CENTRALIZERS OF SEPARABLE SUBALGEBRAS

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0. INTRODUCTION

There has been some interest recently in the relationship between the structure of an algebra R and the centralizer of an appropriate subset A of R (denoted by $C_R(A)$). In particular, I. N. Herstein and L. Neuman have considered the case when A consists of a single element $a \in R$ such that a^n is in the center of R for some positive integer n. They have shown that if R is semiprime and $C_R(a)$ is simple or semisimple Artinian, then R itself must also be simple or semisimple Artinian. This result was extended by M. Cohen [2], who showed that if $C_R(a)$ is a Goldie ring, then R must also be a Goldie ring.

The intent of this paper is to show that these results can be extended to the situation where A is any finite-dimensional separable subalgebra of R. That is, we show that if R is semiprime and $C_R(A)$ is either simple, semisimple, or semisimple Artinian, then so is R.

Further results are obtained on the relationships between the ideals, zero-divisors, and Jacobson radical of $C_R(A)$ and those of R. The result on zero-divisors has now been used by Cohen to show that if R is semiprime and $C_R(A)$ is a Goldie ring, then so is R [3]. We discuss these results in more detail at the end of the paper.

We note that centralizers of separable subalgebras arise naturally as fixed-point sets of automorphism groups, as follows: Let R be a ring whose center k is a field, and let G be a finite group of inner automorphisms of R as a k-algebra such that the order of G is relatively prime to the characteristic of k. For each $\tau \in G$, choose an $x_\tau \in R$ that induces τ. If A is the subalgebra of R generated by the x_τ, then $C_R(A)$ is precisely the ring of fixed points R^G of G acting on R. The algebra A is separable, since it is a homomorphic image of a twisted group algebra $k_t[G]$, which is separable. (For details, see [7].)

This relationship was used in [7] to show that if R^G satisfies a polynomial identity (PI), then R also satisfies an identity, where G is as described above. For, it was first shown that if A is a finite-dimensional separable subalgebra of a k-algebra R, and $C_R(A)$ satisfies a PI, then R satisfies a PI.

1. PRELIMINARIES

In all that follows, unless otherwise stated, R will denote an algebra over a field k, and A will denote a finite-dimensional, separable k-subalgebra of R. By J(R) we denote the Jacobson radical of R, and by N(R) the lower nil (prime) radical of R. A ring R is said to be semiprime of N(R) = (0); equivalently, R has no nonzero nilpotent ideals. $C_R(A)$ denotes the centralizer of A in R.

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We first list several known results that will be used repeatedly. The first follows from the classical theory of algebras [1].

PROPOSITION 1.1. *There exists a finite separable field extension* K *of* k *such that* $A \otimes_k K = K_{n_1} \oplus \cdots \oplus K_{n_\ell}$, *where* K_{n_i} *denotes the* n_i-by-n_i *matrix ring over* K.

We refer to K as a *splitting field* for A, and we say that A is *split* if A is already a direct sum of complete matrix rings over k.

PROPOSITION 1.2 [4, p. 112]. *If* R *has a unit element, and* A *is central simple over* k *with the same unit element as* R, *then* $R = A \otimes_k C_R(A)$.

PROPOSITION 1.3 [6, p. 252]. *If* K *is a separable field extension of* k, *then* $J(R \otimes_k K) = J(R) \otimes_k K$.

PROPOSITION 1.4. *If* K *is a finite separable field extension of* k, *then* $N(R \otimes_k K) \subseteq N(R) \otimes_k K$.

Proof. It was pointed out in [7, (see Proposition 2)] that $R \otimes K$ is semiprime whenever R is. Thus, since $R/N(R)$ is semiprime, $R/N(R) \otimes K$ is semiprime. But $R/_{N(R)} \otimes K \cong (R \otimes K)/(N(R) \otimes K)$; by [6, p. 194], $N(R \otimes K)$ is the intersection of all ideals M such that $(R \otimes K)/M$ has no nonzero nilpotent ideals. Thus $N(R \otimes K) \subseteq N(R) \otimes K$.

