, closed, normal subgroup of G has a minimal closed, normal subgroup.

The minimal condition in (b) is certainly satisfied by finite groups. Thus finite semisimple groups can be defined by condition (a) alone. We find that (b) is also satisfied by discrete groups with certain finiteness conditions, for instance, periodic FC-groups (locally normal groups). Thus, if G is a periodic FC-group and Q is its W-radical, then G/Q is semisimple.

This last result leads us to the study of topological groups with an analogous compactness condition. An element in a topological group G is called *bounded* if its class of conjugates is relatively compact. The set of all bounded elements of G, denoted by B(G), forms a normal subgroup of G. An element in G is called *periodic* if it generates a relatively compact subgroup of G. The set of all periodic elements of G is denoted by P(G). In Section 3, we shall describe the structure of a totally disconnected, locally compact group with the compactness condition, $G = P(G) = \overline{B(G)}$, through the study of its W-radical Q and the factor group G/Q. This can also be

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considered as a continuation of the authors' previous paper [5]. In Section 2 we develop some results on locally projectively soluble groups, to prove that the W-radical Q is locally projectively soluble.

We shall follow [5] and [6] in notation and terminology.

1. W-SOLUBLE TOPOLOGICAL GROUPS

Let us consider, in a topological group G, a well-ordered ascending chain

$$e = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots$$

of closed, normal subgroups such that

(1) $A_\beta = \overline{\bigcup_{\alpha<\beta} A_\alpha}$ if β is a limiting ordinal and

(2) $A_\beta / A_{\beta-1}$ is a nontrivial abelian subgroup if $\beta - 1$ exists. Such a chain is called a W-*chain* of G. A subgroup of G is called a W-*subgroup* of G if it is a member of a W-chain.

(1.1) PROPOSITION. *Let* f *be a continuous homomorphism of a topological group* G *onto a topological group* H. *If* N *is a* W-*subgroup of* G, *then* $\overline{f(N)}$ *is a* W-*subgroup of* H.

Proof. Let $e = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\gamma$ be a W-chain of G such that $A_\gamma = N$. We construct a W-chain of H leading up to $\overline{f(N)}$ by a transfinite argument. Let $B_0 = e$. Suppose μ is an ordinal such that, for each $\lambda < \mu$, a closed, normal subgroup B_λ of H contained in $\overline{f(N)}$ is determined. If μ is limiting, let $B_\mu = \overline{\bigcup_{\lambda<\mu} B_\lambda}$. Otherwise, consider the factor group $\overline{f(N)}/B_{\mu-1}$.

If $\overline{f(N)}/B_{\mu-1}$ is nontrivial, let β be the first ordinal such that $f(A_\beta)B_{\mu-1}/B_{\mu-1}$ is nontrivial. We claim that $f(A_\beta)B_{\mu-1}/B_{\mu-1}$ is abelian. Assume β is limiting; then $A_\beta = \overline{\bigcup_{\alpha<\beta} A_\alpha}$. This implies $f(A_\beta) \subseteq B_{\mu-1}$, which is a contradiction. Hence $\beta - 1$ exists. By assumption, $A_\beta/A_{\beta-1}$ is abelian. It follows that $f(A_\beta)/f(A_{\beta-1})$ is abelian. Since $f(A_{\beta-1}) \subseteq B_{\mu-1}$, $f(A_\beta)B_{\mu-1}/B_{\mu-1}$ is abelian. Now let B_μ be the closure of $f(A_\beta)B_{\mu-1}$ in H. Clearly, B_μ is a closed, normal subgroup of H contained in $\overline{f(N)}$, and $B_\mu/B_{\mu-1}$ is a nontrivial abelian group. Thus the transfinite argument is complete, and there exists an ordinal ρ such that $\overline{f(N)} = B_\rho$.

(1.2) PROPOSITION. *Let* N *be a* W-*subgroup of a topological group* G. *If* K *is a nontrivial normal subgroup of* G *contained in* N, *then* K *contains a nontrivial normal abelian subgroup of* G.

Proof. Let $e = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\gamma$ be a W-chain of G such that $A_\gamma = N$. There exists a least ordinal β such that $K \cap A_\beta \neq e$. We claim that $K \cap A_\beta$ is abelian. If β is limiting, then $A_\beta = \overline{\bigcup_{\alpha<\beta} A_\alpha}$. Since $K \cap A_\alpha = e$ for all $\alpha < \beta$, $K \cap A_\beta$ is a central subgroup of A_β, and hence abelian. If $\beta - 1$ exists, then $A_\beta/A_{\beta-1}$ is abelian. Therefore $(K \cap A_\beta)A_{\beta-1}/A_{\beta-1}$ is abelian. Since $(K \cap A_\beta)A_{\beta-1}/A_{\beta-1}$ is algebraically isomorphic to $(K \cap A_\beta)/(K \cap A_\beta) \cap A_{\beta-1}$ and $K \cap A_{\beta-1} = e$, $K \cap A_\beta$ is again abelian.

