LIFTING OF OPERATORS THAT COMMUTE WITH SHIFTS

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1. INTRODUCTION

Throughout this paper, all Hilbert spaces are assumed to be complex, all operators are assumed to be linear and bounded, and all subspaces are assumed to be closed.

If S is a unilateral shift on a Hilbert space $\mathscr{H}$, $\mathscr{M}$ is a subspace of $\mathscr{H}$ that is invariant under S^*, and T is an operator on $\mathscr{M}$ commuting with S^*, then there exists an operator T_1 on $\mathscr{H}$, also commuting with S^*, that satisfies the condition $\|T_1\| = \|T\|$, and the restriction $T_1 \mid \mathscr{M}$ coincides with T. This particular result has been proved in different ways by several authors [2], [4], [7]. Here we shall consider a related lifting question that is in the nature of an adjoint problem.

If $\mathscr{M}$ is a subspace of $\mathscr{H}$ that is invariant under S and T is an operator on $\mathscr{M}$ commuting with S, does there exist an operator T_1 on $\mathscr{H}$ that also commutes with S and is such that $T_1 \mid \mathscr{M} = T$? Unlike the lifting problem for co-isometries above, this problem has no solution unless a subsidiary condition is satisfied. For each integer $k \geq 1$, the operator $P_k = I - S^k S^{*k}$ is an orthogonal projection on $\mathscr{H}$. If T_1 is an operator on $\mathscr{H}$ commuting with S, then $P_k T_1 P_k = P_k T_1$. Thus, for all $u \in \mathscr{H}$ and all $k \geq 1$, we have

$$\|P_k T_1 u\| = \|P_k T_1 P_k u\| \leq \|T_1\| \; \|P_k u\| .$$

In particular, if $T = T_1 \mid \mathscr{M}$, then for all $u \in \mathscr{M}$, T must satisfy the inequality $\|P_k T u\| \leq \|T_1\| \; \|P_k u\|$. Thus, if an operator T on $\mathscr{M}$ has an extension to $\mathscr{H}$ that commutes with S, then the value

$$\alpha = \sup_{k \geq 1} \sup_{u \in \mathscr{M}} \frac{\|P_k T u\|}{\|P_k u\|} \tag{1.1}$$

must necessarily be finite. Note that by (1.1) $\alpha \geq \|T\|$, while by the remarks above, $\alpha \leq \|T_1\|$ for any extension T_1. L. B. Page [5] conjectured that the condition (1.1) is also sufficient to ensure that T has an extension T_1 to all of $\mathscr{H}$ such that $T_1 S = S T_1$; he conjectured further that if T_1 is an extension of minimal norm, then $\|T_1\| = \alpha$, and he was able to prove this conjecture in several special cases. For shifts of finite multiplicity, the conjecture was proved in [6] by means of a Hardy space model. The object of this paper is to prove this conjecture in the general form given here.

We shall prove our results by reducing the problem to the lifting theorem for co-isometries.

Received December 24, 1974.
This work was supported by National Research Council Grant A-7352.

Michigan Math. J. 22 (1975).

2. RESULTS

THEOREM 1. *Suppose that* S *is a unilateral shift on a Hilbert space* $\mathscr{H}$, *that* $\mathscr{M}$ *is a subspace of* $\mathscr{H}$ *invariant under* S, *and that* T *is an operator on* $\mathscr{M}$ *commuting with* S. *Then, if in addition*

$$\alpha = \sup_{k \geq 1} \sup_{u \in \mathscr{M}} \frac{\|P_k T u\|}{\|P_k u\|} \tag{2.1}$$

is finite, there exists an operator T_1 *on* $\mathscr{H}$ *such that*

$$T_1 S = S T_1, \quad T_1 \mid \mathscr{M} = T, \quad \text{and } \|T_1\| = \alpha .$$

In the same manner that Theorem 4 of [4] was extended to Theorem 5 on intertwining operators, Theorem 1 above can be extended to an apparently more general result. For i = 1, 2, let S_i be a unilateral shift on the Hilbert space $\mathscr{H}_i$, and put $P_{ik} = I - S_i^k S_i^{*k}$, for $k \geq 1$.