PROPOSITION 1.5 [7, Lemma 2]. *If* A *is split and* $\overline{R}$ *is any homomorphic image of* R, *then* $C_{\overline{R}}(\overline{A}) = \overline{C_R(A)}$. *That is, the image of the centralizer is the centralizer of the image* $\overline{A}$ *of* A.

The general method of proof is to reduce the problem to the case when A is split, and then, if necessary, to reduce it to the case when A is central simple.

2. IDEALS

In this section, we show that if R is semiprime, ideals of R contract to ideals of $C_R(A)$.

We begin with the analogue of [5, Lemma 1]. For the first few lemmas, we do not need the previous assumptions about R and A.

LEMMA 2.1. *Let* N *be the lower nil radical of a ring* R. *Let* A *be a subring of* R *with a unit* f, *and say that* $C_R(A)$ *contains a unit* e. *Then* $e + N$ *is a unit for* R/N.

Proof. First note that e acts as the identity on A. For, certainly $f \in C_R(A)$, and thus $ef = fe = f$. But by assumption, $fa = af = a$, for all $a \in A$. Thus $ea = efa = fa = a$. Similarly, $ae = a$.

Now choose any $x \in R$, and consider the element $y = (1 - e)x(1 - e)$. Then $ay = ya = 0$ for each $a \in A$, since $a = ea = ae$. Thus $y \in C_R(A)$. But then $ey = y$. Since certainly $ey = 0$, we see that $y = 0$; that is, $(1 - e)x(1 - e) = 0$ for all $x \in R$. This says that $((1 - e)R)^2 = (0)$, and therefore $(1 - e)R$ is a nilpotent right ideal of R. But then $(1 - e)R \subseteq N$; that is, $x - ex \in N$ for all $x \in R$. Similarly, $x - xe \in N$, for all x. But this says precisely that $e + N$ is a unit element in R/N.

Note that if R is semiprime, Lemma 1 says that R must have a unit element.

The next lemma is well known.

LEMMA 2.2. *Let* R *be a* k*-algebra, and let* A *and* B *be two* k*-subalgebras of* R. *Let* K *be a finite Galois extension of* k, *and consider* $A \otimes_k K$ *and* $B \otimes_k K$ *as subalgebras of* $R \otimes_k K$. *Then, if* $(A \otimes K) \cap (B \otimes K) \neq (0)$, $A \cap B \neq (0)$.

Proof. Write $A_1 = A \otimes K$, $B_1 = B \otimes K$, $R_1 = R \otimes K$, and let G be the Galois group of K over k. Extend G to act on R_1, via $(r \otimes \alpha)^\sigma = r \otimes \alpha^\sigma$, for all $r \in R$, $\alpha \in K$, and $\sigma \in G$. Then $A_1^\sigma = A_1$ and $B_1^\sigma = B_1$, for all $\sigma \in G$.

Now choose $x = \sum_{i=1}^n r_i \otimes \alpha_i \in A_1 \cap B_1$, where the α_i are linearly independent over k and $r_i \in A$ for each i. We shall show that $r_i \in B$ for each i. Now

$$(x\alpha_i)^\sigma = \sum_{j=1}^n r_j \otimes (\alpha_j \alpha_i)^\sigma \in A_1 \cap B_1 .$$

Summing this on σ and letting $\gamma_{ij} = \sum_\sigma (\alpha_i \alpha_j)^\sigma = \mathrm{tr}(\alpha_i \alpha_j) \in k$, we see that for $i = 1, 2, \cdots, n$,

$$\sum_j r_j \otimes \gamma_{ji} = \sum_j r_j \gamma_{ji} \otimes 1 \in A_1 \cap B_1 \subseteq B \otimes K .$$

Since K is a Galois extension of k, the matrix (γ_{ij}) is invertible, and thus each r_j is in B.