(1.3) THEOREM. *There exists, in every topological group* G, *a unique maximal* W-*subgroup* Q *of* G. *Furthermore, the factor group* G/Q *has no nontrivial* W-*subgroups.*

Proof. First we construct a W-chain of G by a transfinite argument. Let $A_0 = e$. Suppose β is an ordinal such that A_α is determined for each $\alpha < \beta$. If β is limiting, let $A_\beta = \overline{\bigcup_{\alpha<\beta} A_\alpha}$. Otherwise, let $A_\beta / A_{\beta-1}$ be a nontrivial, closed, normal, abelian subgroup of $G/A_{\beta-1}$ (if any exists at all). The chain must stop at some ordinal γ where G/A_γ has no nontrivial, normal, abelian subgroups. Let $Q = A_\gamma$; then Q is a W-subgroup of G.

Suppose H is any W-subgroup of G. If H is not contained in Q, then, by Proposition (1.1), the closure of HQ/Q in G/Q is a nontrivial W-subgroup of G/Q. By Proposition (1.2), HQ/Q contains a nontrivial, normal, abelian subgroup of G/Q, a contradiction. Hence H is contained in Q. It follows that Q is the unique maximal W-subgroup of G. This completes the proof of the theorem.

In view of the theorem above, we define the W-*radical* of a topological group to be the unique maximal W-subgroup of the group. A topological group is called W-*soluble* if it coincides with its W-radical. It is easy to see that a topological group G is W-soluble if and only if there exists a W-chain $\{A_\alpha\}_{0 \leq \alpha \leq \gamma}$ of G such that $A_\gamma = G$.

(1.4) PROPOSITION. *The* W-*radical of a topological group is the intersection of all closed normal subgroups* N *of* G *such that* G/N *has no nontrivial, normal, abelian subgroups.*

Proof. It suffices to show that if N is a closed normal subgroup of G such that G/N has no nontrivial, normal, abelian subgroups, then N contains the W-radical Q of G. Suppose Q is not contained in N; then the closure of QN/N in G/N is a nontrivial W-subgroup of G/N, by Proposition (1.1). But this implies that G/N has a nontrivial, normal, abelian subgroup, which contradicts the assumption on N.

The following two propositions follow easily from Propositions (1.1) and (1.2).

(1.5) PROPOSITION. *Every continuous homomorphic image of a* W-*soluble group is* W-*soluble.*

(1.6) PROPOSITION. *Every nontrivial normal subgroup of a* W-*soluble group* G *contains a nontrivial, normal, abelian subgroup of* G.

(1.7) PROPOSITION. *A topological group* G *is* W-*soluble if and only if each of its nontrivial continuous homomorphic images has a nontrivial, normal, abelian subgroup.*

Proof. The necessity of the proposition follows from Propositions (1.5) and (1.6).

Suppose each nontrivial, continuous homomorphic image of G has a nontrivial, normal, abelian subgroup. Let Q be the W-radical of G. Then G/Q has no nontrivial, normal, abelian subgroups (Theorem (1.3)). This implies that G/Q is trivial and hence G is W-soluble.

(1.8) PROPOSITION. *Every normal subgroup of a* W-*soluble group is* W-*soluble.*

Proof. Let G be a W-soluble group, and let H be a normal subgroup of G. Since the W-radical Q of H is a topologically characteristic subgroup of H (see [6] for the definition of a topologically characteristic subgroup of a topological group), it is normal in G. We show that $H \subseteq \overline{Q}$. In fact, if $H \not\subseteq \overline{Q}$, then $H\overline{Q}/\overline{Q}$ is a nontrivial normal subgroup of $G/\overline{Q}$. Since $G/\overline{Q}$ is W-soluble, $H\overline{Q}/\overline{Q}$ contains a nontrivial, normal, abelian subgroup, by Proposition (1.6). It follows that $H/Q = H/H \cap \overline{Q}$ has a nontrivial, normal, abelian subgroup, which is a contradiction.

We conclude this section with a structure theorem, which in a sense tells the degree of commutativity of a W-soluble group.

(1.9) THEOREM. *Let* G *be a* W-*soluble group, and let* A *be a maximal normal abelian subgroup of* G. *Then* G/A *can be embedded algebraically in the group of automorphisms on a nilpotent group of class at most two.*

Proof. Let H be the centralizer of A in G. Clearly, H contains A. First suppose A = H. For each element g in G, let $\phi(g) = I_g \mid A$. Then ϕ is a homomorphism of G into $\mathscr{A}(A)$ with kernel A. Hence G/A is embedded in $\mathscr{A}(A)$.

