THE SIMILARITY PROBLEM FOR REPRESENTATIONS OF A B*-ALGEBRA

Bruce A. Barnes

1. INTRODUCTION

Throughout this paper, A is a B*-algebra and π is a continuous representation of A on a Hilbert space H. The representation π is not assumed to be adjoint-preserving. Let B(H) denote the algebra of all bounded linear operators on H. In this paper we consider the following question concerning the representation π.

*Do there exist a *-representation* γ *of* A *on* H *and an operator* V *with* $V, V^{-1} \in B(H)$ *such that* $\pi = V^{-1}\gamma V$?

If such a representation γ and operator V exist, then π is said to be *similar* to γ. The question stated above for representations of a B*-algebra had its origins in a related question concerning group representations of a locally compact group G.

If δ *is a strongly continuous, uniformly bounded representation of* G *on* H, *do there exist a unitary representation* τ *of* G *on* H *and an operator* V *with* $V, V^{-1} \in B(H)$ *such that* $\delta = V^{-1}\tau V$?

In [6], J. Dixmier answered the second question affirmatively for the case where G is abelian. More generally, the answer is known to be affirmative whenever G is an amenable group [9, Theorem 3.4.1]. However, there do exist strongly continuous, uniformly bounded representations of some important locally compact groups that are not similar to any unitary representation of the group. An example is provided by L. Ehrenpreis and F. I. Mautner in [8]. The corresponding question for representations of a B*-algebra remains open.

The group question and the algebra question are closely related. If δ is a strongly continuous, uniformly bounded representation of G on H, then δ lifts in the usual way to a continuous representation δ_1 of $L^1(G)$ on H. The algebra $L^1(G)$ has a largest B*-norm

$$\|f\| = \sup\{\|\gamma(f)\| : \gamma \text{ is a *-representation of } L^1(G)\},$$

and $\|f\|_1 \geq \|f\|$ for $f \in L^1(G)$ ($\|f\|_1$ denotes the usual L^1-norm of f). If δ_1 is continuous with respect to $\|\cdot\|$, then δ_1 extends to a continuous representation π of $C^*(G)$ on H, where $C^*(G)$ denotes the B*-algebra that is the completion of $L^1(G)$ in the norm $\|\cdot\|$. It is easy to verify that if π is similar to a *-representation of $C^*(G)$ on H, then δ is similar to a unitary representation of G on H. In the other direction, if π is a continuous representation of A on H such that the restriction of π to the group of unitaries in A is similar to a unitary representation, then π is similar to a *-representation of A on H.

Received January 30, 1975.
This research was partially supported by NSF grant GP-28250.

Michigan Math. J. 22 (1975).

The similarity question for representations of a general B*-algebra first appeared in the literature in 1955 in a paper by R. V. Kadison [11]. In [11], Kadison derived very general necessary and sufficient conditions for a continuous representation of a B*-algebra to be similar to a *-representation. In [1], [2], and [3], the present author also considered the similarity question for representations of a B*-algebra and some related questions. The proof of the main theorem in [3] is simplified in this paper. The best results on the question are due to J. Bunce. He proves in [4] that if A is strongly amenable (in the sense of B. E. Johnson [10]), then the similarity question for representations of A has an affirmative answer. Particular examples of strongly amenable B*-algebras are the GCR (post-liminaire) algebras of I. Kaplansky and uniformly hyperfinite algebras.

In this paper, we show that if π is a continuous representation of A on H such that the representations π and $a \to \pi(a^*)^*$ have a common cyclic vector, then there exist a self-adjoint operator U (perhaps unbounded) with domain in H and dense range in H, and a *-representation γ of A on H such that $\pi = U^{-1}\gamma U$ on the domain of U. In particular, this holds whenever π is irreducible. The *-representation γ is unique in the sense that if π is similar to a *-representation of A on H, then π is similar to γ.

