ON GENERALIZED VALUATION RINGS

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1. INTRODUCTION

All rings in this note are commutative with identity, and modules are unital. A number of generalizations of the notion of valuation ring to rings with zero divisors can be found in the literature. In this note, we adopt the following definition, which is equivalent to one proposed by I. Kaplansky in [2, p. 479]: a valuation ring is a ring whose ideals are linearly ordered by set inclusion. This notion of valuation ring has proved to be a useful tool in varied contexts (for examples, see [4], [5], and [7]). We define the valuation semigroup of a valuation ring to be the semigroup of principal ideas under ideal multiplication. In Section 2, we characterize value semigroups abstractly as the Rees factor semigroups of extended positive cones of totally ordered abelian groups. This result may be of independent interest to semigroup theorists. Moreover, our results make it possible to associate with each valuation ring a value group that coincides with the usual notion in the case of domains. These value groups are examined in Section 3.

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2. VALUE SEMIGROUPS

In analogy with value semigroups, we shall use the additive notation for the *value semigroup* R^* of the valuation ring R. Specifically, if $x, y \in R$, then $xR = x^*$, $xyR = x^* + y^*$, $1^* = 0$ (0 denotes the identity of the semigroup), and $0^* = \infty$. Incidentally, the map $*$ is a Hahn valuation on R in the sense of [6]. The *extended positive cone* of a totally ordered abelian group is the usual positive cone together with the disjoint element ∞, where $\infty + a = a + \infty = \infty$ for all a.

Following A. Clifford [3], we define the *natural ordering* on a commutative additive semigroup S by the rule: if $a, b \in S$, then $a \leq b$ whenever $a = b$ or there exists an element $c \in S$ such that $a + c = b$. The semigroup S is *naturally totally ordered* (abbreviation: n.t.o.) if the natural ordering on S is total, that is, if for every pair of elements $a, b \in S$, exactly one of the conditions $a < b$, $a = b$, and $b < a$ is valid for the pair. Note that the natural ordering is preserved by homomorphisms, and that it is compatible with addition. As usual, a nonempty set $I \subseteq S$ is an *ideal* of S if $I + S \subseteq I$. If I is an ideal of S, then the *Rees factor semigroup* (denoted by S/I) is the semigroup whose underlying set is $(S - I) \cup \{\infty\}$, where $\infty \notin S$, and whose addition is given as follows. If the elements a and b belong to S/I and $a + b$ is defined in S and belongs to $S - I$, then the sum in S/I is the same; otherwise, $a + b = \infty$. We shall say that the semigroup S is *segmental* provided that

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(i) S is n.t.o.,

(ii) S has a greatest element ∞, and

(iii) if a, b $\in$ S and a + b = b, then b = ∞.

Also, if S is a semigroup with (additive) identity 0 and S - $\{0\}$ is a segmental subsemigroup, then we say that S is 0-*segmental.*

Before we can state our main result on semigroups, we need the following lemma. Fortunately, the hard work has already been done for us by Clifford, in the fundamental paper [3]. One need only notice that the condition (iii) of the definition of segmental semigroups is simply the conclusion of Lemma 2.2 of [3, p. 636]. A careful scrutiny shows that Lemmas 2.3 and 2.4 as well as virtually all of Section 3, except the claims concerning the Archimedean property, remain valid for segmental semigroups. Therefore we may reformulate Theorem 3 of [3, p. 640] as follows.

LEMMA. *Let* S *be a nontrivial segmental semigroup. Then there exists a* n.t.o. *commutative semigroup* T, *without identity element, in which the cancellation law holds, and such that the following two conditions are satisfied.*

(A) T *contains an ideal* T_∞ *such that* S *is isomorphic to* T/T_∞.

(B) T *is generated by the complement* T - T_∞ *of* T_∞ *in* T. *Moreover, if* S - $\{\infty\}$ *is identified with* T - T_∞, *then* T *is uniquely determined to within an isomorphism leaving fixed the elements of* S - $\{\infty\}$.

THEOREM 1. *The following are equivalent conditions on the semigroup* S.

(1) S $\cong$ R^* *for some valuation ring* R.

(2) S *is a* 0-*segmental semigroup.*

(3) S $\cong$ T/T_∞, *where* T *is the extended positive cone of a totally ordered abelian group and* T_∞ *is an ideal of* T.

Proof. (1) implies (2). We may assume that S = R^*. Let S' = S - $\{0\}$, so that S' is just the subsemigroup of proper principal ideals of R. If x^*, y^* $\in$ S', then either xR $\subset$ yR, xR = yR, or yR $\subset$ xR. If xR $\subset$ yR, for instance, then we may write x = yr for a nonunit r $\in$ R. Hence $x^* = y^* + r^*$, and $y^* < x^*$ in the natural ordering. It follows that S' is n.t.o. Therefore S is a 0-segmental semigroup, as required.