On the other hand, suppose $A \neq H$; then, by Proposition (1.6), H/A contains a nontrivial, normal, abelian subgroup of G/A. Let B/A be a maximal normal abelian subgroup of G/A contained in H/A. Since B is contained in H, it is easy to see that B is a nilpotent group of class at most two. Now we show that the centralizer K of B in G is equal to A. Since B is in the centralizer H of A, A is contained in K. Suppose K/A is nontrivial, then it contains a nontrivial, normal, abelian subgroup B_1/A of G/A. Since $B_1 \subseteq K$ and B/A and B_1/A are both abelian, BB_1/A is abelian. By the maximality of B/A, $BB_1/A = B/A$. This implies that $BB_1 = B$ and hence $B_1 \subseteq B$. It follows that B_1 is contained in the center Z(B) of B. But Z(B) = A, since $A \subseteq Z(B)$ and A is a maximal normal abelian subgroup of G. Hence $B_1 \subseteq A$, a contradiction. Thus K = A. The map that sends each g to $I_g \mid B$, is a homomorphism of G into $\mathscr{A}(B)$ with kernel A. Hence G/A is isomorphic to a subgroup of $\mathscr{A}(B)$.

2. LOCALLY PROJECTIVELY SOLUBLE GROUPS

We call a topological group G *projectively soluble* if each neighborhood of its identity contains a normal subgroup H such that the factor group G/H is soluble. A topological group is called *locally projectively soluble* if each of its finitely generated subgroups is projectively soluble. It is not hard to see that subgroups and continuous homomorphic images of projectively soluble (locally projectively soluble) groups are also projectively soluble (locally projectively soluble). The closure of a projectively soluble subgroup in a topological group is again projectively soluble.

(2.1) LEMMA. *A relatively compact, locally projectively soluble subgroup in a locally compact, totally disconnected group is projectively soluble.*

Proof. Let G be a locally compact, totally disconnected group, and let H be a locally projectively soluble subgroup of G with compact closure $\overline{H}$. Let K be an arbitrary open normal subgroup of $\overline{H}$. Then $\overline{H} = HK$ and $\overline{H}/K$ is finite. Since $H/H \cap K$ is isomorphic to $HK/K = \overline{H}/K$, $H/H \cap K$ is also finite. As a continuous homomorphic image of a locally projectively soluble group H, $H/H \cap K$ is soluble. Hence $\overline{H}/K$ is soluble. Since $\overline{H}$ has arbitrarily small open normal subgroups, it is projectively soluble.

(2.2) LEMMA. *Every compact subgroup of a locally compact, totally disconnected group is contained in an open compact subgroup.*

Proof. Let H be a compact subgroup of a locally compact, totally disconnected group G. Let K be any compact open subgroup of G. Then $N = \bigcap_{h \in H} h^{-1} K h$ is a compact open subgroup of G such that $h^{-1} N h \subseteq N$ for all h in H. This implies HN = NH and HN is an open compact subgroup containing H.

A topological group is called *topologically locally finite* if each of its finitely generated subgroups is relatively compact.

(2.3) LEMMA. *Let* G *be a topologically locally finite, locally compact, totally disconnected group. If* H *is a locally projectively soluble subgroup of* G, *then* $\overline{H}$ *is also locally projectively soluble.*

Proof. Let F be the closure of a finitely generated subgroup of $\overline{H}$. Then F is a compact subgroup of $\overline{H}$, and by Lemma (2.2), F is contained in an open compact subgroup K of $\overline{H}$. The subgroup $H \cap K$ is relatively compact, and it is dense in K. By Lemma (2.1), $\overline{H \cap K} = K$ is projectively soluble. As a subgroup of K, F is also projectively soluble.

(2.4) LEMMA. *Let* G *be a topologically locally finite, locally compact, totally disconnected group,* H *a locally projectively soluble subgroup, and* N *a normal, locally projectively soluble subgroup of* G. *Then* HN *is also a locally projectively soluble subgroup of* G.

Proof. Let $F = \overline{\langle x_1, x_2, \cdots, x_r \rangle}$ be the closure of a finitely generated subgroup of HN. For $i = 1, 2, \cdots, r$, let $x_i = h_i n_i$, where $h_i \in H$ and $n_i \in N$. If

$$L = \overline{\langle h_1, \cdots, h_r, n_1, \cdots, n_r \rangle},$$

then $F \subseteq L$, and hence it is sufficient to show that L is projectively soluble. Let $H_1 = \overline{\langle h_1, \cdots, h_r \rangle}$ and $L_1 = \overline{L \cap N}$. Then $L = H_1 L_1$. Since, by assumption, G is topologically locally finite, the subgroups, L, L_1, and H_1 are all compact. Let K be any open normal subgroup of L. Then $H_1/H_1 \cap K$ and $L_1/L_1 \cap K$ are both finite and soluble. This implies that both $H_1 K/K$ and $L_1 K/K$ are soluble. It follows that

$$(H_1 K/K)(L_1 K/K) = H_1 L_1/K = L/K$$

is soluble.

