GEODESIC SPHERES AND SYMMETRIES IN NATURALLY REDUCTIVE SPACES

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INTRODUCTION

In [2], the author and H. K. Nickerson proved that in a naturally reductive pseudo-Riemannian homogeneous space, the geodesic symmetries are divergence-preserving (volume-preserving up to sign). The proof is based on a complicated combinatorial identity; the paper [2] also contains a proof, due to N. Wallach, which is much simpler but not obviously applicable to all naturally reductive spaces. In Section 1 of the present paper, we prove an algebraic result, and in Section 2, we use it to extend Wallach's proof to all naturally reductive spaces. We find other restrictions on the geometry of naturally reductive spaces (respectively, harmonic spaces); in particular, in Section 3, assuming a positive-definite metric, we show that all sufficiently small geodesic spheres have antipodally symmetric (respectively, constant) mean and scalar curvatures. There is overlap here with work of S. Tachibana and T. Kashiwada [6].

1. ALGEBRAIC PRELIMINARIES

Let $\mathfrak{g}$ be a Lie algebra over a field F of characteristic different from 2, and suppose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a vector-space decomposition, where $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Let P and K denote projection on $\mathfrak{p}$ and $\mathfrak{k}$, respectively, and let B be a symmetric, bilinear F-valued form on $\mathfrak{p}$ satisfying the condition

$$B(P[V, Y], Z) + B(Y, P[V, Z]) = 0 \quad \text{for } V \in \mathfrak{g},\ Y, Z \in \mathfrak{p}. \tag{1}$$

LEMMA. *For* $X \in \mathfrak{p}$, *the map* $\operatorname{ad} X \circ K \circ \operatorname{ad} X\colon \mathfrak{p} \to \mathfrak{p}$ *is* B-*symmetric.*

Proof. For X, Y, Z $\in \mathfrak{p}$, we have the relations

$$\begin{aligned} B([X, K[X, Y]], Z) &= -B(X, [Z, K[X, Y]]) \\ &= -B(X, P[Z, [X, Y]]) + B(X, P[Z, P[X, Y]]), \\ -B(X, P[Z, [X, Y]]) &= B(X, P[X, [Y, Z]]) + B(X, P[Y, [Z, X]]) \\ &= B(X, P[Y, P[Z, X]]) + B(X, P[Y, K[Z, X]]) \\ &= -B(P[X, Y], P[X, Z]) + B([X, K[X, Z]], Y). \end{aligned}$$

Remark 1. In the situation of Section 2, the lemma is equivalent to the curvature identity $\langle R(X, Y)X, Z \rangle = \langle Y, R(X, Z)X \rangle$.

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PROPOSITION 1. *If* $X, Y, Z \in \mathfrak{p}$ *and* n *is a nonnegative integer, then* $B(P\circ(\mathrm{ad}\,X)^n Y, Z) = (-1)^n B(Y, P\circ(\mathrm{ad}\,X)^n Z)$. *Here* $(\mathrm{ad}\,X)^0$ *is the identity map* I.

The proof is by induction on n. The result is clear for $n = 0, 1$; we therefore suppose that $n > 1$ and that the result holds for all $m < n$. Then, using also the previous lemma, we have the equations

$$\begin{aligned} &B(P\circ(\mathrm{ad}\,X)^n Y, Z) \\ &\quad = B(P\circ(\mathrm{ad}\,X)^{n-1}\circ P\circ \mathrm{ad}\,X(Y), Z) + B(P\circ(\mathrm{ad}\,X)^{n-2}\circ \mathrm{ad}\,X\circ K\circ \mathrm{ad}\,X(Y), Z) \\ &\quad = (-1)^n B(Y, P\circ \mathrm{ad}\,X\circ P\circ(\mathrm{ad}\,X)^{n-1}Z) + (-1)^{n-2}B(Y, \mathrm{ad}\,X\circ K\circ \mathrm{ad}\,X\circ P\circ(\mathrm{ad}\,X)^{n-2}Z) \\ &\quad = (-1)^n B(Y, P\circ \mathrm{ad}\,X\circ(P+K)\circ(\mathrm{ad}\,X)^{n-1}Z) = (-1)^n B(Y, P\circ(\mathrm{ad}\,X)^n Z). \end{aligned}$$

In this calculation, note that $K\circ \mathrm{ad}\,X\circ P = K\circ \mathrm{ad}\,X$.