We now resume our assumptions on R and A.

THEOREM 2.1. *If* U *is an ideal of* R, *then either* $U \cap C_R(A) \neq (0)$ *or* AU (*and thus* UA) *is nilpotent.*

Proof. Let K be a finite Galois extension of k that is a splitting field for A, and consider R_1, A_1, and $U_1 = U \otimes K$ as in Lemma 2.2. The algebra A_1 is now split, and U_1 is an ideal of R_1. Also, $C_{R_1}(A_1) = C_R(A) \otimes K$. Now, if $A_1 U_1$ is nilpotent, certainly AU is nilpotent, and by Lemma 2.2, $U \cap C_R(A) \neq (0)$ if $U_1 \cap C_{R_1}(A_1) \neq (0)$. Thus, it will suffice to prove the theorem when A is split.

Let e be a primitive central idempotent of A. Then eAe is central simple over k (since A is split) and $C_{eRe}(eAe) = eC_R(A)e$. In addition, e is the unit element for both eAe and eRe. Thus, by Proposition 1.2, $eRe \cong eAe \otimes_k eC_R(A)e$. Assume for the moment that $eUe \neq (0)$. Then eUe is an ideal of eRe, and therefore (since eAe is central simple) $eUe \cong eAe \otimes_k \bar{V}$, where V is a nonzero ideal of $eC_R(A)e$. But $eC_R(A)e \subseteq C_R(A)$, and thus

$$V \subseteq eUe \cap eC_R(A)e \subseteq U \cap C_R(A);$$

that is, $U \cap C_R(A) \neq (0)$.

We have shown that if $U \cap C_R(A) = (0)$, then $eUe = (0)$ for every primitive central idempotent e of A. Let f be the unit element of A. Then $f = e_1 + \cdots + e_k$, where the e_i are the primitive central idempotents of A. Since $e_i U e_i = (0)$, $fUf = \sum_{i \neq j} e_i U e_j$. But then

$$(fUf)^k = \sum e_{i_1} U e_{i_2} U \cdots U e_{i_{k+1}} = (0),$$

since in every set $\{e_{i_1}, e_{i_2}, \cdots, e_{i_{k+1}}\}$ at least two of the elements e_{i_j} are the same. This implies that $(fU)^{k+1} = (0)$, since $f^2 = f$. But then AU is nilpotent, since $AU = fAU \subseteq fU$, and the theorem is proved.

COROLLARY 2.1. *If* R *has a unit element, and* A *has the same unit element as* R, *then each ideal of* R *either intersects* $C_R(A)$ *nontrivially or is nilpotent.*

Proof. The proof of the theorem shows that if $U \cap C_R(A) = (0)$ and f is the unit of A, then fU is nilpotent.

COROLLARY 2.2. *If* R *is semiprime and* $U \neq (0)$ *is an ideal of* R, *then* $U \cap C_R(A) \neq (0)$.

Proof. If $U \cap C_R(A) = (0)$, it follows that $AU = (0)$ and $UA = (0)$, since R contains no nilpotent ideals. But this says that $U \subseteq C_R(A)$, a contradiction.

COROLLARY 2.3. *If* R *is semiprime and* $C_R(A)$ *is simple, then* R *itself must be simple.*

Proof. Let f be the unit element in A. Then f is a central idempotent in $C_R(A)$, and therefore f is the unit element for $C_R(A)$, since $C_R(A)$ is simple. Thus, by Lemma 2.1, f is the unit element for R.

Now, if U is a nonzero ideal of R, then $U \cap C_R(A)$ is a nonzero ideal of $C_R(A)$, by Corollary 2.2. Thus $U \cap C_R(A) = C_R(A)$, and therefore $f \in U$. This gives $U = R$.