(2) implies (3). Suppose that S is 0-segmental, and let S' = S - $\{0\}$. Since the case in which S = $\{0, \infty\}$ is trivial, we may assume that S - $\{0, \infty\}$ is nonempty. Apply the Lemma to the segmental semigroup S' to obtain a cancellative n.t.o. semigroup T' with S' $\cong$ T'/I_∞, say. Since T' is cancellative, T' can be embedded in the usual way in its "group of differences" G. Identify T' with its image in G, and consider the subset P = T' $\cup$ $\{0\}$ of G. Notice that 0 $\in$ P, and that if both a and -a belong to P, then a = 0 since T' is a subsemigroup of G. Hence P is a positive cone that makes G into a partially ordered group. Furthermore, if g is a nonzero element of G, say g = a - b for suitable a, b $\in$ T', then either a - b or b - a belongs to P, since T' is n.t.o. Therefore, P determines a linear ordering that makes G into a totally ordered abelian group. Let T be the extended positive cone of G. Then the set $T_\infty = I_\infty \cup \{\infty\}$ is an ideal of T such that T/T_∞ is isomorphic to the semigroup S' with a zero adjoined, which is precisely S.

(3) implies (1). Suppose that S = T/T_∞, where T is the extended positive cone of a totally ordered abelian group G. It is a classical theorem of Krull [3] that there exists a valuation ring V whose value group is G. Let I be the ideal of V

consisting of elements of V whose value is an element of T_∞. We complete the proof by observing that the natural map $V \to V/I$ induces a semigroup isomorphism between $(V/I)^*$ and T/T_∞. Since this fact merits additional comment, we isolate it in the following remark.

Remark. The operation * is functorial, since every ring homorphism $\phi: R \to S$ of valuation rings induces a semigroup homorphism $\phi^*: R^* \to S^*$ of 0-segmental semigroups by the formula $\phi^*(x^*) = \phi(x)^*$. In particular, if ϕ is a surjective map, then $(\ker \phi)^*$ is an ideal of R^*. Moreover, if $x \notin \ker \phi$, then $\ker \phi \subseteq xR$, since R is a valuation ring. Hence the correspondence between principal ideals xR, $x \in R - \ker \phi$, and $\phi(x)R$ is one-to-one and order-perserving. Consequently, ϕ^* induces an isomorphism between the Rees factor semigroup $R^*/(\ker \phi)^*$ and S^*. This is one indication that value semigroups might be more useful from a functorial point of view than value groups, even when we are dealing exclusively with domains.

3. VALUE GROUPS

Let R be a valuation ring. Then by Theorem 1 there is an extended positive cone T of the totally ordered abelian group G and ideal T_∞ for which $R^* \cong T/T_\infty$. Moreover, the uniqueness part of the lemma asserts that there is exactly one such T (up to an isomorphism), subject to the condition that T is generated by $R^* - \{\infty\}$, where suitable identifications are made. Consequently, the group G is uniquely determined by R, and the positive cone of G is generated by $R^* - \{\infty\}$ (as a subset of T). The group G is defined to be the *value group* of R, and we shall write $\Gamma(R) = G$. Notice that if R is a domain, then T is isomorphic to the extended positive cone of the usual value group of R, whence $\Gamma(R)$ is order-isomorphic to the usual value group of R. Therefore, nothing new is introduced in the case of valuation domains. The following result sheds some light on the general situation.

THEOREM 2. *Let* P *be the minimal prime ideal of the valuation ring* R. *Then there is an exact sequence of totally ordered abelian groups*

$$0 \to \Gamma(R/P) \xrightarrow{\alpha} \Gamma(R) \xrightarrow{\beta} A \to 0$$

such that α *is an order-preserving map of* $\Gamma(R/P)$ *onto a convex subgroup of* $\Gamma(R)$, *and* A *(with the natural ordering induced by* β*) is an Archimedean group. Moreover, if the sequence splits, then* $\Gamma(R)$ *is order-isomorphic to* $A \oplus \Gamma(R/P)$, *where the direct sum has the lexicographic ordering.*

Proof. If R is already a domain, then $P = 0$ and the result is trivial. Suppose therefore that $P \neq 0$. Let T denote the extended positive cone of $\Gamma(R)$, and let T_∞ be an ideal of T such that $R^* = T/T_\infty$. Then P^* is an ideal of R^* such that $R^* - P^*$ is a subsemigroup of T/T_∞ and therefore of T itself. Let I be the ideal of T corresponding to P^* under the natural map $T \to T/T_\infty$; then it follows that T is the disjoint union of the ideal I and the subsemigroup $R^* - P^* = S$. Furthermore, S is convex in the sense that if $0 \leq x \leq s$ for $x \in T$ and $s \in S$, then $x \in S$. Consequently, the group of differences of S is a convex (or isolated) subgroup H of the group of differences of $T - \{\infty\}$, and the latter group is simply $\Gamma(R)$. But from the Remark it is clear that $R^*/P^* = (R/P^*)$, where the latter semigroup is the extended positive cone of $\Gamma(R/P)$. Therefore the nonextended positive cone of $\Gamma(R/P)$ is the semigroup $R^* - P^*$. This implies that $H = \Gamma(R/P)$, and this gives us the first part of the exact sequence. Now form the factor group $\Gamma(R)/\Gamma(R/P) = A$. Since $\Gamma(R/P)$ is a convex subgroup of $\Gamma(R)$, we can make A into a totally ordered abelian group by letting the positive cone of A be the image of T in A. But the semigroup T is generated by the subset $T - T_\infty$, whence $(T + A)/A$ is generated by 0 and the image of $I - T_\infty$. Let a and b be elements of the form