(2.5) LEMMA. *Let* G *be a compact, totally disconnected group and* H *a closed normal subgroup of* G. *If both* H *and* G/H *are projectively soluble, then* G *is also projectively soluble.*

Proof. Let K be an arbitrary open normal subgroup of G. Then KH/H is also an open normal subgroup of G/H. Since G/H is projectively soluble, the factor group (G/H)/(KH/H) is soluble. Hence G/KH is soluble. The group KH/K is isomorphic to $H/H \cap K$. But $H/H \cap K$ is soluble, since H is projectively soluble and $H \cap K$ is an open normal subgroup in H. This implies that KH/K is soluble. Now G/K, as an extension of the soluble group KH/K by the soluble group G/KH is itself soluble.

(2.6) THEOREM. *Let* G *be a topologically locally finite, locally compact, totally disconnected group. Then there exists, in* G, *a unique maximal closed, normal, locally projectively soluble subgroup* L. *Furthermore, the factor group* G/L *contains no nontrivial normal locally projectively soluble subgroups.*

Proof. The union of the members in an ascending chain of locally projectively soluble subgroups in G is obviously locally projectively soluble. By Lemma (2.3), the closure of the union is also locally projectively soluble. This implies the existence of a maximal closed, normal, locally projectively soluble subgroup L. The uniqueness of the subgroup L follows from Lemma (2.4).

To show that G/L contains no nontrivial normal locally projectively soluble subgroups, assume that H is a normal subgroup of G such that H contains L and H/L is locally projectively soluble. Let F be the closure of a finitely generated subgroup in H. Then FL/L is compact and projectively soluble. Since $F/(F \cap L)$ is topologically isomorphic to FL/L, it is also projectively soluble. As a compact subgroup in the locally projectively soluble group L, $F \cap L$ is projectively soluble (Lemma (2.1)). Now, by Lemma (2.5), we see that F is projectively soluble. Consequently, H is locally projectively soluble and is contained in L. This completes the proof of the theorem.

3. PERIODIC GROUPS WITH DENSE BOUNDED PARTS

An element in a topological group G is called *bounded* if its class of conjugates is relatively compact. The set of all bounded elements of G, denoted by $B(G)$, forms a topologically characteristic subgroup of G, and it is called the *bounded part* of G. Locally compact groups with dense bounded parts have been studied in the paper [5]. There, it was proved that every locally compact group with dense bounded part is an extension of a compact group by a direct product of a vector group and a locally compact, totally disconnected group with dense bounded part. Hence the problem of finding the structure of locally compact groups with dense bounded parts is, in a sense, reduced to the totally disconnected case. In a topological group G, an element is called *periodic* if it is contained in a compact subgroup. The set of all periodic elements in G is called the *periodic part* of G, and it is denoted by $P(G)$. If $G = P(G)$, then G is called *periodic*. It has been proved in [5] that $P(G)$ forms an open, topologically characteristic subgroup of G, and the factor group $G/P(G)$ is a discrete pure abelian group, whenever G is a locally compact, totally disconnected group with dense bounded part. In this section, we make an attempt to describe the structure of a periodic, locally compact, totally disconnected group G with dense bounded part, through the study of its W-radical Q and its factor group G/Q.

(3.1) LEMMA. *Let G be a periodic, locally compact group with dense bounded part, and let B be a compact neighborhood of some point in G. Then B generates a compact open subgroup of G.*

Proof. Let H be the subgroup generated by B. Clearly, H is an open subgroup of G. Hence

$$\overline{H \cap B(G)} \subseteq H \cap \overline{B(G)} = H \cap G = H.$$

Since $B(H) \supseteq H \cap B(G)$, it follows that $\overline{B(H)} = H$. As a compactly generated, locally compact group with dense bounded part, H is an $\overline{FC}$-group; that is, $B(H) = H$ (see Proposition 4 in [5]). Since H is periodic, it follows from Theorem 1 of [4] that H is compact.

(3.2) PROPOSITION. *If G is a periodic, locally compact group with dense bounded part, then every compactly generated subgroup of G is relatively compact; in particular, G is topologically locally finite.*

Proof. Let K be the subgroup generated by a compact subset C. Let B be a compact neighborhood of the identity. Then CB is a compact subset with nonempty interior. By Lemma (3.1), CB generates a compact open subgroup H of G. Clearly, K is contained in H and is therefore relatively compact.