3. RADICALS AND SEMISIMPLICITY

In this section we show that the radical of $C_R(A)$ is simply the radical of R intersected with $C_R(A)$.

LEMMA 3.1. $J(R) \cap C_R(A) \subseteq J(C_R(A))$.

Proof. We shall show that $J(R) \cap C_R(A)$ is a quasi-regular ideal of $C_R(A)$. Now, if $x \in J(R) \cap C_R(A)$, then x has a quasi-inverse y in R. But since $x \in C_R(A)$ and $x + y + xy = 0$, it follows that $y \in C_R(A)$ also. This completes the proof.

THEOREM 3.1. *If* $C_R(A)$ *is semisimple, then* J(R) *is nilpotent.*

Proof. By Lemma 3.1, $J(R) \cap C_R(A) = (0)$, and thus by Theorem 2.1, $AJ(R)$ and $J(R)A$ are nilpotent.

First consider the case when A is split. Let N_0 be the ideal of R generated by $AJ(R)$ and $J(R)A$. Clearly, N_0 is nilpotent. Let $\overline{R} = R/N_0$. Since $N_0 \cap A = (0)$ and $N_0 \cap C_R(A) = (0)$, we see that $\overline{A} \cong A$ and $\overline{C_R(A)} \cong C_R(A)$. By Proposition 1.5, $\overline{C_R(A)} = C_{\overline{R}}(\overline{A})$, and thus $C_{\overline{R}}(\overline{A})$ is semisimple. Now in $\overline{R}$, since $\overline{A}\,\overline{J(R)} = (0)$ and $\overline{J(R)}\,\overline{A} = (0)$, we have the relation $\overline{J(R)} \subseteq C_{\overline{R}}(\overline{A})$. But $\overline{J(R)}$ is a quasi-regular ideal, and therefore $\overline{J(R)} \subseteq J(C_{\overline{R}}(\overline{A})) = (0)$. That is, $J(R) \subseteq N_0$, and thus J(R) is nilpotent.

Now consider the general case. Let K be a splitting field for A, and consider $R_1 = R \otimes K$, $A_1 = A \otimes K$, and $C_{R_1}(A_1) = C_R(A) \otimes K$. Since $C_R(A)$ is semisimple, $C_{R_1}(A_1)$ is semisimple, by Proposition 1.3. Thus, the hypotheses are preserved in R_1. Now, by the previous paragraph, $J(R_1)$ is nilpotent. Since $J(R_1) = J(R) \otimes K$ (Proposition 1.3 again), it follows that J(R) is nilpotent.

We next prove a converse to Theorem 3.1.

THEOREM 3.2. *If* R *is semisimple, then* $C_R(A)$ *is semisimple.*

Proof. Arguing as at the end of the proof of the previous theorem, we may assume that A is split. Let f denote the unit element of A.

We first show that the algebra $f\,C_R(A) = f\,C_R(A)f$ is semisimple. Now fRf is semisimple, by [4, Theorem 1.3.3], and $C_{fRf}(A) = f\,C_R(A)$, since f is a central idempotent in $C_R(A)$. Let e be a primitive central idempotent of A; then eAe is central and simple, and in fact $eAe \cong k_n$, since A is split. Now, by Proposition 1.2,

$$eRe = eAe \otimes_k e\,C_R(A)\,e \cong (e\,C_R(A)\,e)_n\,.$$

Since eRe is semisimple, $e\,C_R(A)\,e$ is semisimple. But f, the unit in A, may be written as $f = e_1 + \cdots + e_k$, where the e_i are the primitive central idempotents in A. Since the e_i are also central orthogonal idempotents in $C_R(A)$,

$$f\,C_R(A) = \sum_i e_i\,C_R(A) = \sum_i e_i\,C_R(A)\,e_i\,,$$

and the last member is a finite direct sum of semisimple rings. Thus $f\,C_R(A)$ is semisimple.

