QUASIALGEBRAIC OPERATORS, COMPACT PERTURBATIONS, AND THE ESSENTIAL NORM

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1. INTRODUCTION AND NOTATION

Quasialgebraic operators generalize algebraic operators, in the same way that quasinilpotent operators generalize nilpotent operators. An element T of a Banach algebra is *quasialgebraic* if one can find a sequence $\{p_n\}$ of monic polynomials with deg $p_n = d(n)$, such that $\lim_n \|p_n(T)\|^{1/d(n)} = 0$. This concept was first introduced by P. R. Halmos in [14], where he related it to the potential-theoretic notion of capacity. As a starting point for our paper, we rely on some of his observations and techniques to focus on this question: if T is a bounded operator on a Hilbert space, and the coset $\nu(T)$ in the Calkin algebra is quasialgebraic, does it contain a compact perturbation of T that is quasialgebraic with respect to the same sequence of polynomials?

More precisely, W. B. Arveson has asked: if $\{p_n\}$ is a sequence of monic polynomials of degrees d(n) such that $\lim_n \|p_n(\nu(T))\|^{1/d(n)} = 0$, does there exist a compact K such that $\lim_n \|p_n(T+K)\|^{1/d(n)} = 0$? Like other questions involving the structure of the Calkin algebra, it is recalcitrant. But it deserves attention, for an affirmative answer would imply two previous results: for a Hilbert space operator T, C. L. Olsen has proved that if $p(\nu(T)) = 0$, for some polynomial p, then there is a compact K with $p(T+K) = 0$ [17]; T. T. West has shown that if $\lim_n \|[\nu(T)]^n\|^{1/n} = 0$, then there is a compact K such that $\lim_n \|(T+K)^n\|^{1/n} = 0$ [24, Theorem 7.5]. In other words, an algebraic coset in the Calkin algebra contains an algebraic operator, and a quasinilpotent coset contains a quasinilpotent operator.

If the answer to Arveson's question is yes, then a quasialgebraic coset in the Calkin algebra must contain a quasialgebraic operator. In fact, we show that even more is true, by observing that a coset and every element in it must have the same capacity [Section 2]. These results were also obtained independently by David S. G. Stirling [22]. Complications arise when we insist that some compact perturbation of T be quasialgebraic with respect to the *same sequence of polynomials* as $\nu(T)$. However, if the sequence $\{p_n\}$ of monic polynomials has a subsequence of bounded degree, then, using Halmos's techniques and Olsen's theorem, we can easily answer Arveson's question. In any case, an application of a theorem of J. G. Stampfli [21] enables us to answer a weakened version of the question [Section 3]; that is, if a Hilbert space operator T is such that $\lim_n \|p_n(\nu(T))\|^{1/d(n)} = 0$ for a sequence $\{p_n\}$ of monic polynomials with deg $p_n = d(n)$, then there exist a compact K and a sequence $\{s(n)\}$ of positive integers such that

$$\lim_n \|[p_n(T+K)]^{s(n)}\|^{1/d(n)s(n)} = 0.$$

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The most difficult aspect of this problem seems to involve those operators whose norms cannot be appropriately related to their spectral radii. On the other hand, if some compact perturbation of an operator T is subnormal, a much stronger statement is true: there exists a compact K such that $\|p(T+K)\|$ is the essential norm of p(T) for *all* polynomials p; by the *essential norm* of an operator T we mean the norm $\|\nu(T)\|$ of its image in the Calkin algebra [Section 4]. We show that an analogous result holds for any essentially normal operator. For an operator T unrelated to the family of subnormal operators, we offer a more modest conjecture (still much stronger than Olsen's result): corresponding to each polynomial p, there exists a compact K such that $\|p(T+K)\|$ is the essential norm of p(T) (where K now depends on p).

It is well known that this conjecture holds for $p(z) = z$, so that for each Hilbert space operator T, there is a compact K with $\|T+K\|$ equal to the essential norm of T [11], [15]. This result is due to I. C. Gohberg and M. G. Kreĭn. We establish our conjecture for $p(z) = z^2$, under the additional assumption that either T is a partial isometry, or else TT^* commutes with T^*T. Finally, in Section 5, we are able to show that $\inf_{K \text{ compact}} \|p(T+K)\|$ equals the essential norm of p(T) for every polynomial p if either T or T^* is quasitriangular.