$$a = m_1 p_1 + \cdots + m_r p_r \quad \text{and} \quad b = n_1 q_1 + \cdots + m_s q_s,$$

where the m_i and n_j are positive integers and the p_i and q_j belong to $I - T_\infty$. Then the image of p_1 under the natural map $T \to T/T_\infty = R^*$ belongs to P^*. Since P is the minimal prime ideal of R, it is also the prime radical of R; therefore every element of P is nilpotent. Hence some positive multiple of each element of P^* is ∞. Thus some positive multiple of p_1, say mp_1, belongs to T_∞. It follows that $mp_1 > q_i$ for all i, so that $(n_1 + \cdots + n_s)\, ma > b$. We now see, by passing to the factor group A, that A is an Archimedean group, as required.

The last assertion of the Theorem is true for each split exact sequence $0 \to H \to G \to G/H \to 0$, where H is a convex subgroup and G/H has the induced quotient ordering. To see this, we write $G = K \oplus H$ with K order-isomorphic to G/H, and we deduce from the convexity of K that $k_1 + h_1 \leq k_2 + h_2$ if and only if $k_1 < k_2$ or $k_1 = k_2$ and $h_1 \leq h_2$.

COROLLARY. *If* R *is a valuation ring with minimal prime ideal* P *such that* $P^2 \subset P$, *then* $\Gamma(R)$ *is order-isomorphic to* $\mathbb{Z} \oplus \Gamma(R/P)$, *where the direct sum has the lexicographic ordering.*

Proof. The condition that $P^2 \subset P$ translates into $P^* + P^* \subset P^*$. We saw in the proof of Theorem 2 (adopt the same notation) that the positive cone of A consists of 0 and the semigroup generated by the image $I - T_\infty$ in A. The fact that $P^* + P^* \subset P^*$ implies that $I + I \subset I$. Since the complement of I in T is the convex subsemigroup S, it is easily seen that $I + H = I$. Hence the strict inequality $I + I \subset I$ is preserved in passage to $\Gamma(R)/H = A$. Therefore, A is an Archimedean group whose nonextended positive cone is $C \cup \{0\}$, where the inequality $C + C \subset C$ is satisfied. Every Archimedean group is order-isomorphic to an additive subgroup of the reals, as is well known. But C has a least element, for else every $c \in C$ could be written as $c = (c - a) + c \in C + C$ for some $a \in A$ with $0 < a < c$. Therefore A must be order-isomorphic to the ring $\mathbb{Z}$ of rational integers, as required.

Our final result shows that the value group of a ring that is a homomorphic image of a valuation domain has a nice interpretation in terms of value groups of domains.

THEOREM 3. *Let* I *be a proper ideal of the valuation domain* V, *and let* P *be the largest prime ideal of* V *contained in* I. *Then* $\Gamma(V/I)$ *is order-isomorphic to* $\Gamma(V/P)$.

Proof. Let $T = V/I$. We may assume that I is not prime, for otherwise there is nothing to prove. To simplify notation, we may suppose that $P = 0$ and $I \neq 0$. Then $S' = R^* - \{0\}$ is nontrivial segmental semigroup, by Theorem 1, and the finite elements of S do not form a subsemigroup. In addition, if

$$T = V^* - \{0, \infty\} \quad \text{and} \quad T_\infty = I^* \cap T,$$

we see from the Remark that $S \cong T/T_\infty$. Identify the finite elements of S with corresponding elements in T. We claim that T is generated by the finite elements of S. To see this, let Q be the minimum prime ideal of V that contains I, so that $I \subset Q$. Fix an element $x \in P - I$, and let y be an arbitrary nonzero element of I. Then there exists an element $r_1 \in R$ such that $xr_1 = y$. If $r_1 \in I$, then there exists

$r_2 \in R$ such that $xr_2 = r_1$. Were it possible to obtain in this fashion an infinite sequence $r_1, r_2, \cdots$ of elements of I, we could write $y = x^n r_{n-1}$ for all positive integers n. However, the image of x in R/yR belongs to the unique minimal prime of R/yR; therefore the image of x is nilpotent. Thus for some integer n we have the relation $x^n \in yR$, say $x^n = ys$ for suitable $s \in R$. But then $y = ysr_{n-1}$, whence r_{n-1} is a unit. This contradiction shows that for some positive integer n we must have the relation $r_n \notin I$. Hence $y^* = (n+1)x^* + r_n^*$, which establishes the claim. It follows from the uniqueness claim of the Lemma that $\Gamma(R)$ is order-isomorphic to the group of differences of T, which is simply $\Gamma(V)$.

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