Now, $C_R(A) = C_1 \oplus C_2$, where

$$C_1 = f\,C_R(A) \quad \text{and} \quad C_2 = (1 - f)\,C_R(A) = \{x - fx \mid x \in C_R(A)\}$$

(the "1" is only a formal device). It is easy to verify that $C_2 = (1 - f)R(1 - f)$, and thus C_2 is semisimple, since R is semisimple (use the same proof as in [4, Theorem 1.3.3]).

Since $C_R(A)$ is the direct sum of two semisimple rings, $C_R(A)$ must be semisimple.

COROLLARY 3.1. *If* R *is semiprime, then* $C_R(A)$ *is semisimple if and only if* R *is semisimple.*

Proof. Apply Theorems 3.1 and 3.2.

COROLLARY 3.2. $J(R) \cap C_R(A) = J(C_R(A))$.

Proof. Because of Lemma 3.1, it is enough to show that $J(C_R(A)) \subseteq J(R)$. As in the previous arguments, Proposition 1.3 implies that it will suffice to show this for the case when A is split.

Consider $\overline{R} = R/J(R)$. By Proposition 1.5, $C_{\overline{R}}(\overline{A}) = \overline{C_R(A)}$. Since $\overline{R}$ is semisimple, $C_{\overline{R}}(\overline{A})$ is semisimple, by Theorem 3.2. Now $\overline{J(C_R(A))}$ is a quasi-regular ideal of $\overline{C_R(A)}$, which is semisimple, and thus $\overline{J(C_R(A))} = (0)$. This proves $J(C_R(A)) \subseteq J(R)$.

We now prove an analogous result for the lower nil radical. The proof is very similar.

THEOREM 3.3. *If* R *is semiprime, then* $C_R(A)$ *is semiprime.*

Proof. The proof follows that of Theorem 3.2 almost exactly. $C_R(A) = C_1 \oplus C_2$, where

$$C_1 = f\,C_R(A) \quad \text{and} \quad C_2 = (1 - f)R(1 - f)\,.$$

C_2 is semiprime since R is, and we need only show that C_1 is semiprime. As before, under the assumption that A is split, we get the equation

$$eRe = eAe \otimes_k eC_R(A)e .$$

If N were a nilpotent ideal of $eC_R(A)e$, the set $eAe \otimes N$ would be a nilpotent ideal of the semiprime ring eRe. Thus $eC_R(A)e$ is semiprime, and therefore $fC_R(A)$ is semiprime. This proves that $C_R(A)$ is semiprime when A is split.

However, if A is not split, consider R_1, A_1, and $C_{R_1}(A_1) = C_R(A) \otimes K$ as before. By Proposition 1.4, R_1 is semiprime, since R is semiprime; certainly, if $C_{R_1}(A_1)$ is semiprime, then $C_R(A)$ is semiprime. Thus the proof for the split case suffices.

4. SEMISIMPLE ARTINIAN RINGS

The main purpose of this section is to determine the structure of R when $C_R(A)$ is semisimple Artinian. Before proceeding, we note that when K is a finite separable extension of k, then R is Artinian if and only if $R \otimes_k K$ is Artinian. It then follows, by virtue of Proposition 1.3, that R is semisimple Artinian if and only if $R \otimes K$ is semisimple Artinian.

LEMMA 4.1. *Assume that* R *is semiprime and that* A *is split. Then, if* $C_R(A)$ *is simple Artinian,* R *is simple Artinian.*

Proof. First, R is simple, by Corollary 2.3. Now A must be simple also, since every central idempotent of A is also a central idempotent in $C_R(A)$, which is simple. It also follows that the unit element in A is a unit element for $C_R(A)$, and thus, by Lemma 2.1, it must also be a unit element for R.

Since A is split, A is a central simple k-algebra, and thus $R = A \otimes_k C_R(A)$, by Proposition 1.2. Since A is finite-dimensional and $C_R(A)$ is Artinian, R must also be Artinian.