2. NATURALLY REDUCTIVE SPACES

Let M be a naturally reductive, pseudo-Riemannian homogeneous space. Thus we have a connected Lie group G with Lie algebra $\mathfrak{g}$, a closed subgroup K with Lie subalgebra $\mathfrak{k}$ such that $M = G/K$, and a vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ that is $\mathrm{Ad}\,K$ invariant. The projection $\pi: G \to G/K$ takes $e \in G$ to $0 = \pi(e) \in M$, and π_* is used to identify $\mathfrak{p}$ with T_0M. Finally, M has a G-invariant pseudo-Riemannian metric $\langle\ ,\ \rangle$, and this induces a nondegenerate symmetric bilinear form B on $\mathfrak{p}$ that satisfies condition (1). Summarizing some notation and results from [2], we let U be a symmetric neighborhood around zero in $\mathfrak{p}$ such that the map $\psi(X) = \pi(\exp X)$ is a diffeomorphism of U onto an open neighborhood of 0 in M. Since M is naturally reductive, ψ is the exponential Exp_0 for the pseudo-Riemannian structure. With $P: \mathfrak{g} \to \mathfrak{p}$ denoting projection, define $A_Z: \mathfrak{p} \to \mathfrak{p}$ for $Z \in U$ by the equation

$$A_Z = P\circ\frac{I - e^{-\mathrm{ad}\,Z}}{\mathrm{ad}\,Z} = \sum_{j\geq 0}\frac{(-1)^j}{(j+1)!}P\circ(\mathrm{ad}\,Z)^j. \tag{2}$$

The differential of the exponential exp for the Lie group is computed as in [3, page 95], and is given by the formula

$$(d\exp)_Z = d(L_{\exp Z})_e\circ\frac{I - e^{-\mathrm{ad}\,Z}}{\mathrm{ad}\,Z} \qquad (Z \in \mathfrak{g}).$$

Identifying the vector space $\mathfrak{p}$ with each of its tangent spaces and considering $\exp Z$ as a mapping of M, we obtain the relation

$$(d\psi)_Z = (d\exp Z)_0\circ A_Z. \tag{3}$$

Let ω be a volume form for the pseudo-Riemannian structure defined in a neighborhood of $0 \in M$, set $\tilde{\omega} = (\psi^*\omega)_0$, and take $\tilde{\omega}$ as a volume form on $\mathfrak{p}$, invariant under translation and invariant up to sign under the map $Z \to -Z$. Then

$$(\psi^*\omega)_Z = (\det A_Z)\tilde{\omega}_Z. \tag{4}$$

Also, since $\exp(-Z)$ is an isometry on M, we obtain the formula

(5) $$\langle (d\psi)_Z Y_1, (d\psi)_Z Y_2 \rangle_{\psi(Z)} = B(A_Z Y_1, A_Z Y_2) \quad \text{for } Y_i \in T_Z \mathfrak{p}.$$

Let A_Z^t denote the transpose of A_Z with respect to B; that is, let

(6) $$B(A_Z Y_1, Y_2) = B(Y_1, A_Z^t Y_2) \quad \text{for } Y_i \in \mathfrak{p}.$$

We now apply Proposition 1 and conclude that $A_Z^t = A_{-Z}$ (it was to obtain this relation that N. Wallach's argument, given in [2], needed to restrict itself to a special class of naturally reductive space). Combining these, we have the following result.

PROPOSITION 2. *If* M *is a naturally reductive, pseudo-Riemannian homogeneous space, then the geodesic symmetries of* M *are divergence-preserving (volume-preserving up to sign). Further, the space* M *is harmonic (in the sense of* [5]) *if and only if* $\det A_Z$ *is a function of* $B(Z, Z)$ *alone, and* M *is locally symmetric if and only if* A_Z *and* A_{-Z} *commute for all* $Z \in U$.

Remark 2. If $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we find $A_Z = A_{-Z}$, and therefore M is locally-symmetric (as is well-known).