Remark. Let G be a periodic, locally compact, totally disconnected group with dense bounded part. From Proposition (3.2), G is topologically locally finite.

Hence, in view of Theorem (2.6), we can speak of the unique maximal closed, normal, locally projectively soluble subgroup of G.

(3.3) THEOREM. *Let* G *be a topologically semisimple, locally compact group with dense bounded part. Then* G *can be embedded algebraically in the group of automorphisms on a weak direct product of nonabelian, compact, simple groups.*

Proof. Let $R = \overline{\sum S_i}$ be the TCR-radical of G, where each S_i is a closed, nonabelian, topologically simple subgroup of G (see [6] for notation and terminology). Since the centralizer of $\sum S_i$ in G is trivial, we can embed G in $\mathscr{A}\left(\sum S_i\right)$ by assigning each element g of G to the restriction of I_g to $\sum S_i$. It is sufficient to show that each S_i is compact. We claim that $B(G) \cap S_i \neq e$ for all i. Suppose there exists an index i such that $B(G) \cap S_i = e$. Then $B(G) \cap g^{-1} S_i g = e$ for all $g \in G$. Since, by Proposition (3.3) of [6], each $g^{-1} S_i g$ is still a factor of $\sum S_i$, it follows that the family $\{g^{-1} S_i g \mid g \in G\}$ is direct and each $g^{-1} S_i g$ is normal in $T = \sum_{g \in G} g^{-1} S_i g$.

Let $D = B(G) \cap T$. Since $D \cap g^{-1} S_i g = e$ for all $g \in G$, D is contained in the center of T, and hence it is trivial. But this implies that T is a central subgroup of G, since T is normal in G and B(G) is dense in G. Thus we have arrived at a contradiction. Therefore $B(G) \cap S_i \neq e$ for all i. Since $B(S_i) \supseteq B(G) \cap S_i$ and S_i is topologically simple, it follows from Theorem (1.10) of [6] that S_i is compact and algebraically simple. The proof of the theorem is complete.

Let G be a periodic, locally compact, totally disconnected group with dense bounded part. Let Q be the W-radical of G. One may ask the following question. Is G/Q topologically semisimple? We show this is so when G is discrete.

(3.4) THEOREM. *If* G *is a periodic* FC-*group and* Q *is the* W-*radical of* G, *then* G/Q *is semisimple.*

Proof. From Theorem (1.3), we see that G/Q has no nontrivial normal abelian subgroups. Let N be any nontrivial normal subgroup of G/Q. Since G is a periodic FC-group, so is G/Q. This implies that N contains a nontrivial finite normal subgroup of G/Q. Thus N has a minimal normal subgroup. It follows from Proposition (4.3) of [6] that G/Q is semisimple.

Although the answer to the question is positive when G is discrete, we shall have two examples showing that G/Q is not necessarily topologically semisimple even when G is compact and totally disconnected. Let us first prove two lemmas.

(3.5) LEMMA. *Let* S *be a nonabelian simple group, and let* F *be an arbitrary group. Then the wreath product* $G = S \circledw F$ *is semisimple.*

Proof. For each $f \in F$, let S_f be a copy of S, and let $\hat{f}$ be the automorphism on $\prod_{f \in F} S_f$ defined by

$$(\hat{f}(a))_{f'} = a_{ff'} \qquad \text{for all } a \in \prod_{f \in F} S_f \text{ and } f' \in F.$$

Then G can be expressed as the semidirect product $\left(\prod_{f \in F} S_f\right) \circledS F$, where F acts on $\prod_{f \in F} S_f$ through the homomorphism $f \to \hat{f}$. Clearly, $\prod_{f \in F} S_f \times e$ is contained in the TCR-radical of G (considering G as a discrete group). It suffices to show that the centralizer Z of $\prod_{f\in F} S_f \times e$ in G is trivial. Let $(a, b) \in Z$, where $a \in \prod_{f\in F} S_f$ and $b \in F$; then

$$(a, b)(x, e) = (a\hat{b}(x), b) = (x, e)(a, b) = (xa, b)$$

for all $(x, e) \in \prod_{f\in F} S_f \times e$. This implies that $a\hat{b}(x) = xa$ for all $x \in \prod_{f\in F} S_f$. Therefore $a_f(\hat{b}(x))_f = a_f x_{bf} = x_f a_f$ for all $x \in \prod_{f\in F} S_f$ and $f \in F$. If b is not equal to e, then $bf \neq f$. Let x be such that $x_f \neq e$ and $x_{bf} = e$; then $a_f = x_f a_f$. But this implies $x_f = e$, which is a contradiction. Hence b must be e. Consequently, $(a, b) = (a, e)$ lies in the center of $\prod_{f\in F} S_f \times e$ and must be trivial.