THEOREM 2. *Suppose that for* i = 1, 2, *the operator* S_i *is a unilateral shift on the Hilbert space* $\mathscr{H}_i$, *that* $\mathscr{M}$ *is a subspace of* $\mathscr{H}_1$ *invariant under* S_1, *and that* X *is an operator from* $\mathscr{M}$ *to* $\mathscr{H}_2$ *such that* $S_2 X = X S_1$. *Then, if in addition*

$$\alpha = \sup_{k \geq 1} \sup_{u \in \mathscr{M}} \frac{\|P_{2k} X u\|}{\|P_{1k} u\|} \tag{2.2}$$

is finite, there exists an operator Y *from* $\mathscr{H}_1$ *to* $\mathscr{H}_2$ *such that* $S_2 Y = Y S_1$, $Y \mid \mathscr{M} = X$, *and* $\|Y\| = \alpha$.

The deduction of Theorem 2 from Theorem 1 requires only a standard construction; therefore we shall prove Theorem 2 here.

Proof of Theorem 2. If $\mathscr{H}$ denotes the product Hilbert space $\mathscr{H}_1 \times \mathscr{H}_2$, then S, where $S(u_1, u_2) = (S_1 u_1, S_2 u_2)$ for $(u_1, u_2) \in \mathscr{H}$, is a unilateral shift on $\mathscr{H}$ and $\mathscr{M}_1 = \mathscr{M} \times \mathscr{H}_2$ is a subspace of $\mathscr{H}$ invariant under S. If we put

$$T(u_1, u_2) = (0, X u_1)$$

for $(u_1, u_2) \in \mathscr{M}_1$, then TS = ST and T satisfies (2.1) with α given by (2.2). Hence by Theorem 1, T has an extension T_1 to all of $\mathscr{H}$ that commutes with S, and such that $\|T_1\| = \alpha$. Since

$$0 \times \mathscr{H}_2 \subset \mathscr{M}_1 \quad \text{and} \quad T_1(0 \times \mathscr{H}_2) = T(0 \times \mathscr{H}_2) = 0,$$

T_1 must have the form $T_1(u_1, u_2) = (Z u_1, Y u_1)$, where Y is an operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ that is an extension of X. Now TS = ST, and so from the equations

$$(Z S_1 u_1, Y S_1 u_1) = TS(u_1, u_2) = ST(u_1, u_2) = (S_1 Z u_1, S_2 Y u_1)$$

we deduce that $Y S_1 = S_2 Y$. Thus, by the remarks in the Introduction, Y satisfies $Y S_1 = S_2 Y$ and so $\|Y\| \geq \alpha$. But $\|Y\| \leq \|T\ \| = \alpha$, whence $\|Y\| = \alpha$, as asserted.

3. PRELIMINARY EXTENSIONS

Let U be the minimal unitary dilation of S, and let $\mathscr{L}$ be the Hilbert space on which U acts; then $\mathscr{H} \subset \mathscr{L}$ and $U \mid \mathscr{H} = S$. If $\mathscr{M}$ is an S-invariant subspace of $\mathscr{H}$ and T is an operator on $\mathscr{M}$ commuting with S, then T has a unique extension to an operator $T^{\vee}$ on the subspace $\mathscr{M}^{\vee} = \bigvee_{n=0}^{\infty} U^{*n}\mathscr{M}$, such that $T^{\vee}$ commutes with U and $\|T^{\vee}\| = \|T\|$. This extension $T^{\vee}$ is given explicitly by

$$T^{\vee} u = \lim_{k \to \infty} U^{*k} T m_k,$$

where $m_k \in \mathscr{M}$ and $\lim_{k\to\infty} U^{*k} m_k = u \in \mathscr{M}^{\vee}$. For constructions of this type, see [1], [3]. Now $\mathscr{M}^{\vee} \cap \mathscr{H}$ is an S-invariant subspace of $\mathscr{H}$, and $T^{\vee} \mid \mathscr{M}^{\vee} \cap \mathscr{H}$ is an extension of T, but unfortunately $T^{\vee}(\mathscr{M}^{\vee} \cap \mathscr{H})$ need not be in $\mathscr{H}$. Indeed, in the counterexamples of [5], [6], it is precisely at this point that the obstruction to lifting occurs. The hypothesis (2.1) will ensure that $T^{\vee}(\mathscr{M}^{\vee} \cap \mathscr{H}) \subset \mathscr{H}$.