At this point we introduce some notation. Let ϕ be the universal representation of A on the universal representation space H_ϕ [7, p. 43]. We denote the w.o. (weak operator) closure of $\phi(A)$ in $B(H_\phi)$ by B. If X is a Banach space and T is a linear operator on X, let $\mathscr{D}(T)$, $\mathscr{R}(T)$, and $\mathscr{N}(T)$ denote the domain, range, and null space of T, respectively. Let X* denote the dual space of X. If $x \in X$ and $f \in X^*$, we use the notation $\langle x, f \rangle = f(x)$.

2. THE RESULTS

There is a standard lifting theorem to the effect that a *-representation of a B*-algebra A lifts to a *-representation of B, the von Neumann enveloping algebra of A [7, 12.1.5]. We prove a generalization of this theorem. Our original proof was patterned after a proof of a related result of P. G. Spain in [16].

THEOREM 1. *Let* ρ: $A \to B(X)$ *be a continuous representation of* A *on a reflexive Banach space* X. *Then there exists a representation* $\tilde{\rho}$: $B \to B(X)$ *such that* $\tilde{\rho}(\phi(a)) = \rho(a)$ *for all* $a \in A$, *and* $\tilde{\rho}$ *is continuous as a map from* B *in the* w.o. *topology into* B(X) *in the* w.o. *topology. Furthermore, the set* $\{T \in \tilde{\rho}(B): \|T\| \leq r\}$ *is* w.o. *closed for all* $r > 0$.

Proof. If $x \in X$ and $f \in X^*$, then $a \to \langle \rho(a)x, f \rangle$ is a continuous linear functional on A. Then, by [7, Corollaire (12.1.3)(ii)], there exists a unique w.o. continuous linear functional $\omega[x, f]$ on B with the property that

$$\omega[x, f](\phi(a)) = \langle \rho(a)x, f \rangle \qquad (a \in A).$$

Also, by [7, Corollaire (12.1.3)(iii)] the norm of $\omega[x, f]$ on B is the same as the norm of $a \to \langle \rho(a)x, f \rangle$ on A. Therefore

$$|\omega[x, f](b)| \leq \|\rho\| \; \|x\| \; \|f\| \; \|b\|$$

whenever $x \in X$, $f \in X^*$, $b \in B$. It follows that for each $b \in B$, the form $\omega[x, f](b)$ is continuous and bilinear on $X \times X^*$. Hence, since X is reflexive, there exists a unique operator $\tilde{\rho}(b) \in B(X)$ such that

$$\langle \tilde{\rho}(b)x, f \rangle = \omega[x, f](b)$$

whenever $x \in X$, $f \in X^*$, and $b \in B$. That $\tilde{\rho}$ is linear and continuous as a map from B with the w.o. topology into B(X) with the w.o. topology follows immediately. Since

$$\tilde{\rho}(b) = \rho(\phi^{-1}(b)) \qquad (b \in \phi(A)),$$

$\tilde{\rho}$ is an algebra homomorphism on $\phi(A)$. Fix $b \in \phi(A)$ and $c \in B$. Let $\{c_\lambda\}$ be a net in $\phi(A)$ such that $c_\lambda \to c$ in the w.o. topology in $B(H_\phi)$. Then $bc_\lambda \to bc$ in the w.o. topology, so that

$$\tilde{\rho}(bc_\lambda) \to \tilde{\rho}(bc) \quad \text{and} \quad \tilde{\rho}(b)\tilde{\rho}(c_\lambda) \to \tilde{\rho}(b)\tilde{\rho}(c)$$

in the w.o. topology in B(H). Since $\tilde{\rho}(bc_\lambda) = \tilde{\rho}(b)\tilde{\rho}(c_\lambda)$, we have the relation $\tilde{\rho}(bc) = \tilde{\rho}(b)\tilde{\rho}(c)$. Essentially the same argument establishes that

$$\tilde{\rho}(bc) = \tilde{\rho}(b)\tilde{\rho}(c) \qquad \text{whenever } b, c \in B.$$

Thus, $\tilde{\rho}$ is a representation of B on X.