Remark. The subgroup $\prod_{f\in F} S_f \times e$ of G mentioned in the proof of the lemma above is, in fact, the TCR-radical of G. Let $R_0 = \prod_{f\in F} S_f \times e$. Suppose R_0 is not equal to the TCR-radical R of G; then $R = R_0 \times T$ for some nontrivial normal subgroup T of R. But this implies T is in the centralizer of R_0 and must be trivial.

(3.6) LEMMA. *Let* A *be an abelian group that has an element of order greater than two, and let* F *be an arbitrary group. Then every normal abelian subgroup of the wreath product* $G = A \circledw F$ *is contained in the base subgroup of* G.

Proof. As in the proof of Lemma (3.5), G can be expressed as the semidirect product $\left(\prod_{f\in F} A_f\right) \circleds F$, where F acts on $\prod_{f\in F} A_f$ through the homomorphism $f \to \hat{f}$. Let N be a normal abelian subgroup of G. We show that N is contained in the base subgroup $\prod_{f\in F} A_f \times e$ of G. Let $(a, f) \in N$ and $(x, e) \in \prod_{f\in F} A_f \times e$. Since N is normal,

$$(x, e)(a, f)(x^{-1}, e) = (xa\hat{f}(x^{-1}), f)$$

is still an element of N. Thus we have the relation

$$(a, f)(xa\hat{f}(x^{-1}), f) = (a\hat{f}(x)\hat{f}(a)\hat{f}(\hat{f}(x^{-1})), f^2) = (xa\hat{f}(x^{-1}), f)(a, f) = (xa\hat{f}(x^{-1})\hat{f}(a), f^2).$$

This implies $\hat{f}(x)\hat{f}(x) = x\hat{f}(\hat{f}(x))$. Hence

$$(\hat{f}(x)\hat{f}(x))_e = x_f^2 = (x\hat{f}(\hat{f}(x)))_e = x_e x_{f^2}.$$

Now suppose $f \neq e$. If $f^2 \neq e$, let x be such that $x_f = x_{f^2} = e$ and $x_e \neq e$. But this is contradictory to the equation $x_f^2 = x_e x_{f^2}$. If $f^2 = e$, let x be such that $x_e = e$ and $x_f^2 \neq e$. Again we have a contradiction. Hence $f = e$.

(3.7) *Example.* Let S be a nonabelian, finite, simple group. Define a sequence $\{S_n\}$ of finite semisimple groups as follows: $S_1 = S$, $S_{n+1} = S \circledw S_n$ for $n \geq 1$. For each $n \geq 1$, let ϕ_n be the natural projection of S_{n+1} onto S_n. Now let G be the projective limit of $\{S_n\}$ induced by $\{\phi_n\}$. Then G is a closed subgroup of the complete direct product $\prod_{n\geq 1} S_n$ with the product topology. Clearly, G is a compact, totally disconnected group. We show that the TCR-radical R and the W-radical Q of G are both trivial. From this it follows that G/Q is not topologically semisimple. Since $S_{n+1} = S \circledw S_n$ for all $n \geq 1$, every element s_{n+1} in S_{n+1} is of the unique form $x_{n+1}s_n$, where $s_n \in S_n$. Hence s_{n+1} can also be written as

$x_{n-1}x_n \cdots x_1$, where $x_m x_{m-1} \cdots x_1 = s_m \in S_m$ for $m \leq n+1$. Now it is not hard to see that the elements in G are of the form

$$(x_1, x_2x_1, \cdots, x_nx_{n-1}\cdots x_1, \cdots).$$

For $n \geq 1$, let K_n be the closed normal subgroup of $\prod_{n\geq 1} S_n$ consisting of elements whose first n components are identities. Put $H_n = G \cap K_n$. Then H_n is a closed normal subgroup of G. Let B_n be the subgroup of G consisting of elements of the form

$$(x_1, x_2x_1, \cdots, x_nx_{n-1}\cdots x_1, e\,x_nx_{n-1}\cdots x_1, eex_n\cdots x_1, \cdots).$$

Then it is easy to see that the mapping ψ_n sending $(x_1, x_2x_1, \cdots, x_nx_{n-1}\cdots x_1, \cdots)$ into

$$(x_1, x_2x_1, \cdots, x_nx_{n-1}\cdots x_1, ex_nx_{n-1}\cdots x_1, eex_nx_{n-1}\cdots x_1, \cdots)$$

is a continuous homomorphism of G onto B_n with kernel equal to H_n. Now let A be a normal abelian subgroup of G; then $\psi_n(A)$ is also a normal abelian subgroup of B_n. Since B_n is semisimple, $\psi_n(A)$ is trivial. Hence A lies in the kernel H_n of ψ_n, for each n. It follows that A is trivial, because the intersection of all H_n is trivial. This proves that the W-radical Q of G is trivial.

