RANK-ONE COMMUTATORS AND HYPERINVARIANT SUBSPACES

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Let $\mathscr{X}$ be a complex Banach space of dimension greater than one, and let $\mathscr{L}(\mathscr{X})$ denote the algebra of all bounded linear operators on $\mathscr{X}$. A subspace $\mathscr{M}$ of $\mathscr{X}$ is said to be *hyperinvariant* for an operator T in $\mathscr{L}(\mathscr{X})$ if $(0) \neq \mathscr{M} \neq \mathscr{X}$ and $T'\mathscr{M} \subset \mathscr{M}$ for every operator T' in $\mathscr{L}(\mathscr{X})$ that commutes with T. The following remarkable result was obtained by V. Lomonosov in [2] (see also [3] and [4, Section 8.2]).

LOMONOSOV'S THEOREM. *If* T *is a nonscalar element of* $\mathscr{L}(\mathscr{X})$ *(that is, if* $T \neq \lambda$*) and* T *commutes with a nonzero compact operator, then* T *has a hyperinvariant subspace.*

More recently, J. Daughtry, using the Lomonosov technique as developed in [3] and [4, Section 8.2], proved that if $T \in \mathscr{L}(\mathscr{X})$ and there exists a compact operator K in $\mathscr{L}(\mathscr{X})$ such that TK - KT has rank one, then T has a nontrivial invariant subspace. The purpose of this note is to show that the stronger version of Daughtry's theorem obtained by replacing the work "invariant" by the word "hyperinvariant" is valid, and moreover, that the hyperinvariant subspaces that arise do so for rather elementary reasons. Our central observation is the following theorem.

THEOREM 1. *Suppose that* S *and* T *are operators in* $\mathscr{L}(\mathscr{X})$ *such that* $R = TS - ST$ *has rank one and such that both* kernel S *and the quotient space* $\mathscr{X}/(\text{range } S)^-$ *are nonzero and finite-dimensional. Then there exists a complex number* λ *such that either* $T - \lambda$ *has a nontrivial kernel or the range of* $T - \lambda$ *is not dense in* $\mathscr{X}$. *Consequently,* T *has a hyperinvariant subspace.*

Proof. Let x be a nonzero vector such that $Sx = 0$, and consider the subspace $\mathscr{M} = \bigvee_{n=0}^{\infty} T^n x$. Clearly, $\mathscr{M}$ is nonzero and invariant under T. If $\mathscr{M}$ is finite-dimensional, then T has an eigenvalue, the corresponding eigenspace is a hyperinvariant subspace for T, and the theorem is proved. (Note that T is not a scalar, since R has rank one.) Thus we may assume that $\mathscr{M}$ is infinite-dimensional. Since the kernel of S is finite-dimensional, it is impossible that $\mathscr{M} \subset$ kernel S. Thus there exists a smallest positive integer $n_0 \geq 1$ such that $S(T^{n_0}x) \neq 0$. Hence

$$RT^{n_0 - 1}x = -ST^{n_0}x \neq 0,$$

and since R has rank one, it follows that the range of R is contained in the range of S. It follows immediately that $\mathscr{R} = (\text{range } S)^-$ is an invariant subspace for T and thus that $T^*\mathscr{R}^0 \subset \mathscr{R}^0$, where $\mathscr{R}^0$ denotes the annihilator of $\mathscr{R}$ in $\mathscr{X}^*$. Since $\dim(\mathscr{X}/\mathscr{R}) = \dim \mathscr{R}^0$, and since $\mathscr{X}/\mathscr{R}$ is nonzero and finite-dimensional by hypothesis, it follows that T^* has an eigenvalue $\bar{\lambda}$, and hence that

$$[\text{range}(T - \lambda)]^0 = \text{kernel}(T^* - \bar{\lambda})$$

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is nonzero. Thus the range of $T - \lambda$ is not dense in $\mathscr{X}$, and $[\text{range}(T - \lambda)]^-$ is a hyperinvariant subspace for T. The proof is complete.

The following corollary is an extension of the results of Lomonosov and Daughtry.

THEOREM 2. *Suppose that* K *is a nonzero compact operator in* $\mathscr{L}(\mathscr{X})$ *and that* T *belongs to* $\mathscr{L}(\mathscr{X})$ *and satisfies the condition* $TK - KT = R$, *where* $\text{rank } R \leq 1$. *Then* T *has a hyperinvariant subspace.*

Proof. Suppose not. Then the commuting algebra $\mathscr{A}$ of T is transitive, and thus by Theorem 2 of [3] there exist a nonzero vector x in $\mathscr{X}$ and an operator T' in $\mathscr{A}$ such that $T'Kx = x$. It follows that $T(T'K) - (T'K)T = T'R$, and there are two cases to consider. If $T'R = 0$, then T commutes with the nonzero compact operator $T'K$, and thus it has a hyperinvariant subspace, by Lomonosov's Theorem. If $T'R \neq 0$, then $T'R$ has rank one, and we also have the relation

$$T(T'K - 1) - (T'K - 1)T = T'R.$$

By virtue of the standard Riesz theory of compact operators, $\text{kernel}(T'K - 1)$ is a nonzero finite-dimensional subspace of $\mathscr{X}$, and the quotient space

$$\mathscr{X}/[\text{range}(T'K - 1)]^-$$

is also nonzero and finite-dimensional. Hence, by Theorem 1, T has a hyperinvariant subspace, which is a contradiction. Thus the theorem is proved.

If one could improve Theorem 2 by replacing the hypothesis that $\text{rank } R \leq 1$ by the weaker hypothesis that $\text{rank } R \leq 2$, then one would have solved affirmatively the hyperinvariant-subspace problem for operators on a complex Banach space. Indeed, if K has rank one, then for all T in $\mathscr{L}(\mathscr{X})$, the commutator $TK - KT$ has rank at most two. We conjecture that Theorem 2 remains true under the hypotheses that $\text{rank } R = n$ and that $\text{rank } K > n/2$. We also conjecture that Theorem 1 remains true under the hypotheses that $\text{rank } R = n > 0$ and that $\dim(\text{kernel } S)$ and $\dim(\mathscr{X}/[\text{range } S]^-)$ are greater than or equal to n.

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