Throughout the paper, $\mathscr{B}(\mathscr{H})$ will denote the algebra of bounded linear operators on a separable, complex Hilbert space. Let $\mathscr{K}$ denote the closed two-sided ideal in $\mathscr{B}(\mathscr{H})$ of compact operators, and let $\nu: \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})/\mathscr{K}$ be the natural homomorphism onto the C*-algebra quotient, the *Calkin algebra*. Define the *essential norm* of $T \in \mathscr{B}(\mathscr{H})$ to be

$$\inf_{K\in\mathscr{K}} \|T+K\| = \|\nu(T)\|,$$

the norm of its image in the Calkin algebra. We use $\sigma(T)$ to denote the spectrum of T in $\mathscr{B}(\mathscr{H})$, while $\sigma(\nu(T))$ will denote the spectrum of $\nu(T)$ in $\mathscr{B}(\mathscr{H})/\mathscr{K}$, commonly referred to as the *essential spectrum* of T. Let r(T) and $r(\nu(T))$ denote the spectral radii of T and $\nu(T)$, respectively. The *Weyl spectrum* of T, which is $\bigcap_{K\in\mathscr{K}} \sigma(T+K)$, will be denoted by $\sigma_w(T)$. A polynomial is *monic* if its leading coefficient is one. If T is an element in a Banach algebra and $p(T) = 0$ for every polynomial p, then T is *algebraic;* if $\lim_n \|T^n\|^{1/n} = 0$, then T is *quasinilpotent.* For general facts about Hilbert space operators, the reader is referred to [12].

2. LIFTING QUASIALGEBRAIC ELEMENTS OF THE CALKIN ALGEBRA

Corresponding to each element T of a Banach algebra, define the *capacity* of T by

$$\operatorname{cap} T = \lim_n \left\{ \inf_{\substack{p \text{ monic} \\ \deg p = n}} \|p(T)\| \right\}^{1/n}.$$

This limit does exist [14, p. 857]. Define the *capacity* of a compact set $X \subset \mathbb{C}$ to be

$$\operatorname{cap} X = \lim_n \left\{ \inf_{\substack{p \text{ monic} \\ \deg p = n}} \sup\{ |p(z)| : z \in X\} \right\}^{1/n}.$$

As in [14], define the *spectral capacity* of T to be cap $\sigma(T)$. Note that in the definition of cap $\sigma(T)$, the expression $\sup\{|p(z)|: z \in X\}$ can be replaced by $r(p(T))$. Halmos proved that T is quasialgebraic if and only if cap T = cap $\sigma(T) = 0$. In potential theory, the number cap X is known as Tchebycheff's constant, as transfinite diameter, and by other names. Much is known about sets of capacity zero, and this translates immediately to facts about quasialgebraic operators. We shall refer to [23] for this information.

First we state a precise comparison between various spectra associated with $T \in \mathscr{B}(\mathscr{H})$. In addition to $\sigma_w(T)$ and $\sigma(\nu(T))$ specified in the introduction, define the *left essential spectrum* of T to be

$$\sigma_\ell(\nu(T)) = \{\lambda: \nu(T) - \lambda \text{ is not left-invertible in the Calkin algebra}\},$$

and the *Browder spectrum* of T to be $\sigma(T) \setminus \sigma_{00}(T)$, where $\sigma_{00}(T)$ denotes the set of isolated eigenvalues of finite multiplicity. If X is a compact set of complex numbers, a *hole* in X is a bounded component of $\mathbb{C} \setminus X$. Now the boundary of $\sigma(\nu(T))$ is contained in $\sigma_\ell(\nu(T))$, and $\sigma_\ell(\nu(T))$ is compact [10, Theorem 3.1]; clearly, $\sigma_\ell(\nu(T)) \subset \sigma(\nu(T))$. It is an elementary topological fact that this implies $\sigma(\nu(T))$ is obtained from $\sigma_\ell(\nu(T))$ by filling in some holes. This method of argument establishes the following result (see [10, Theorem 2.4] and [19, Theorem 1]).

THEOREM 2.1. *If* $T \in \mathscr{B}(\mathscr{H})$, *then*

$$\sigma_\ell(\nu(T)) \subseteq \sigma(\nu(T)) \subseteq \sigma_w(T) \subseteq \sigma(T) \setminus \sigma_{00}(T),$$

and each of these compact sets is obtained by filling in some holes of the preceding set.

The next result shows that Weyl's theorem holds for every quasialgebraic operator [4].

THEOREM 2.2. *For a quasialgebraic* $T \in \mathscr{B}(\mathscr{H})$,

$$\sigma_\ell(\nu(T)) = \sigma(\nu(T)) = \sigma_w(T) = \sigma(T) \setminus \sigma_{00}(T).$$

Proof. A compact set of capacity zero can contain no continuum [23, p. 56], and hence it must be totally disconnected. In particular, it has no holes. Since cap $\sigma(T) = 0$, the result follows from Theorem 2.1.

Theorem 2.3 and Corollary 2.4 have also been obtained independently by Stirling [22, pp. 51-54].