Now, if $r > 0$, let $W_r = \{T \in \tilde{\rho}(B): \|T\| \leq r\}$. By [5, Lemma 5.3], $\tilde{\rho}(B)$ is a norm-closed subalgebra of B(X). Suppose T is contained in the w.o. closure of W_r. Choose a net $\{T_\lambda\}$ such that $T_\lambda \to T$ in the w.o. topology. By the open-mapping theorem, there exists a bounded net $\{b_\lambda\} \subset B$ such that $\tilde{\rho}(b_\lambda) = T_\lambda$ for all λ. Therefore there exist a subnet $\{c_\delta\}$ of $\{b_\lambda\}$ and some $b \in B$ such that $c_\delta \to b$ in the w.o. topology. Hence

$$\tilde{\rho}(b) = \text{w.o.} \lim \tilde{\rho}(c_\delta) = T.$$

This completes the proof.

Let ρ be a continuous representation of A on a reflexive Banach space X. In the remainder of this paper, we simplify our notation by denoting the extension $\tilde{\rho}$ constructed in Theorem 1 also by ρ. This should cause no confusion.

Next we use Theorem 1 to simplify considerably the proof of the main result in [3].

THEOREM 2. *Let* X *be a reflexive Banach space, and let* ρ *be a continuous, irreducible representation of* A *on* X. *Assume that there exists a nonempty subset* S *of* B *such that*

$$W = \bigcap \{\mathcal{N}(\rho(b)): b \in S\}$$

is a nonzero, finite-dimensional subspace of X. *Then* X *is a Hilbert space in an equivalent norm, and* ρ *is similar to a *-representation of* A *on this Hilbert space.*

Proof. Let

$$L = \{b \in B: \rho(b)W = \{0\}\}.$$

By Theorem 1, L is a w.o.-closed left ideal of B. Therefore, by [15, Proposition 1.10.1], there exists a projection $p \in B$ such that $L = B(1 - p)$. This implies that $\rho(p)w = w$ for all $w \in W$. Also, since $S \subset L$ and $\rho(b)\rho(p) = 0$ for all $b \in S$, we see that $\mathcal{R}(\rho(p)) = W$. Thus $\rho(p)$ has finite-dimensional range, so that the algebra

$\rho(pBp)$ is finite-dimensional. A nonzero idempotent e in a complex normed algebra D is minimal if $eDe = \{\lambda e: \lambda \text{ complex}\}$. Standard Wedderburn theory implies that there exists $q \in pBp$ such that $\rho(q)$ is a minimal idempotent of the finite-dimensional algebra $\rho(pBP)$. Thus $\rho(q)$ is a minimal idempotent of $\rho(B)$, since

$$\rho(q)\,\rho(B)\,\rho(q) = \rho(qBq) = \rho(qpBpq) = \{\lambda\rho(q): \lambda \text{ complex}\}.$$

Therefore $B/\ker(\rho)$ contains a minimal idempotent, so that the result follows from [3, Corollary 2.3].

Recall from the Introduction that π denotes a continuous representation of A on a Hilbert space H. If δ is a representation of A on H, and C is a *-subalgebra of A, let δ_C denote the restriction of δ to the algebra C.

Now we state and prove the main result.

THEOREM 3. *Assume that there exists* $x_0 \in H$ *such that both* $\pi(A)x_0$ *and* $\pi(A)^*x_0$ *are dense in* H. *Then there exist a *-representation* γ *of* A *on* H *and a self-adjoint operator* U *(possibly unbounded) with the properties* $U \geq 0$, U *is one-to-one, and* $\mathscr{R}(U)$ *is dense, such that for all* $x \in \mathscr{D}(U)$ *and all* $a \in A$

$$\pi(a)x = U^{-1}\gamma(a)Ux.$$

Furthermore, if C *is a closed *-subalgebra of* A *such that* π_C *is similar to a *-representation of* A, *then* π_C *is similar to* γ_C.