Remark 3. Our results imply further restrictions on the geometry of M. For simplicity, suppose B is positive definite, $\{Z_1, \cdots, Z_n\}$ is an orthonormal basis for $\mathfrak{p}$, and $\{z^1, \cdots, z^n\}$ is the corresponding normal coordinate system on $\psi(U)$; for $Z = \sum z^k Z_k$, define $a_i^j(Z) = a_i^j(z^1, \cdots, z^n)$ by the equation

(7) $$A_Z Z_i = \sum_{j=1}^{n} a_i^j(Z) Z_j.$$

Then the metric coefficients of M in the normal coordinate system are given by the formula

(8) $$g_{ij} = \sum_{k=1}^{n} a_i^k a_j^k.$$

Thus the eigenvalues of the matrix (g_{ij}) are the same at $\psi(Z)$ and $\psi(-Z)$. Equivalently, if $\langle\ ,\ \rangle^o$ denotes the flat metric on $\psi(U)$ induced by $d\psi$ from B on $\mathfrak{p}$, then the eigenvalues of $\langle\ ,\ \rangle$ with respect to $\langle\ ,\ \rangle^o$ are the same at $\psi(Z)$ and $\psi(-Z)$. Note that for a local isometry f of $(M, \langle\ ,\ \rangle)$ with fixed point 0, these eigenvalues are also the same at p and f(p), but in general there is no such f taking $\psi(Z)$ to $\psi(-Z)$.

Remark 4. For future reference, note that since $\exp(-Z)$ is an isometry translating the geodesic $t \to \psi(tZ)$ along itself, we have the equation

(9) $$\mathrm{Ric}_{\psi(Z)}((d\psi)_Z Z, (d\psi)_Z Z) = \mathrm{Ric}_0(Z, Z).$$

3. GEODESIC SPHERES

Now let $(M, \langle\ ,\ \rangle)$ be any Riemannian manifold with normal coordinate system $(z^1, \cdots, z^n)$, centered and orthogonal at $0 \in M$. Let s be the geodesic distance from 0, and let X be the unit tangent vector field along geodesic rays emanating from 0 given by $sX = \sum z^k \frac{\partial}{\partial z^k}$. Define (1, 1)-tensor fields Λ and Π by

$\Lambda(Y) = \nabla_Y(sX)$ and $\Pi(Y) = R(X, Y)X$ (see [1]). Let $\mathscr{S}(s)$ be the geodesic sphere of radius s centered at 0. Since X gives a unit normal field to $\mathscr{S}(S)$, the shape operator σ and the second fundamental form h of $\mathscr{S}(s)$ in M are given by

$$(10) \qquad h(Y_1, Y_2) = \langle \sigma(Y_1), Y_2 \rangle, \qquad \sigma(Y) = -\nabla_Y X = -\frac{1}{s}\Lambda(Y),$$

for vectors Y tangent to $\mathscr{S}(s)$. Note that $\Lambda(X) = X$. Thus we can compute the mean curvature $\mathscr{H}$ of $\mathscr{S}(s)$ by

$$(11) \qquad (n-1)\mathscr{H} = \operatorname{Tr} \sigma = \frac{1}{s}(1 - \operatorname{Tr} \Lambda),$$

a formula also essentially found by Tachibana and Kashiwada [6].

We can also compute the trace of Λ by

$$\operatorname{Tr} \Lambda = \operatorname{div} sX = \sum_k \left(\frac{\partial}{\partial z^k} z^k + z^k \operatorname{div} \frac{\partial}{\partial z^k} \right) = n + \sum_{i,k} z^k \Gamma^i_{ik} = n + sX \cdot \log \sqrt{g},$$

where Γ^i_{ik} are Christoffel symbols and g is the determinant of the coefficients of the metric tensor in the normal coordinate system. Therefore we have proved the following result.