The method of proof now follows that of [5, Theorem 5]. Our next lemma was established in the course of proving [5, Theorem 5].

LEMMA 4.2 (Herstein and Neumann). *Let* R *be a semiprime ring with* 1. *Assume that* $1 = e_1 + \cdots + e_k$, *where the elements* e_i *are orthogonal idempotents, and that* $e_i R e_i$ *is simple Artinian for* $i = 1, \cdots, k$. *Then* R *is semisimple Artinian.*

THEOREM 4.1. *Let* R *be semiprime. Then, if* $C_R(A)$ *is semisimple Artinian,* R *is semisimple Artinian.*

Proof. By the remarks preceeding Lemma 4.1, it will suffice to prove the theorem when A is split.

Let $T = C_R(A) = e_1 T \oplus \cdots \oplus e_k T$, where each e_i is a central idempotent in T and each $e_i T$ is simple Artinian. The unit element 1 of T is also the unit element for R (by Lemma 2.1), and $1 = e_1 + \cdots + e_k$.

Let $R_i = e_i R e_i$. It is easy to verify that each R_i is semiprime and that $C_{R_i}(e_i A) = e_i C_R(A) e_i = e_i T$. Now, since e_i centralizes A, the algebra $e_i A$ (as a homomorphic image of A) is also a split separable k-algebra. Since $C_{R_i}(e_i A)$ is simple Artinian, we may apply Lemma 4.1 to the ring R_i to see that R_i is simple Artinian. If we apply Lemma 4.2, it follows that R is semisimple Artinian.

COROLLARY 4.1. *If* R *is semiprime and* $C_R(A)$ *is simple Artinian, then* R *is simple Artinian.*

This follows from Theorem 4.1 and Corollary 2.3.

We now prove a converse to Theorem 4.1.

THEOREM 4.2. *If* R *is semisimple Artinian, then* $C_R(A)$ *is semisimple Artinian.*

Proof. By the remarks preceeding Lemma 4.1, it will suffice to prove the theorem when A is split.

Since R is semisimple Artinian, $1 \in R$, and $1 = e_1 + \cdots + e_k$, where the e_i are central idempotents and each $e_i R$ is simple Artinian.

Now $1 \in C_R(A)$, and therefore $C = C_R(A) = \sum_i \oplus e_i C$. Thus, to show that C is semisimple Artinian, it will suffice to show that each $e_i C$ is semisimple Artinian. Now $e_i A = e_i A e_i$ is separable and finite-dimensional, and $C_{e_i R}(e_i A) = e_i C$. Thus, by looking at the rings $e_i R$, we see that we need only prove the theorem when R is simple.

Let f be the unit element in A, and let e be any primitive central idempotent in A. Since A is split, eAe is central simple over k, and as before,

$$eRe = eAe \otimes_k e\, C_R(A)\, e,$$

by virtue of Proposition 1.2. Since R is simple Artinian, eRe is simple Artinian, and thus $e\, C_R(A)\, e = e\, C_R(A)$ is simple Artinian. Now $f = g_1 + \cdots + g_m$, where the g_i are the primitive central idempotents in A. Thus $f\, C_R(A) = \sum_j g_i C_R(A)$ is semisimple Artinian.

Now, as in Theorem 3.2, $C = C_1 \oplus C_2$, where

$$C_1 = f\, C_R(A) \quad \text{and} \quad C_2 = (1 - f)\, C_R(A) = (1 - f)\, R\, (1 - f).$$

Again, C_2 is simple Artinian, since R is. Thus C, being a finite direct sum of simple Artinian rings, must be semisimple Artinian.

COROLLARY 4.2. *Let* R *be semiprime. Then* $C_R(A)$ *is semisimple Artinian if and only if* R *is semisimple Artinian.*

This follows from Theorems 4.1 and 4.2.