THEOREM 2.3. *For each* $T \in \mathscr{B}(\mathscr{H})$,

$$\text{cap } \sigma(\nu(T)) = \text{cap } \sigma_w(T) = \text{cap } \sigma(T).$$

Proof. From the definition, we see that capacity is a monotone set function; therefore

$$\text{cap } \sigma(\nu(T)) \le \text{cap } \sigma_w(T) \le \text{cap } \sigma(T) \setminus \sigma_{00}(T) \le \text{cap } \sigma(T).$$

Since $\sup\{|p(z)|: z \in \sigma(\nu(T))\}$ must occur on the boundary of $\sigma(\nu(T))$, filling in holes of $\sigma(\nu(T))$ does not increase capacity. Thus

$$\text{cap } \sigma(\nu(T)) = \text{cap } \sigma_w(T) = \text{cap } \sigma(T) \setminus \sigma_{00}(T).$$

Now, any countable set has capacity zero, and adding a set of capacity zero does not increase capacity [23, pp. 53, 57]. Thus cap $\sigma(T) = \text{cap}\, \sigma(T) \setminus \sigma_{00}(T)$, and the result follows.

Using Halmos's theorem that capacity and spectral capacity of an operator are the same, and his characterization of quasialgebraic operators as those having capacity zero, we immediately obtain the following result.

COROLLARY 2.4. *For* $T \in \mathscr{B}(\mathscr{H})$ *and* $K \in \mathscr{K}$, $\text{cap}\, \nu(T) = \text{cap}\, T = \text{cap}\,(T + K)$. *Thus* T *is quasialgebraic if and only if* $\nu(T)$ *is quasialgebraic and if and only if* $T + K$ *is quasialgebraic for every compact* $K \in \mathscr{B}(\mathscr{H})$.

3. CONSTRUCTION OF AN OPERATOR T + K CORRESPONDING TO A PRESCRIBED SEQUENCE $\{p_n\}$

THEOREM 3.1. *Let* $T \in \mathscr{B}(\mathscr{H})$ *be such that* $\lim_n \|p_n(\nu(T))\|^{1/d(n)} = 0$, *for a sequence* $\{p_n\}$ *of monic polynomials with* $\deg p = d(n)$. *If* $\{p_n\}$ *has a subsequence of bounded degree, then there exists* $K \in \mathscr{K}$ *with* $\lim_n \|p_n(T + K)\|^{1/d(n)} = 0$.

Proof. Halmos shows that the hypothesis implies $\nu(T)$ is in fact algebraic [14, p. 857]. Let p be the monic polynomial of least degree such that $p(\nu(T)) = 0$, and let $\deg p = m$. Then there is a $K \in \mathscr{K}$ with $p(T + K) = 0$ [17, Theorem 2.4].

Arguing as in [14, p. 857], we set $\|q\| = \|q(\nu(T))\|$ to define a norm on the m-dimensional vector space of all polynomials with degree $q < m$. In particular, there is a constant $\alpha > 0$ such that

$$\max |\text{coefficients of } q| < \alpha \|q(\nu(T))\|,$$

whenever degree $q < m$. We can represent each p_n in the form $p_n = s_n p + q_n$ ($\deg q_n < m$). Then

$$\begin{aligned}\lim_n \|p_n(T + K)\|^{1/d(n)} &= \lim_n \|q_n(T + K)\|^{1/d(n)} \\ &\leq \lim_n \{m\alpha \|q_n(\nu(T))\| \cdot \max_{k \leq m} \|(T + K)^k\|\}^{1/d(n)} = 0.\end{aligned}$$

Thus the result is proved.

Stampfli has shown that for each $T \in \mathscr{B}(\mathscr{H})$, there is a $K \in \mathscr{K}$ with $\sigma(T + K) = \sigma_w(T)$ [21]. Using this, and techniques similar to those of [14, Theorem 3], we get a modification of Arveson's conjecture for every $T \in \mathscr{B}(\mathscr{H})$.

THEOREM 3.2. *Suppose that* $T \in \mathscr{B}(\mathscr{H})$ *and that* $\lim_n \|p_n(\nu(T))\|^{1/d(n)} = 0$, *for a sequence* $\{p_n\}$ *of monic polynomials* ($\deg p_n = d(n)$). *Then there exist a* $K \in \mathscr{K}$ *and a sequence* $\{s(n)\}$ *of positive integers such that*

$$\lim_n \|(p_n(T + K))^{s(n)}\|^{1/d(n)s(n)} = 0.$$

Proof. Applying Stampfli's result, we have an element $K \in \mathscr{K}$ with

$$\sigma(T + K) = \sigma_w(T) = \sigma(\nu(T))$$

(the latter inequality follows from Theorem 2.2 and Corollary 2.4, since $\nu(T)$ is quasialgebraic). Thus

$$r(p_n(T+K)) = r(p_n(\nu(T))) \leq \|p_n(\nu(T))\|,$$

so that $\lim_n r(p_n(T+K))^{1/d(n)} = 0$. For fixed n,

$$\|(p_n(T+K))^k\|^{1/k} \to r(p_n(T+K)), \quad \text{as } k \to \infty.$$

Given $\varepsilon_n > 0$, we choose $s(n)$ with

$$\|(p_n(T+K))^{s(n)}\|^{1/s(n)} < r(p_n(T+K)) + \varepsilon_n.$$

Now simply choose ε_n small enough so that

$$\|(p_n(T+K))^{s(n)}\|^{1/s(n)d(n)} < (r(p_n(T+K)) + \varepsilon_n)^{1/d(n)} \to 0$$

as $n \to \infty$. The theorem is proved.