Proof. Define a functional β on B by the equation

$$\beta(a) = (\pi(a)x_0, x_0) \qquad (a \in B).$$

Since π is w.o.-continuous by Theorem 1, β is a normal functional on the von Neumann algebra B. By the polar-decomposition theorem for normal functionals [7, p. 240], there exists a partial isometry $u \in B$ such that $\alpha = u^* \cdot \beta$ is a positive functional and $\beta = u \cdot \alpha$. Let $x_1 = \pi(u^*)^* x_0$. With this notation,

$$\alpha(a) = (\pi(a)x_0, x_1) \qquad (a \in B).$$

Also, $(\pi(a)x_0, x_0) = \beta(a) = (\pi(u)\pi(a)x_0, x_1)$ for all $a \in B$. Since $\pi(A)x_0$ is dense in H, it follows that $\pi(u)^*x_1 = x_0$. Thus $\pi(a)^*x_0 = \pi(ua)^*x_1$ $(a \in B)$. By hypothesis, $\pi(A)^*x_0$ is dense in H, and so we have proved that

$$\text{(1)} \qquad \pi(B)^*x_1 \text{ is dense in } H.$$

Now define a form $\langle \cdot , \cdot \rangle$ with domain $D = \pi(B)x_0$ by

$$\langle \pi(a)x_0, \pi(b)x_0 \rangle = \alpha(b^*a) \qquad (a, b \in B).$$

Using (1), we can easily verify that $\pi(a)x_0 = 0$ if and only if $\alpha(a^*a) = 0$. This implies that the form $\langle \cdot , \cdot \rangle$ is well-defined. Since α is a positive functional, this form is symmetric and nonnegative. Let V be the self-adjoint positive operator associated with the closure of this form as in [12, Theorem 2.23, p. 331]. Let U be the positive square root of V. By [12, Corollary 2.27, p. 332], D is a core of U. This means, by definition [12, p. 166], that if $x \in \mathscr{D}(U)$, then there exists $\{x_n\} \subset D$ such that $x_n \to x$ and $Ux_n \to Ux$. In particular, UD is dense in $\mathscr{R}(U)$. Since $V = U^2$, we see that $\mathscr{R}(V) \subset \mathscr{R}(U)$. We use this and (1) to prove that UD is dense in H. It suffices to show that $\mathscr{R}(V)$ is dense in H. For all $a, b \in B$, we have the relations

$$(\pi(a)x_0, V\pi(b)x_0) = \langle \pi(a)x_0, \pi(b)x_0 \rangle = \alpha(b^*a) = (\pi(a)x_0, \pi(b^*)^*x_1).$$

Since $\pi(A)x_0$ is dense in H, it follows that $V\pi(b)x_0 = \pi(b^*)^*x_1$ for all $b \in B$. Therefore $\mathscr{R}(B)$ is dense, by (1). We have verified that

$$U\pi(A)x_0 \text{ is dense in } H. \tag{2}$$

Note that U is one-to-one on $\mathscr{D}(U)$, since $\mathscr{R}(U)$ is dense in H.

Next we define the *-representation γ. Let

$$L = \{a \in B: \alpha(a^*a) = 0\} = \{a \in B: U\pi(a)x_0 = 0\} = \{a \in B: \pi(a)x_0 = 0\}.$$

Form the quotient space

$$B/L = \{a + L: a \in B\}.$$

As usual, B/L is an inner-product space with inner product

$$\langle a + L, b + L \rangle = \alpha(b^*a) \qquad (a, b \in B).$$

Let H_α be the Hilbert-space completion of this inner-product space, and let δ_α be the *-representation of B on H_α determined in the usual fashion by the equation $\delta_\alpha(b)(a + L) = ba + L$ $(a, b \in B)$. Define a map $W_0: B/L \to H$ by

$$W_0(a + L) = U\pi(a)x_0.$$

Since for all $a \in B$,

$$\langle a + L, a + L \rangle = \alpha(a^*a) = \|U\pi(a)x_0\|^2,$$

it follows that W_0 extends to a unitary map W of H_α onto the closure of $U\pi(A)x_0$ in H. By (2), W is a unitary map of H_α onto H. Define a *-representation $\gamma: B \to B(H)$ by

$$\gamma(a) = W\delta_\alpha(a)W^{-1} \qquad (a \in B).$$

If $a, b \in B$, then

$$\gamma(a)U(\pi(b)x_0) = W\delta_\alpha(a)(b + L) = W(ab + L) = U\pi(a)(\pi(b)x_0).$$