Finally, we consider the situation when R is not necessarily semiprime.

THEOREM 4.3. *Assume that* $C_R(A)$ *is semisimple Artinian. Then* R *has a nilpotent ideal* N *such that* R/N *is semisimple Artinian.*

Proof. By Theorem 3.1, J(R) is nilpotent. Since R/J(R) is semisimple, J(R) = N is the desired ideal. It only remains to show that R/N is Artinian. For this, it suffices to show that $R/N \otimes K$ is Artinian, for each finite extension K of k.

Let K be a splitting field for A. Then

$$R/_N \otimes K = R \otimes K/_{N \otimes K} = R \otimes K/_{J(R \otimes K)},$$

since $J(R) \otimes K = J(R \otimes K)$, by Proposition 1.3. Thus, if $R_1 = R \otimes K$, it suffices to show that $R_1/J(R_1)$ is Artinian.

Now R_1, A_1, and $C_{R_1}(A_1) = C_R(A) \otimes K$ satisfy the hypotheses on A and $C_R(A)$. In $\overline{R}_1 = R_1/J(R_1)$,

$$C_{\overline{R}_1}(\overline{A}_1) = \overline{C_{R_1}(A_1)} \cong C_{R_1}(A_1),$$

since A_1 is split. Thus we may apply Theorem 4.1 to the ring $\overline{R}_1$, and therefore $\overline{R}_1$ is semisimple Artinian. The theorem is now proved.

COROLLARY 4.3. *Assume that* $C_R(A)$ *is simple. Then* J(R) *is nilpotent and* R/J(R) *is simple. Moreover, if* $C_R(A)$ *is simple Artinian,* R/J(R) *is simple Artinian.*

Proof. If e is the unit element for A, then $e \in C_R(A)$, and therefore e is the unit element for $C_R(A)$. Thus, by Lemma 2.1, e + N is a unit element in R/N, where N is the lower nil radical of R. Now, since $C_R(A)$ is semisimple, J(R) = N is nilpotent, by Theorem 3.1.

Let U be any ideal of R. If $U \cap C_R(A) = (0)$, then eU is nilpotent, by Theorem 2.1. Thus $eU \subseteq N$, hence $\overline{eU} = (0)$ in $\overline{R} = R/N$. But $\bar{e}$ is the unit element in $\overline{R}$, and thus $\overline{U} = (0)$. On the other hand, if $U \cap C_R(A) \neq (0)$, then $U \cap C_R(A) = C_R(A)$, by the simplicity of $C_R(A)$. This means that $e \in U$, and thus in $\overline{R}$, $\overline{U} = \overline{R}$. We have shown that N is a maximal ideal of R, and thus R/N is simple.

The rest of the proof follows directly from Theorem 4.3.

5. ZERO-DIVISORS

We say that an element x is *regular* in R if x is not a zero-divisor in R. In this section, we show that if x is regular in $C_R(A)$, then x is regular in R, provided that R is semiprime.

LEMMA 5.1. *Let* R *be semiprime, and assume that* $x \in C_R(A)$ *is regular in* $C_R(A)$. *Then*

(1) $xa \neq 0$, *for all* $a \in A$ $(a \neq 0)$,

(2) *if* x *is not regular in* R, *then* $xz = 0$, *for some* $z \in fRf$ $(z \neq 0)$, *where* f *is the unit element of* A.

Proof. (1) Say that $xa = 0$, for some $a \in A$. Then $xAaA = (0)$, since $x \in C_R(A)$. But AaA is an ideal of A, and thus it contains a nonzero element z in the center of A. But then $z \in C_R(A)$ and $xz = 0$, a contradiction.