Note that $\lim_n r(p_n(T))^{1/d(n)} = 0$ does not imply $\lim_n \|p_n(T)\|^{1/d(n)} = 0$. To see this, consider a quasinilpotent T with $p_n(T) = T^n + T$.

4. COMPACT PERTURBATIONS AND THE REALIZATION OF THE ESSENTIAL NORM OF p(T)

An operator $T \in \mathscr{B}(\mathscr{H})$ is *subnormal* if T is unitarily equivalent to the restriction of a normal operator (on some Hilbert space) to a closed invariant subspace; T is *hyponormal* if $T^*T \geq TT^*$; and T is *seminormal* if either T or T^* is hyponormal. Each condition is more general than the preceding.

THEOREM 4.1 (N. Salinas [19]). *Let $T \in \mathscr{B}(\mathscr{H})$ be hyponormal. Then there is a compact normal K that commutes with T and such that $\sigma(T+K)$ is the Weyl spectrum of T. In addition, T + K is hyponormal, subnormal, normal, or positive whenever T is.*

Proof. According to Coburn's generalization of Weyl's theorem [7], $\sigma(T) = \sigma_w(T) \cup \sigma_{00}(T)$. Now, $\sigma_{00}(T)$ is a sequence $\{\lambda_n\}$, which converges to the compact set $\sigma_w(T)$. The finite-dimensional eigenspace $\mathscr{E}_n$ associated with λ_n reduces T. For each λ_n, choose $\mu_n \in \sigma_w(T)$ so that $|\lambda_n - \mu_n| \to 0$ as $n \to \infty$; set $K = \sum_n (\mu_n - \lambda_n) E_n$, where E_n is the projection with range $\mathscr{E}_n$. Then K is clearly compact and commutes with T, and $\sigma(T+K) = \sigma_w(T)$.

We conjecture that every $T \in \mathscr{B}(\mathscr{H})$ and every polynomial p, it is possible to find $K \in \mathscr{K}$ such that $\|p(T+K)\|$ is the essential norm of p(T). If $T \in \mathscr{B}(\mathscr{H})$ and $\|p(T)\|$ is determined by $\sigma(T)$, a stronger statement is true:

THEOREM 4.2. *If $T \in \mathscr{B}(\mathscr{H})$ is subnormal, then there exists a compact normal operator K that commutes with T and such that $\|p(T+K)\| = \|p(\nu(T))\|$ for all polynomials p. If T is merely seminormal, then there exists such a K satisfying the condition $\|(T+K)^n\| = \|\nu(T)\|$ for $n = 1, 2, \cdots$.*

Proof. Choose K as in Theorem 4.1. Stampfli has shown that the norm of each hyponormal operator is equal to its spectral radius [20]. If T + K is subnormal, then for each polynomial p, p(T + K) is subnormal. Thus

$$\|p(T+K)\| = r(p(T+K)) = \sup\{|p(\lambda)| : \lambda \in \sigma_w(T)\}$$
$$= \sup\{|p(\lambda)| : \lambda \in \sigma(\nu(T))\} = r(p(\nu(T))) \leq \|p(\nu(T))\|,$$

since $\sigma_w(T)$ is obtained by filling in some holes of $\sigma(\nu(T))$. It is obvious that $\|p(T+K)\| \geq \|p(\nu(T))\|$.

If T is seminormal, then by taking adjoints we see that it suffices to consider the case where T is hyponormal. Then $T+K$ is hyponormal, so that $\|T+K\| = r(T+K)$. Although $(T+K)^n$ may not be hyponormal, it is easily seen that $\|(T+K)^n\| = r((T+K)^n)$. Indeed, choose $\lambda \in \sigma(T+K)$, with $|\lambda| = \|T+K\|$. Then $\lambda^n \in \sigma((T+K)^n)$, so that

$$|\lambda|^n \geq \|(T+K)^n\| \geq r((T+K)^n) \geq |\lambda^n|.$$

Thus the argument above for subnormal T applies to polynomials $p(T) = T^n$ $(n = 1, 2, \cdots)$, for hyponormal T.