Next let R be the TCR-radical of G. Then $\psi_n(R)$ is contained in the TCR-radical of B_n. This implies every element of R is of the form

$$(\overbrace{e, e, \cdots, e}^{(n-1)}, x_nee\cdots e, x_{n+1}x_nee\cdots e, \cdots).$$

Since n is arbitrary, R must be trivial.

(3.8) *Example.* Let A be a finite, abelian group that has an element of order greater than two. Define a sequence $\{A_n\}$ of soluble groups as follows: $A_1 = A$, $A_{n+1} = A \circledw A_n$ for $n \geq 1$. Following the same scheme as that in the last example, we obtain a projective limit G of the soluble groups A_n.

Again we show that the W-radical Q and the TCR-radical R of G are trivial. For each $n \geq 1$, define H_n, B_n, ψ_n as in the last example. Since B_n is soluble, $\psi_n(R)$ is trivial. This implies that R is trivial. Next, assume H is a normal abelian subgroup of G. Then $\psi_n(H)$ is also a normal abelian subgroup of B_n. By Lemma (3.6), $\psi_n(H)$ lies in the base subgroup of B_n. This implies that every element of Q is of the form

$$(\overbrace{e, e, \cdots, e}^{(n-1)}, x_nee\cdots e, x_{n+1}x_nee\cdots e, \cdots).$$

Since n is arbitrary, Q must be trivial.

(3.9) THEOREM. *Let* G *be a periodic, locally compact, totally disconnected group with dense bounded part. If* G *has no nontrivial, normal, abelian subgroups, then for each neighborhood* V *of the identity in* G, *there exists a compact normal subgroup* K *of* G *such that* $K \subseteq V$ *and* G *can be embedded algebraically in* $\mathscr{A}\left(\prod S_i \times K\right)$, *where each* S_i *is a finite, nonabelian, simple group.*

Proof. Suppose V is a neighborhood of the identity in G. There exists a compact open subgroup F of G contained in V. Let $\mathscr{K}$ be the family of normal subgroups of G contained in F. Let $\{K_\alpha\}$ be an increasing chain of elements in $\mathscr{K}$. Then the union of the members in the chain is still an element of $\mathscr{K}$. Hence there exists a maximal element K in $\mathscr{K}$. Moreover, K is unique, because the product of two elements in $\mathscr{K}$ is also an element in $\mathscr{K}$. Clearly, K is a compact, normal subgroup of G. Let H be the centralizer of K in G. Since, by assumption, G has no nontrivial, normal, abelian subgroups, $H \cap K = e$.

Suppose H = e. Then G is embedded algebraically in $\mathscr{A}(K)$ through the homomorphism $g \to I_g \mid K$.

Suppose $H \neq e$. We show that H is topologically semisimple. In fact, let E be any nontrivial, closed, topologically characteristic subgroup of H. Then E is a normal subgroup of G. If E meets B(G) trivially, then E is central. This contradicts the assumption of the theorem. Hence $E \cap B(G) \neq e$. Since G is periodic, E contains at least one nontrivial, compact, normal subgroup of G. Let $\{L_\alpha\}$ be a decreasing chain of nontrivial, compact normal subgroups of G contained in E. We claim that the intersection J of all L_α is not contained in F and hence is nontrivial. Suppose $J \subseteq F$. Then the family $\{L_\alpha^c\}$ together with F forms an open covering of G. Fix any L_{α_0}. There exist $L_{\alpha_1}, L_{\alpha_2}, \cdots, L_{\alpha_n}$ such that

$$L_{\alpha_0} \subseteq L_{\alpha_1}^c \cup L_{\alpha_2}^c \cup \cdots \cup L_{\alpha_n}^c \cup F .$$

This implies $\bigcap_{i=0}^n L_{\alpha_i} \subseteq F$. Let L_{α_j} be minimal among $L_{\alpha_0}, L_{\alpha_1}, \cdots, L_{\alpha_n}$. Then $L_{\alpha_j} = \bigcap_{i=0}^n L_{\alpha_j}$, and it is contained in F. Since K is the unique maximal normal subgroup of G contained in F, L_{α_j} must lie in K. But this is impossible, because L_{α_j} also lies in H and $H \cap K = e$. Consequently, E contains at least one minimal compact normal subgroup M of G. As a minimal closed normal subgroup of G, M is topologically characteristically simple. By Corollary (2.7) in [6], M is a complete direct product of isomorphic finite simple groups. Clearly, such simple groups are nonabelian. Hence the TCR-radical of E is nontrivial. From Proposition (4.1) of [6], we conclude that H is topologically semisimple.