PROPOSITION 3. *The geodesic symmetry at* 0 *is divergence-preserving in a neighborhood of* 0 *if and only if each sufficiently small geodesic sphere centered at* 0 *has antipodally symmetric mean curvature. Further* (Tachibana and Kashiwada [6]), M *is harmonic at* 0 *if and only if each sufficiently small geodesic sphere centered at* 0 *has constant mean curvature.*

Let $\mathscr{K}$ be the scalar curvature of $\mathscr{S}(s)$, and K the scalar curvature of M. The Gauss equation yields the formula

$$(12) \qquad \mathscr{K} = K - 2\operatorname{Ric}(X, X) - \operatorname{Tr} \sigma^2 + (\operatorname{Tr} \sigma)^2,$$

and, since $\Lambda(X) = X$, we see that $\operatorname{Tr} \sigma^2 = \frac{1}{s^2}(\operatorname{Tr} \Lambda^2 - 1)$. For any vector field Y, we also have the relation

$$\begin{aligned} R(Y, sX)sX &= \nabla_Y \nabla_{sX}(sX) - \nabla_{sX} \nabla_Y(sX) - \nabla_{[Y,sX]}(sX) \\ &= -(\nabla_{sX}\Lambda)(Y) + \Lambda(Y) - \Lambda(\Lambda(Y)). \end{aligned}$$

This yields the basic differential equation [1], [4], [5]

$$(13) \qquad \operatorname{Tr} \Lambda^2 = \operatorname{Tr} \Lambda - sX \cdot \operatorname{Tr} \Lambda + s^2 \operatorname{Tr} \Pi.$$

Remark 4 shows that if M is a naturally reductive, Riemannian homogeneous space, then $\operatorname{Tr} \Pi = -\operatorname{Ric}(X, X)$ is antipodally symmetric on geodesic spheres centered at 0, while if M is harmonic, it is an Einstein space and $\operatorname{Tr} \Pi$ is constant on geodesic spheres centered at 0. The same holds for $\operatorname{Tr} \Lambda$ and $sX \cdot \operatorname{Tr} \Lambda$, thus by (13) for $\operatorname{Tr} \Lambda^2$. In both cases, K is constant. Therefore we have proved the following result.

PROPOSITION 4. *If* M *is a naturally reductive, Riemannian homogeneous space, each sufficiently small geodesic sphere has antipodally symmetric mean and scalar curvatures. If* M *is harmonic, each sufficiently small geodesic sphere has constand mean and scalar curvatures.*

Remark 5. If M is a naturally reductive, Riemannian homogeneous space, and we identify $\mathfrak{p}$ and $T_{\psi(Z)}M$ via $(d\psi)_Z$, then at $\psi(Z)$ we find that

$$\Lambda = \frac{1}{2} A_Z^{-1} \circ (A_{-Z}^{-1} \circ P \circ e^{\operatorname{ad} Z} + P \circ e^{-\operatorname{ad} Z} \circ A_Z^{-1}) \circ A_Z, \tag{14}$$

$$-\operatorname{sh}(Y_1, Y_2) = \frac{1}{2} B(Y_1, P \circ e^{\operatorname{ad} Z} \circ A_Z(Y_2)) + \frac{1}{2} B(P \circ e^{\operatorname{ad} Z} \circ A_Z(Y_1), Y_2) \tag{15}$$

for all Y_i orthogonal to Z in $\mathfrak{p}$. To prove this, note that the components of Λ in the normal coordinate system $(z^1, \cdots, z^n)$ are

$$\Lambda_i^j = \delta_i^j + \frac{1}{2} (sX \cdot g_{ir}) g^{rj}.$$

Using the Euclidean coordinate system $(z^1, \cdots, z^n)$ on $\mathfrak{p}$, we can apply sX to $\operatorname{End}(\mathfrak{p}, \mathfrak{p})$-valued functions on $\mathfrak{p}$ and obtain the equations

$$sX \cdot (P \circ (\operatorname{ad} Z)^j) = j P \circ (\operatorname{ad} Z)^j, \tag{16}$$

$$sX \cdot A_Z = -A_Z + P \circ e^{-\operatorname{ad} Z}. \tag{17}$$

Now use (8) and $A_Z^t = A_{-Z}$ to obtain (14) and (15).

It is useful to note that with the identifications indicated,

$$d(\exp(-Z)) \colon T_{\psi(Z)} M \to T_0 M$$

is given by A_Z.

Remark 6. If M is a naturally reductive, Riemannian homogeneous space, then ad hoc calculations (omitted) using formulas in [1] and [2] indicate that Tr Λ^3 is not in general antipodally symmetric, although we have not constructed a specific counterexample.

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