COROLLARY 4.3. *If some compact perturbation of* $T \in \mathcal{B}(\mathcal{H})$ *is subnormal (seminormal), then there exists a compact* K *such that* $\|p(T+K)\|$ *is the essential norm of* $p(T)$ *for all polynomials* p (*for all* $p(z) = z^n$, $n = 1, 2, \cdots$).

An operator $T \in \mathcal{B}(\mathcal{H})$ is called *essentially normal* if $T^*T - TT^*$ is compact (so that $\nu(T)$ is normal) [5]. In general, T may not be a compact perturbation of a normal operator.

COROLLARY 4.4. *If* $T \in \mathcal{B}(\mathcal{H})$ *is essentially normal, then there exists a compact* K *such that* $\|p(T+K)\|$ *is the essential norm of* $p(T)$ *for all polynomials* p.

Proof. L. Brown, R. G. Douglas, and P. A. Fillmore have shown that if T is an essentially normal operator on $\mathcal{H}$, then there exist subnormal operators S_1 and S_2 such that T is unitarily equivalent to some compact perturbation of $S_1 \oplus S_2^*$ [5, p. 123]. The result follows from this and the preceding corollary.

Let $\mathcal{P}(\nu(T))$ denote the closed subalgebra of the Calkin algebra generated by $\nu(T)$. Corollary 4.4 implies that if T is essentially normal, then there exists an isometric isomorphism σ from $\mathcal{P}(\nu(T))$ into $\mathcal{B}(\mathcal{H})$ such that the composition $\nu\sigma$ is the identity on $\mathcal{P}(\nu(T))$. This should be contrasted with the results of Brown, Douglas, and Fillmore, who show that an analogous *-isomorphism into $\mathcal{B}(\mathcal{H})$ from the C^*-subalgebra generated by $\nu(T)$ in the Calkin algebra exists only when T is a compact perturbation of a normal operator [5, p. 85].

We can now answer Arveson's question for any operator similar to a normal operator.

COROLLARY 4.5. *If* $T \in \mathcal{B}(\mathcal{H})$ *is a spectral operator of scalar type and there exists a sequence* $\{p_n\}$ *of monic polynomials with* $\deg p_n = d(n)$ *and with* $\lim_n \|p_n(\nu(T))\|^{1/d(n)} = 0$, *then there exists a* $K \in \mathcal{K}$ *such that*

$$\lim_n \|p_n(T+K)\|^{1/d(n)} = 0.$$

Proof. By definition, T is similar to a normal N, say $T = SNS^{-1}$. Choose K with $\|p(N+K)\| = \|p(\nu(N))\|$ for all p. Then $T + SKS^{-1}$ satisfies the conclusion.

The reader may wonder why the preceding corollary was not stated for all T similar to a subnormal operator. In answer, we remark that a seminormal quasialgebraic operator is necessarily normal. If cap $\sigma(T) = 0$, then $\sigma(T)$ has Lebesgue measure zero [23, p. 58], and C. R. Putnam has proved that a seminormal operator whose spectrum has measure zero is normal [18]. On the other hand, non-diagonalizable normal quasialgebraic operators do exist: for example, multiplication by $f(x) = x$ on $L^2(C, \mu)$, where C is a Cantor set of finite logarithmic measure, and μ is induced by a homeomorphism to a Cantor set of nonzero Lebesgue measure [23, p. 66].

The next result involves no assumption related to normality. An operator $U \in \mathcal{B}(\mathcal{H})$ is a *partial isometry* if U^*U is a projection. We adopt the standard notation $|T| = (T^*T)^{1/2}$.

THEOREM 4.6. *If* $U \in \mathcal{B}(\mathcal{H})$ *is a partial isometry, then there exists a compact* K *such that* $\|(U + K)^2\|$ *is equal to the essential norm* $\|\nu(U)^2\|$.

Proof. Let $U^*U = E$ and $UU^* = F$, where E and F are projections of infinite rank. Then $U^2 = UEFU$, and $\|\nu(EF)\| = \|\nu(U^2)\|$.

Let $\mathcal{E}$ and $\mathcal{F}$ denote the ranges of E and F, respectively. We can decompose $\mathcal{H}$ into an orthogonal sum of six closed subspaces as follows:

$$\mathcal{H} = \mathcal{F} \oplus \mathcal{F}^\perp = (\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3) \oplus (\mathcal{F}_3' \oplus \mathcal{F}_1' \oplus \mathcal{F}_2'),$$

where

$$\mathcal{F}_1 = \mathcal{F} \cap \mathcal{E}, \qquad \mathcal{F}_1' = \mathcal{F}^\perp \cap \mathcal{E}^\perp,$$

$$\mathcal{F}_2 = \mathcal{F} \cap \mathcal{E}^\perp, \qquad \mathcal{F}_2' = \mathcal{F}^\perp \cap \mathcal{E},$$

$$\mathcal{F}_3 = \mathcal{F} \ominus \mathcal{F}_1 \ominus \mathcal{F}_2, \qquad \mathcal{F}_3' = \mathcal{F}^\perp \ominus \mathcal{F}_1' \ominus \mathcal{F}_2'.$$