Now let $R = \overline{\sum S_i}$ be the TCR-radical of H, where each S_i is a closed, nonabelian, topologically simple subgroup. From the fact that $\overline{B(G)} = G$, it follows that each S_i is compact (see the proof of Theorem (3.3)). As a compact, totally disconnected, simple group, each S_i is a finite group. Let Z be the centralizer of $\left(\sum S_i\right) K$ in G. Then Z is contained in the intersection of H and the centralizer $Z_G\left(\sum S_i\right)$ of $\sum S_i$ in G. Since $Z_G\left(\sum S_i\right) = Z_G(R)$, it follows that Z is contained in $Z_H(R)$ and is therefore trivial. Now it is easy to see that G can be embedded algebraically in $\mathscr{A}\left(\sum S_i \times K\right)$ through the mapping $g \to I_g \mid \left(\sum S_i\right) K$. The proof of the theorem is complete.

(3.10) THEOREM. *Let* G *be a periodic, locally compact, totally disconnected group with dense bounded part. If* G *is* W*-soluble, then* G *is locally projectively soluble and, for each neighborhood* V *of the identity in* G*, there exists a compact normal subgroup* K *of* G *contained in* V *such that every compact normal subgroup of* G/K *is soluble.*

Proof. Assume that G is W-soluble. Since G is a periodic, locally compact, totally disconnected group with dense bounded part, it is topologically locally finite by Proposition (3.2). It follows from Theorem (2.6) that G has a unique maximal closed, normal, locally projectively soluble subgroup L and G/L contains no nontrivial, normal, abelian subgroups. By Proposition (1.5), G/L is also W-soluble. This implies that G/L is trivial. Otherwise, G/L would contain a nontrivial, normal, abelian subgroup. Thus G is locally projectively soluble.

Let V be a neighborhood of the identity in G. There exists a compact open subgroup F of G contained in V. As mentioned in the proof of Theorem (3.7), there exists a unique maximal normal subgroup K of G contained in F. Let H/K be a compact normal subgroup of G/K. Then H is also a compact normal subgroup of G. As a compact, totally disconnected, locally projectively soluble group, H is projectively soluble (see Lemma (2.1)). This implies that some nth derived group H_n of H is contained in F. Since H_n is normal in G, H_n is contained in K. Therefore H/K is soluble.

(3.11) THEOREM. *Let* G *be a periodic, locally compact, totally disconnected group with dense bounded part, and let* Q *be its* W-*radical. Then the following statements are true:*

(1) G/Q *can be embedded algebraically in* $\mathscr{A}\left(\coprod S_i \times K\right)$, *where* $\coprod S_i$ *denotes a weak direct product of finite, nonabelian, simple groups and* K *is a compact, totally disconnected group.*

(2) Q *is locally projectively soluble.*

(3) Q *is an extension of a* W-*soluble, periodic, locally compact, totally disconnected group with dense bounded part by an abelian group.*

Proof. Since G/Q has no nontrivial, normal, abelian subgroups, statement (1) follows immediately from Theorem (3.9).

Let L be the unique maximal closed, normal, locally projectively soluble subgroup of G. Then G/L has no nontrivial, normal, abelian subgroups. It follows from Proposition (1.4) that Q is contained in L and is therefore locally projectively soluble. This proves statement (2).

Let Q_B be the W-radical of B(G), and let $R = \overline{Q_B}$. Then $Q_B = R \cap B(G)$. It is not hard to see that R is a W-subgroup of G and hence lies in Q. Let Q_1/R be the W-radical of G/R. Since $(G/R)/(Q_1/R) \cong G/Q_1$ has no nontrivial, normal, abelian subgroups, Q is contained in Q_1. Similarly, since $(G/R)/(Q/R) \cong G/Q$ has no nontrivial, normal, abelian subgroups, Q_1/R is contained in Q/R. It follows that Q/R is exactly the W-radical of G/R. We claim that Q/R is abelian. Since B(G)R/R is dense in G/R, it suffices to show $Q/R \cap B(G)R/R$ is trivial. In fact, if $Q/R \cap B(G)R/R$ is not trivial, then, by Proposition (1.2), B(G)R/R contains a nontrivial, normal, abelian subgroup of G/R. On the other hand, B(G)R/R, being algebraically isomorphic to $B(G)/R \cap B(G) = B(G)/Q_B$, contains no nontrivial, normal, abelian subgroups, and this is a contradiction. The proof of the theorem is complete.

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