CAPACITY AND MEASURE

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1. *Introduction.* A *condenser* in the euclidean space R^n is a pair $E = (A, C)$, where A is open in R^n and C is compact in A. For $p \geq 1$, we define the *p-capacity* of E as

$$\operatorname{cap}_p E = \inf_u \int |\nabla u|^p \, dm,$$

where the infimum is taken over all functions u in $C_0^\infty(A)$ such that $u(x) = 1$ for all $x \in C$. It is well known that if $\operatorname{cap}_p(A_0, C) = 0$ for some bounded A_0, then $\operatorname{cap}_p(A, C) = 0$ for all open sets A containing C. In this case, we write $\operatorname{cap}_p C = 0$, and otherwise $\operatorname{cap}_p C > 0$. The case $p = n$ is particularly important in the theory of quasiregular maps, and here we write $\operatorname{cap} = \operatorname{cap}_n$. If $p > n$, then $\operatorname{cap}_p C = 0$ only in the case $C = \emptyset$.

The capacity of a condenser can also be defined with the aid of moduli of path families. Given a bounded condenser $E = (A, C)$, we let Γ_E be the family of all paths $\alpha\colon [a, b) \to A$ such that $\alpha(a) \in C$ and $\alpha(t) \to \partial A$ as $t \to b$. Then

$$\operatorname{cap}_p E = M_p(\Gamma_E),$$

by W. P. Ziemer [10]. Here M_p denotes the p-modulus. Instead of Γ_E, we may take the family of all paths joining C and ∂A in $A \setminus C$.

In this note we shall give a new proof for the following result: If a compact set $C \subset R^n$ has a finite h-measure for $h(r) = (\log(1/r))^{1-n}$, then $\operatorname{cap} C = 0$. The corresponding result holds for the p-capacity with $h(r) = r^{n-p}$.

The earliest result of this type is due to J. W. Lindeberg [4]. He showed that for $n = p = 2$, $\operatorname{cap} C = 0$ for every compact set C of h-measure zero, $h(r) = (\log(1/r))^{-1}$. This result was extended for sets of finite h-measure by P. Erdős and J. Gillis [2]. A simple proof of their result was given by L. Carleson [1]. His proof is also applicable in higher dimensions. These authors used a potential-theoretic definition for capacity. For $p = 2$, this is equivalent to our definition. For $p \neq 2$, this is no longer true, although there are close connections (see [8, p. 332]). Our results are contained in papers of N. Meyers [6, Theorem 21] and, V. G. Mazja and V. P. Havin [5, Section 7], who formulated them in a very general framework. The present formulation is from H. Wallin [9, Theorem 4.3]. For related results, see [8, Remark on p. 335] and [7, Theorem 4.2].

2. *Notation.* If $C \subset R^n$ and $r > 0$, we let $B(C, r)$ be the set of all x in R^n such that $\operatorname{dist}(x, C) < r$. In particular, $B(x, r)$ is the open ball with center at x and radius r. If C is compact, $E(C, r)$ will denote the condenser $(B(C, r), C)$.

3. LEMMA. *If* $p > 1$ *and* C *is a compact set in* R^n *with* $\operatorname{cap}_p C > 0$, *then* $\lim_{r \to 0} \operatorname{cap}_p E(C, r) = \infty$.

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Proof. Assume $\operatorname{cap}_p C > 0$. Since $\operatorname{cap}_p E(C, r)$ is decreasing in r, it converges to a limit a $(0 < a \leq \infty)$ as $r \to 0$. Suppose that $a < \infty$. Set $\Gamma(r) = \Gamma_{E(C,r)}$. For $0 < s < r$, we let $\Gamma(s, r)$ be the family of all paths joining $\partial B(C, s)$ and $\partial B(C, r)$ in $B(C, r) \setminus \overline{B}(C, s)$. Then

$$M_p(\Gamma(r))^{\frac{1}{1-p}} \geq M_p(\Gamma(s))^{\frac{1}{1-p}} + M_p(\Gamma(s, r))^{\frac{1}{1-p}}$$

(see [3, Theorem 1 (d), p. 178], for example). As $s \to 0$, $M_p(\Gamma(s)) \to a$ and

$$M_p(\Gamma(s, r)) = \operatorname{cap}_p(B(C, r), \overline{B}(C, s)) \to \operatorname{cap}_p E(C, r) = M_p(\Gamma(r)).$$

Hence we obtain the inequality $a \leq 0$, a contradiction.

4. *Notation.* For any function $h: (0, 1) \to (0, \infty)$, we let Λ_h denote the corresponding Hausdorff measure.

5. THEOREM. *Let* $h(r) = (\log(1/r))^{1-n}$, *and let* C *be a compact set in* R^n *such that* $\Lambda_h(C) < \infty$. *Then* cap C = 0.

Proof. By Lemma 3, it suffices to show that $\operatorname{cap} E(C, r) = M(\Gamma(r))$ is bounded for small r. Let $0 < r < 1$, set $a = \Lambda_h(C)$, and choose a countable covering of C with balls $B_i = B(x_i, r_i)$ with $x_i \in C$ and $r_i < r^2$ such that

$$\sum_i \left(\log \frac{1}{r_i}\right)^{1-n} \leq a + 1.$$

Let Γ_i be the family of all paths joining the boundary components of the annulus $B(x_i, r) \setminus \overline{B}_i$. Then $\Gamma(r)$ is minorized by $\bigcup \{\Gamma_i \mid i \in N\}$. By [3, Theorem 1], this implies

$$M(\Gamma(r)) \leq \sum_i M(\Gamma_i) = \omega \sum_i \left(\log \frac{r}{r_i}\right)^{1-n},$$

where ω is the $(n-1)$-area of the unit sphere. Here $r/r_i > r_i^{-1/2}$, whence

$$M(\Gamma(r)) \leq 2^{n-1} \omega \sum_i \left(\log \frac{r}{r_i}\right)^{1-n} \leq 2^{n-1} \omega(a+1).$$

6. *The case* $1 < p < n$. Using the same method, we can show that if $1 < p < n$ and $\Lambda_h(C) < \infty$ with $h(r) = r^{n-p}$, then $\operatorname{cap}_p C = 0$. The proof makes use of the formula

$$M_p(\Gamma) = \omega\left(\frac{\alpha}{a^{-\alpha} - b^{-\alpha}}\right)^{p-1} \qquad \left(\alpha = \frac{n-p}{p-1}\right)$$

for the family Γ of paths joining the boundary components of the spherical annulus $B(b) \setminus \overline{B}(a)$.

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