Now, if we define $\mathcal{E}_3 = \mathcal{E} - \mathcal{F}_1 \ominus \mathcal{F}_2'$, then the subspaces $\mathcal{E}_3$ and $\mathcal{F}_3$ are appropriately related within $\mathcal{F}_3 \oplus \mathcal{F}_3'$, so that we can write E as a 6-by-6 operator matrix

$$E = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P & (P - P^2)^{1/2} & 0 & 0 \\ 0 & 0 & (P - P^2)^{1/2} & I - P & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

relative to our decomposition for $\mathcal{H}$; here $P \in \mathcal{B}(\mathcal{F}_3)$ and $0 \le P \le I$ ($\mathcal{F}_3$ is equivalent to $\mathcal{F}_3'$, and we are suppressing the equivalence operator) [8, p. 180]. Now observe that

$$\mathrm{FEF} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If dim $\mathscr{F}_1$ is infinite, then

$$\|U^2\| = 1 = \|\nu(\mathrm{FEF})\| \leq \|\nu(\mathrm{EF})\| = \|\nu(U^2)\|,$$

and the proof is complete. Hence we may assume that $\mathscr{F}_1$ is finite-dimensional.

Since P is positive, it follows from Theorems 4.1 and 4.2 that there exists a compact operator L such that $P + L \geq 0$ and such that

$$\sigma(P + L) = \sigma_w(P) = \sigma(\nu(P)), \qquad \|P + L\| = \|\nu(P)\|.$$

Let $Q = P + L$, so that $0 \leq Q \leq I$. We can then define a projection

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q & (Q - Q^2)^{1/2} & 0 & 0 \\ 0 & 0 & (Q - Q^2)^{1/2} & I - Q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

(direct computations show that $G^2 = G$ and $G^* = G$). Furthermore, $G - E$ is compact: for $\mathscr{F}_1$ is finite-dimensional, the operator L is compact, and in addition

$$\nu((Q - Q^2)^{1/2} - (P - P^2)^{1/2}) = (\nu(P + L - (P + L)^2))^{1/2} - (\nu(P - P^2))^{1/2} = 0,$$

since ν maps the unique positive square root of a positive operator to the unique positive square root of its image in the Calkin algebra. Set $K = U(G - E)$, so that $U + K = UG$.

Observe that since norm equals spectral radius, it follows from the spectral-mapping theorem that each positive A in $\mathscr{B}(\mathscr{H})$ satisfies the conditions $\|A^{1/2}\| = \|A\|^{1/2}$ and $\|\nu(A)^{1/2}\| = \|\nu(A)\|^{1/2}$. This allows us to compute:

$$\begin{aligned} \|(U + K)^2\| &= \|(UG)^2\| = \|\mathrm{UEGFUG}\| \leq \|\mathrm{GF}\| = \|\,|\mathrm{GF}|\,\| = \|(\mathrm{FGF})^{1/2}\| = \|Q^{1/2}\| \\ &= \|Q\|^{1/2} = \|\nu(P)\|^{1/2} = \|\nu(\mathrm{FEF})\|^{1/2} = \|\,|\nu(\mathrm{EF})|^2\|^{1/2} \\ &= \|\,|\nu(\mathrm{EF})|\,\| = \|\nu(\mathrm{EF})\| = \|\nu(U^2)\|. \end{aligned}$$

Since the inequality $\|(U+K)^2\| \geq \|\nu(U^2)\|$ is obvious, the theorem is proved.

COROLLARY 4.7. *If some compact perturbation of* $T \in \mathscr{B}(\mathscr{H})$ *is a partial isometry, then there exists a compact* K *with* $\|(T+K)^2\| = \|\nu(T)^2\|$.

Note that the proof of Theorem 4.6 differs from the proofs of all our earlier results insomuch as it does not refer to $\sigma(U)$. However, a spectral assumption is implicitly involved; the argument requires that $\sigma(U^*U)$ be equal to $\{0, 1\}$. One might next attempt to extend this result to operators T such that $\sigma(T^*T)$ is finite. In the following theorem we again impose a hypothesis related to normality. Operators with this property have recently been studied in [6] and [16].

THEOREM 4.9. *For* $T \in \mathscr{B}(\mathscr{H})$, *assume that* T^*T *commutes with* TT^*. *Then there is a compact* K *with* $\|(T+K)^2\| = \|\nu(T)^2\|$.