PROPOSITION 1. *If* $\sup_{k\geq 1} \sup_{u\in\mathscr{M}} \frac{\|P_k T u\|}{\|P_k u\|} = \alpha < +\infty$, *then for all* $u \in \mathscr{M}^{\vee}$,

$$\|QT^{\vee} u\| \leq \alpha \|Qu\|. \tag{3.2}$$

Proof. Let $u \in \mathscr{M}^{\vee}$. Choose a sequence $\{m_k\}$, with $m_k \in \mathscr{M}$ and such that $\lim_{k\to\infty} U^{*k} m_k = u$; then

$$\|QU^{*k}Tm_k\| = \|U^{*k}P_kTm_k\| \leq \alpha \|P_k m_k\| = \alpha \|U^{*k}P_k m_k\| = \alpha \|QU^{*k} m_k\|.$$

Now let $k \to \infty$, and we have the inequality $\|QT^{\vee} u\| \leq \alpha \|Qu\|$.

In particular, if $u \in \mathscr{M}^{\vee} \cap \mathscr{H}$, then $Qu = 0$, and so $T^{\vee} u \in \mathscr{H}$. Thus $T^{\vee} \mid \mathscr{M}^{\vee} \cap \mathscr{H}$ is an extension of T.

If $\mathscr{K}$ and Q denote the orthogonal complement of $\mathscr{H}$ in $\mathscr{L}$ and the orthogonal projection of $\mathscr{L}$ onto $\mathscr{K}$, respectively, then the conjugate of U^* on $\mathscr{K}$ is the compression of U to $\mathscr{K}$, that is, $QU \mid \mathscr{K}$. Thus any operator B on $\mathscr{K}$ commuting with QU can, by first taking adjoints in $\mathscr{K}$ and then extending this adjoint as in [1], [3], be dilated to an operator on $\mathscr{L}$ that commutes with U. For the proof of Theorem 1, it will be best to have a precise summary of this observation.

PROPOSITION 2. *If* B *is an operator on* $\mathscr{K}$ *commuting with* QU, *then* B *has a dilation* B_1 *to* $\mathscr{L}$ *that commutes with* U. *For* $u \in \mathscr{L}$, B_1 *is given explicitly by*

$$B_1 u = \lim_{k\to\infty} U^k B Q U^{*k} u, \tag{3.3}$$

and further $\|B_1\| = \|B\|$.

4. PROOF OF THEOREM 1

With the notation of Sections 2 and 3, $Q\mathscr{M}^{\vee}$ is a linear manifold in $\mathscr{K}$. Define an operator B' on $Q\mathscr{M}^{\vee}$ by $B'u = QT^{\vee}y$, where $u \in Q\mathscr{M}^{\vee}$ and $y \in \mathscr{M}^{\vee}$ is such that $Qy = u$. As a consequence of Proposition 1, B' is well defined, linear, and bounded, for

$$\|B' u\| = \|Q T^\vee y\| \leq \alpha \|Qy\| = \alpha \|u\| .$$

Moreover, B' commutes with QU. We extend B' by continuity to an operator on $\overline{QM^\vee}$, the closure of $Q\mathcal{M}^\vee$. This extension will also be denoted by B'. The subspace $\overline{Q\mathcal{M}^\vee}$ is invariant under QU, the operator QU has a co-isometric extension to $\mathcal{K}$, namely QU itself, and B' commutes with QU; thus, by [4, Corollary 4.1], B' has an extension B on $\mathcal{K}$ such that $BQU = QUB$ and $\|B\| = \|B'\| \leq \alpha$. The final extension is by Proposition 2. By that proposition, the operator B has a dilation to an operator B_1 on $\mathcal{L}$ that commutes with U and satisfies the inequality $\|B_1\| = \|B\| \leq \alpha$, and for any $u \in \mathcal{L}$, $B_1 u$ can be computed according to (3.3). If $u \in \mathcal{H}$, then $U^k Q U^{*k} u \in \mathcal{H}$, and so

$$QB_1 u = \lim_{k \to \infty} QU^k BQU^{*k} u = \lim_{k \to \infty} BQU^k QU^{*k} u = 0 .$$