Recall that D is a core of U [12, Corollary 2.27, p. 332]. Thus, if $x \in \mathscr{D}(U)$, then there exists $\{x_n\} \subset D$ such that $x_n \to x$ and $Ux_n \to Ux$. By the previous computation,

$$\gamma(a)Ux_n = U\pi(a)x_n \qquad (n \geq 1),$$

so that for each $a \in B$,

$$\pi(a)x_n \to \pi(a)x, \qquad U\pi(a)x_n = \gamma(a)Ux_n \to \gamma(a)Ux.$$

Since U is closed, it follows that

$$\pi(a)x \in \mathscr{D}(U) \text{ and } U\pi(a)x = \gamma(a)Ux \qquad (x \in \mathscr{D}(U),\ a \in B). \tag{3}$$

Now let C be a closed *-subalgebra of A, and suppose that there exist a *-representation δ of C and an invertible operator $S \in B(H)$ such that

$$S^{-1}\delta(a)S = \pi(a) \qquad (a \in C).$$

Then, for $x \in \mathscr{D}(U)$,

$$\delta(a)Sx = S\pi(a)x = SU^{-1}\gamma(a)Ux.$$

If $y = Ux$, we have the equation

$$\delta(a)SU^{-1}y = SU^{-1}\gamma(a)y.$$

Let $T = SU^{-1}$. Then $\mathscr{D}(T) = \mathscr{R}(U)$, and for $y \in \mathscr{D}(T)$,

$$\delta(a)Ty = T\gamma(a)y \qquad (a \in C). \tag{4}$$

It is a straightforward exercise to verify that T is nonsingular in the sense of [13, p. 285]. Then it follows from (4) and [13, Theorem 1, p. 285] that there exists a unitary operator $R \in B(H)$ such that

$$\gamma(a) = R\delta(a)R^{-1} = (RS)\pi(a)(RS)^{-1}$$

for all $a \in C$. This completes the proof.

COROLLARY 1. *Assume that π is irreducible. Then the conclusions of Theorem* 3 *hold. Let γ be as in the statement of Theorem* 3. *Then π is similar to a *-representation of* A *on* H *if and only if γ is irreducible.*

Proof. Fix $x_0 \in H$ ($x_0 \neq 0$). Since π is irreducible, $\pi(A)x_0$ is dense in H. Suppose $y \in H$ and $y \perp \pi(A)^*x_0$. Then $\pi(A)y \perp x_0$, and since π is irreducible and $x_0 \neq 0$, it follows that $y = 0$. Thus, $\pi(A)^*x_0$ is dense in H. This proves that Theorem 3 holds. We use the notation of Theorem 3 in what follows.

If π is similar to a *-representation of A on H, then by Theorem 3, π is similar to γ. Therefore, in this case, γ is irreducible. Conversely, assume γ is irreducible. Let

$$L = \{a \in A: \gamma(a^*a) = 0\}.$$

By [7, Théorème 2.9.5], L is a maximal modular left ideal of A. Since $U\pi(a^*a) = \gamma(a^*a)U = 0$ for all $a \in L$, we see that

$$0 = (\pi(a^*a)x_0, x_1) = \alpha(a^*a) = \|U\pi(a)x_0\|^2 \qquad (a \in L).$$

Thus $L = \{a \in A: \pi(a)x_0 = 0\}$. Therefore, by [3, Proposition 2.2], π is similar to a *-representation of A on H. Thus π is similar to γ, by Theorem 3.

If C is a commutative B*-algebra and δ is a continuous representation of C on a Hilbert space H, then it follows indirectly from Dixmier's result for representations of abelian groups [6, Théorème 6] or directly from Bunce's result [4, Theorem 1], that δ is similar to a *-representation of C on H. We use this fact several times in the proof of the next corollary.

COROLLARY 2. *Assume that π is irreducible and π(B) contains a subalgebra that is similar to a maximal commutative *-subalgebra of* B(H). *Then π is similar to γ. This is the case if there exists* $b \in B$ *such that π(b) is similar to a self-adjoint operator with a cyclic vector in* H.