Proof. Let $U|T|$ be the polar decomposition for T. Then $T^* = U^*|T^*|$ and $|T^*| = U|T|U^*$. Since $|T|$ is a strong limit of polynomials in T^*T, it commutes with TT^*. Similarly, $|T^*|$ commutes with $|T|$. Thus $A = |T||T^*|$ is a positive operator. Set $\mu = \|\nu(T)^2\|$. Then

$$\|\nu(A)\| = \|\nu(U)\nu(|T||T^*|)\nu(U^*)\| = \|\nu(T^2)\| = \mu .$$

We can assume that $\mu < \|A\|$ $(= \|T^2\|)$, for otherwise the proof is trivial. We now construct S so that $S - T$ is compact and $\|S^2\| = \mu$.

First we isolate those spectral subspaces for A on which $\|A\|$ is large. Let $E(\lambda)$ be the spectral resolution of A. Let $\{\alpha_n\}$ be a sequence of positive numbers with $\alpha_0 = 0$, $\alpha_1 = \|A\| - \mu$, $\alpha_n \searrow 0$. Set

$$E_0 = E([0, \mu]) \quad \text{and} \quad E_n = E((\mu + \alpha_{n+1}, \mu + \alpha_n]) \quad (n \geq 1).$$

Then we claim that E_n is a finite-rank projection for $n \geq 1$: for all $x \in E_n\mathscr{H}$ $(n \geq 1)$,

$$(\mu + \alpha_{n+1})\|x\| \leq \|Ax\| \leq (\mu + \alpha_n)\|x\| .$$

Now, if there were an infinite orthonormal sequence $(x_k) \subset E_n\mathscr{H}$, we would have the inequality $\|Ax_k\| \geq \mu + \alpha_{n+1}$ for each k, and hence $\|\nu(A)\| \geq \mu + \alpha_{n+1}$, a contradiction.

We now define S by altering $|T|$ with a convenient scalar factor on each E_n. Let B be the positive operator

$$B = \sum_{k=0}^{\infty} \frac{\mu}{\mu + \alpha_k} |T| E_k$$

(note that $|T|$ commutes with A, so that E_k commutes with $|T|$ for all k). Set $S = UB$: this is the polar decomposition for S; in particular, $|S^*| = UBU^*$. Also, $T - S = U(|T| - B)$, where

$$|T| - B = \sum_{k=1}^{\infty} \frac{\alpha_k}{\mu + \alpha_k} |T| E_k$$

is compact; therefore $T - S$ is compact.

To show that $\|S^2\| = \mu$, it suffices to show that $\| \, |S^*| \, |S| \, \| \leq \mu$, since clearly $\|S^2\| \geq \mu$, and since

$$\|S^2\| = \| \, |S| \, |S^*| \, \| = \| \, |S^*| \, |S| \, \| .$$

First, observe that

$$\begin{aligned} |S^*| &= UBU^* = \sum_{k=0}^{\infty} \frac{\mu}{\mu + \alpha_k} U \, |T| \, E_k U^* \\ &= \sum_{k=0}^{\infty} \frac{\mu}{\mu + \alpha_k} (U E_k U^*) (U \, |T| \, U^*) (U E_k U^*) \\ &= \sum_{k=0}^{\infty} \frac{\mu}{\mu + \alpha_k} Q_k \, |T^*| \, Q_k , \end{aligned}$$

where the $Q_k = U E_k U^*$ are pairwise orthogonal projections; since

$$U^* U E_0 = U^* U E((0, \mu]) ,$$

we see that $U^* U E_k = E_k$ $(k \geq 1)$. Also,

$$\sum_{k=0}^{\infty} Q_k \, |T^*| \, Q_k = |T^*| ,$$

since

$$|T^*| \, Q_k = U \, |T| \, E_k U^* = U E_k \, |T| \, U^* = Q_k \, |T^*| .$$

Hence for each unit vector $x \in \mathscr{H}$,

$$\begin{aligned} \| \, |S^*| \, |S| \, x\|^2 &= \Big\| \sum_{k,j} \left(\frac{\mu}{\mu + \alpha_j}\right) \left(\frac{\mu}{\mu + \alpha_k}\right) Q_j \, |T^*| \, |T| \, E_k x \Big\|^2 \\ &= \sum_j \left(\frac{\mu}{\mu + \alpha_j}\right)^2 \Big\| \sum_k \frac{\mu}{\mu + \alpha_k} Q_j \, |T^*| \, |T| \, E_k x \Big\|^2 \\ &\leq \sum_j \Big\| \sum_k \frac{\mu}{\mu + \alpha_k} Q_j \, |T^*| \, |T| \, E_k x \Big\|^2 = \Big\| \sum_k \frac{\mu}{\mu + \alpha_k} |T^*| \, |T| \, E_k x \Big\|^2 \\ &\leq \sup_k \left(\frac{\mu}{\mu + \alpha_k}\right)^2 \|AE_k\|^2 \leq \sup_k \left(\frac{\mu}{\mu + \alpha_k}\right)^2 (\mu + \alpha_k)^2 = \mu^2 . \end{aligned}$$

The proof is complete.