(2) Say that x is not regular, and let $I = \{y \in R \mid xy = 0\}$. Then I is a nonzero right ideal of R. Let $W = (1 - f)I(1 - f)$. Now $AW = WA = (0)$, and therefore $W \subseteq C_R(A)$. Since $xW = (1 - f)xI(1 - f) = (0)$ and x is regular in $C_R(A)$, we see that $W = (0)$. Thus $((1 - f)I)^2 = (0)$. Since R is semiprime, $(1 - f)I = (0)$. That is, $y = fy$, for all $y \in I$.

Since R is semiprime, $yRy = fyRfy \neq (0)$, and since $yR \subseteq I$, this means that $fIf \neq (0)$. But, $xfIf = fxIf = (0)$, and thus $xz = 0$, for some $z \in fRf$ $(z \neq 0)$.

THEOREM 5.1. *Let* R *be semiprime. Then, if* $x \in C_R(A)$ *is regular in* $C_R(A)$, x *is regular in* R.

Proof. We first claim that we may assume A is split. For consider $R_1 = R \otimes K$, $A_1 = A \otimes K$, and $C_{R_1}(A_1) = C_R(A) \otimes K$ as before, where K is a splitting field for A. Now, if x is regular in $C_R(A)$, $x \otimes 1$ must be regular in $C_R(A) \otimes K$. For if not, say $(x \otimes 1)y = 0$, and write $y = \sum r_i \otimes k_i$, where the r_i belong to $C_R(A)$ and the k_i are linearly independent over k. Then $(x \otimes 1)y = \sum x r_i \otimes k_i = 0$ implies that $xr_i = 0$, for all i. Since $r_i \in C_R(A)$, it follows that $r_i = 0$ for all i and thus $y = 0$. Thus $x \otimes 1$ is regular in $C_{R_1}(A_1)$. If we can show that $x \otimes 1$ is regular in R_1, surely x is regular in R. Thus, assume that A is split.

By Lemma 5.1, if x is not regular in R, then $xz = 0$, where $0 \neq z \in fRf$ and f is the unit element in A. Consider the ring fRf. Clearly,

$$C_{fRf}(A) = f\,C_R(A)\,f = f\,C_R(A),$$

and $xf \in C_{fRf}(A)$ is regular in $C_{fRf}(A)$ (for, if $z \neq 0$ and $z \in f\,C_R(A)$, then $xfz = xz \neq 0$, since $z \in C_R(A)$). Since fRf is a semiprime ring, we may reduce the problem to the case of the ring fRf; that is, we may assume that the unit in A is the unit element for R.

Assuming that x is not regular in R, let I be the right annihilator of x. Then I is a nonzero right ideal of R. Let e be a primitive central idempotent in A, and consider eI. Since $1 \in A$ and $I \neq (0)$, $eI \neq (0)$ for some such e. Since R is semiprime, it follows that $eIe \neq (0)$. But then $xeIe = exIe = (0)$, since $e \in C_R(A)$. Thus, xe is a zero divisor in the ring eRe.

Now $eRe = eAe \otimes_k e\,C_R(A)\,e$, as before. Consider $1 \otimes xe$. Since it is not regular in eRe, $(1 \otimes xe)y = 0$ for some y. Write $y = \sum a_i \otimes r_i$, where $r_i \in e\,C_R(A)\,e$ and the a_i are linearly independent over k. Then

$$(1 \otimes xe)\,y = \sum a_i \otimes xer_i = (0).$$

Thus $xer_i = 0$, for all i. But $er_i = r_i \in C_R(A)$, and thus $r_i = 0$, since x is regular in $C_R(A)$. This implies that $y = 0$, a contradiction. This proves the theorem.

As we mentioned in the introduction, Cohen has used this last result, along with several other results discussed above, to show that when R is semiprime and $C_R(A)$ is a Goldie ring, then R is also a Goldie ring [3]. The general method of proof is to show first that R can be localized at the set T of regular elements of $C_R(A)$ (which are regular in R, by Theorem 5.1), and then to show that the localization R_T is semisimple Artinian. Since R is an order in R_T, R must be a Goldie ring.

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