Proof. Let C be a maximal commutative *-subalgebra of B(H), and assume that S is an invertible operator in B(H) such that

$$S^{-1} C S \subset \pi(B).$$

By replacing π with the representation $a \to S\pi(a) S^{-1}$, if necessary, we may assume without loss of generality that $C \subset \pi(B)$. Since π is w.o.-continuous, there exists a central projection $q \in B$ such that $\ker(\pi) = (1 - q) B (1 - q)$ [15, Proposition 1.10.5]. Let π_q denote the restriction of π to qBq. Consider the representation

$$\pi_q^{-1}: C \to B(H_\phi).$$

By the remarks preceding the corollary, there exists an invertible operator $W \in B(H_\phi)$ such that

$$c \to W^{-1} \pi_q^{-1}(c) W$$

is a *-representation of C. Then $K = W^{-1} \pi_q^{-1}(C) W$ is a commutative B*-algebra. Define a *-representation δ_1 of K on H by

$$\delta_1(a) = \pi(WaW^{-1}) \qquad (a \in K).$$

Then $a \to \gamma(WaW^{-1})$ is a continuous representation of K on H. As before, this representation is similar to a *-representation δ_2 of K on H. Thus, there exists an invertible operator $R \in B(H)$ such that

$$\delta_2(a) = R^{-1} \gamma(WaW^{-1}) R \qquad (a \in K).$$

Since $U\pi(b) = \gamma(b) U$ for all $b \in B$, we have for all $a \in K$ the equations

$$R^{-1} U \delta_1(a) = R^{-1} U\pi(WaW^{-1}) = R^{-1} \gamma(WaW^{-1}) U$$

$$= R^{-1} \gamma(WaW^{-1}) R R^{-1} U = \delta_2(a) R^{-1} U.$$

It is easy to verify that $R^{-1} U$ is nonsingular [13, p. 285]. Therefore, by [13, Theorem 1, p. 285], there exists a unitary operator $S \in B(H)$ such that

$$\delta_1(a) = S^{-1} \delta_2(a) S \qquad (a \in K).$$

Hence,

$$C = \delta_1(K) = S^{-1} \delta_2(K) S = S^{-1} R^{-1} \gamma(WKW^{-1}) RS.$$

Let $T = RS$. We have shown that $TCT^{-1} \subset \gamma(B)$. If E is a projection contained in $\gamma(B)'$, then $T^{-1} ET$ commutes with C. Therefore $T^{-1} ET \in C$, so that $E \in TCT^{-1} \subset \gamma(B)$. There exists an idempotent $e \in B$ such that $\gamma(e) = E$. Then $U^{-1} \gamma(e) U$ is a bounded idempotent that commutes with $\pi(B)$. By the irreducibility of π, the projection $\gamma(e)$ must be either 0 or the identity operator. Thus, $\gamma(B)'$ consists of scalar multiples of the identity operator. Hence γ is irreducible, and Corollary 1 implies that π is similar to γ.

Now assume that $S \in B(H)$ is self-adjoint with a cyclic vector in H, and that there exists an invertible operator $R \in B(H)$ such that $S \in R\pi(A) R^{-1}$. The map $a \to R\pi(a) R^{-1}$ is a w.o.-continuous representation of B into B(H). Let D be the uniformly closed subalgebra of B(H) generated by S and the identity operator.

Then $D \subset R\pi(B)R^{-1}$, by [5, Lemma 5.3]. Let C be the w.o. closure of D in B(H). By Kaplansky's density theorem, every operator in C is the w.o. limit of a bounded net in D. It follows from Theorem 1 that $C \subset R\pi(B)R^{-1}$. The result then follows from the previous argument if C is a maximal commutative *-subalgebra of B(H). We verify that this is the case. By a standard form of the spectral theorem, there exists a finite regular Borel measure μ on X, the spectrum of S, such that S is unitarily equivalent to the multiplication operator M_ϕ on $L^2(X, \mu)$, where $\phi(t) = t$ ($t \in X$) (see the proof of [14, Theorem 1.6]). Then it follows from [14, Theorem 1.20] that C is unitarily equivalent to $L^\infty(X, \mu)$, and hence that it is a maximal commutative *-subalgebra of B(H).

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University of Oregon
Eugene, Oregon 97403