COROLLARY 4.10. *If* S^*S *commutes with* SS^* *for some compact perturbation* S *of an operator* T, *then there exists a compact* K *with* $\|(T + K)^2\| = \|\nu(T)^2\|$.

5. THE ESSENTIAL NORM OF $p(T)$ AS $\inf \|p(T+K)\|$

An operator $T \in \mathscr{B}(\mathscr{H})$ is *quasitriangular* if there exists an increasing sequence $\{E_n\}$ of finite-rank projections of supremum I such that

$$\lim_n \|E_n^{\perp} T E_n\| = 0,$$

where $E_n^{\perp}$ denotes $I - E_n$. This property was first considered in a paper of W. Arveson and J. Feldman [3] in connection with the existence of invariant subspaces. It was later studied systematically by Halmos [13], who indicated a relationship between quasitriangular and quasialgebraic operators in [14]. Quasitriangular operators have recently been characterized [1], [9] as those operators for which $T - \lambda$ is not a semi-Fredholm operator of negative index for any complex λ; thus the class of T for which either T or T^* is quasitriangular is very large. To prove Theorem 5.2 for this class, we use the following lemma, proved in [2, p. 292].

LEMMA 5.1. *Let* $T \in \mathscr{B}(\mathscr{H})$, *and let* $\{E_n\}$ *be an increasing sequence of finite-rank projections of supremum* I. *Then* $\lim_n \|E_n^{\perp} T E_n^{\perp}\|$ *is the essential norm of* T.

The authors are indebted to the referee for suggesting the following short proof for Theorem 5.2.

THEOREM 5.2. *If* $T \in \mathscr{B}(\mathscr{H})$ *and either* T *or* T^* *is quasitriangular, then*

$$\inf_{K \in \mathscr{K}} \|p(T+K)\| = \|\nu(p(T))\|,$$

for each polynomial p.

Proof. It suffices to assume T is quasitriangular. Choose an increasing $\{E_n\}$ of finite-rank projections of supremum I, with $\lim_n \|E_n^{\perp} T E_n\| = 0$. Since $p(E_n^{\perp} T E_n^{\perp}) = p(T + K_n)$ for some $K_n \in \mathscr{K}$, and in view of the lemma, it suffices to show that

$$\lim_n \|p(E_n^{\perp} T E_n^{\perp})\| = \lim_n \|E_n^{\perp} p(T) E_n^{\perp}\|.$$

It is convenient to consider first an arbitrary polynomial p with constant term zero. Now write $p(T) = p[T(E_n + E_n^{\perp})]$, and expand

$$E_n^{\perp} p(T(E_n + E_n^{\perp})] E_n^{\perp}.$$

Observe that we obtain a sum of terms each of which contains E_n as a factor, plus a summand

$$E_n^{\perp} p(T E_n^{\perp}) E_n^{\perp} = p(E_n^{\perp} T E_n^{\perp})$$

(since p has zero constant term). Now, each of the first-mentioned terms is a product of the form

$$\alpha E_n^{\perp} T X_1 T X_2 \cdots T X_j E_n^{\perp},$$

for some $j \leq \deg p$, where X_k is either E_n or $E_n^{\perp}$, and $\alpha \in \mathbb{C}$. Let k be the smallest index with $X_k = E_n$. Then $X_{k-1} T X_k = E_n^{\perp} T E_n$, so that $\|X_{k-1} T X_k\| \underset{n}{\rightarrow} 0$. Thus the norm of each of these terms converges to zero as $n \rightarrow \infty$, and we are left with

$$\lim \|E_n^{\perp} p(T) E_n^{\perp}\| = \lim_n \|p(E_n^{\perp} T E_n^{\perp})\|,$$

as desired.

Now assume that p is a polynomial with nonzero constant term; that is, let $p(z) = \lambda_0 (z - \lambda_1) \cdots (z - \lambda_m)$, where $\lambda_i \neq 0$, for each i. Let q be the polynomial $q(z) = p(z + \lambda_1)$. Then $q(T - \lambda_1) = p(T)$, and clearly $T - \lambda_1$ is quasitriangular whenever T is quasitriangular.

Since q has no constant term, we can apply the result above to deduce that

$$\inf_{K \in \mathcal{K}} \|p(T + K)\| = \inf_{K \in \mathcal{K}} \|q(T - \lambda_1 + K)\| = \|\nu(q(T - \lambda_1)\| = \|\nu(p(T))\|,$$

and the proof is complete.

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