Thus $B_1 \mathcal{H} \subset \mathcal{H}$. Define the operator T_1 on $\mathcal{H}$ by $T_1 = B_1 \mid \mathcal{H}$; then $T_1 U = U T_1$ and $\|T_1\| \leq \|B_1\| \leq \alpha$. To complete the proof, we need only show that $T_1 \mid \mathcal{M} = T$. For this purpose it will be sufficient to show that $T_1 \mid \mathcal{M}^\vee \cap \mathcal{H} = T^\vee \mid \mathcal{M}^\vee \cap \mathcal{H}$. This, however, is easy; for if $u \in \mathcal{M}^\vee \cap \mathcal{H}$, then $QU^{*k}u \in Q\mathcal{M}^\vee$ and

$$U^k BQU^{*k} u = U^k B' QU^{*k} u = U^k QT^\vee U^{*k} u = U^k QU^{*k} T^\vee u .$$

As $k \to \infty$, the first term in this equation converges to $B_1 u = T_1 u$ and the last term converges to $T^\vee u$. Thus $T^\vee \mid \mathcal{M}^\vee \cap \mathcal{H} = T_1 \mid \mathcal{M}^\vee \cap \mathcal{H}$, as required. That $\|T_1\| = \alpha$ is also clear, for we have shown that $\|T_1\| \leq \alpha$, while in the Introduction we showed that we must necessarily have $\|T_1\| \geq \alpha$.

5. A COUNTEREXAMPLE

Now it is easy to see that the extension of minimal norm is not uniquely defined on any subspace larger than the smallest reducing subspace for S that contains $\mathcal{M}$. It has been suggested to us that the extension of minimal norm is in fact unique on this reducing subspace. For shifts of multiplicity one, this is trivially true [5].

We conclude this paper with an example to show that the conjecture is false for shifts of multiplicity three or more. For this example, we take $\mathcal{H} = H^2 \times H^2 \times H^2$, where H^2 is Hardy's subspace of the space of functions that are analytic in the unit disk $D = \{z;\ |z| < 1\}$. In this setting, S is just pointwise multiplication by z, and if $u = (u_1, u_2, u_3) \in \mathcal{H}$, then

$$(Su)(z) = (zu_1(z),\ zu_2(z),\ zu_3(z)) .$$

The example will consist of a subspace $\mathcal{M} \subset \mathcal{H}$ that is not reducing; indeed, the smallest reducing subspace containing $\mathcal{M}$ is all of $\mathcal{H}$, but $\mathcal{M}$ does contain a reducing subspace. On $\mathcal{H}$ we give two operators T_1 and T_2 such that

$$T_1 \mid \mathcal{M} = T_2 \mid \mathcal{M}, \qquad \|T_1\| = \|T_2\| = \|T_1 \mid \mathcal{M}\|, \qquad \ker(T_1 - T_2) = \mathcal{M},$$

and both T_1 and T_2 commute with S. Thus no extension of $T = T_1 \mid \mathcal{M}$ can be unique on any subspace larger than $\mathcal{M}$.

The subspace $\mathscr{M}$ of $\mathscr{H}$ defined by

$$\{(u_1, zu_2, (1-z)u_2);\ u_1, u_2 \in H^2\}$$

is closed and S-invariant. Define an operator A on $\mathscr{H}$ that commutes with S by $A(u_1, u_2, u_3) = (0, 0, zu_3 + (z-1)u_2)$. Then $\|A\| \leq 2\sqrt{2}$, and with an easy computation, $\ker A = \mathscr{M}$. Let T_1 be the orthogonal projection of $\mathscr{H}$ onto the reducing subspace $H^2 \times 0 \times 0$; that is, let $T_1 u = (u_1, 0, 0)$ for $u = (u_1, u_2, u_3) \in \mathscr{H}$; then $T_1 \mathscr{M} \subset \mathscr{M}$, $T_1 S = S T_1$, and $\|T_1\| = \|T_1 | \mathscr{M}\| = 1$. If we define T_2 on $\mathscr{H}$ by $T_2 = T_1 + 4^{-1} A$, then $T_2 | \mathscr{M} = T_1 | \mathscr{M}$, $T_2 S = S T_2$, and $\|T_2\| = 1$. The smallest reducing subspace that contains $\mathscr{M}$ also contains $S^* \mathscr{M}$, $(I - SS^*)\mathscr{M}$, and all linear combinations of elements of these spaces. In particular, for any $\lambda, \mu, \nu \in C$, it contains

$$(\lambda, \mu, \nu) = (I - SS^*)(\lambda, z(\mu+\nu), (1-z)(\mu+\nu)) + S^*(0, z\mu, (1-z)\mu)$$

and all its translates. Thus the smallest reducing subspace is $\mathscr{H}$ itself.

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