ON APPROXIMATION OF NORMAL OPERATORS BY WEIGHTED SHIFTS

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An intriguing problem in the theory of perturbations of operators on a separable Hilbert space is the following: If an operator has a small self-commutator and there are no points in the spectrum of the operator of nonzero Fredholm index, is that operator necessarily close in norm to a normal operator? That is, is there a norm analogue to the result of L. G. Brown, R. G. Douglas, and P. A. Fillmore on operators with compact self-commutators? If the self-commutator is both small and compact, must the operator differ from a normal operator by a small compact operator?

In this note we show that if a weighted bilateral shift on a separable Hilbert space has a small self-commutator and has no point in its spectrum on which the shift has a Fredholm index other than zero, then that shift is close in norm to a normal operator. If, in addition, the self-commutator is compact, then the difference between the shift and the normal operator is compact and of small norm.

To be precise, we show the following (see Theorem 2).

Let $\{\phi_i\}$ $(-\infty < i < \infty)$ be an orthonormal basis for H. Let S be the weighted bilateral right shift defined by $S(\phi_i) = a_i \phi_{i+1}$. Let $\|S\| = 1$. Now assume that $\{a_i\}$ satisfies the following two conditions:

Condition 1 (no nonzero index).

$$\liminf_{i \to -\infty} |a_i| \leq \limsup_{i \to \infty} |a_i|, \qquad \liminf_{i \to \infty} |a_i| \leq \limsup_{i \to -\infty} |a_i|.$$

Condition 2 (small self-commutator).

For some $\varepsilon < 1/256$, the inequality

$$\big\| a_i| - |a_{i+1} \big\| < \varepsilon,$$

holds for each index i.

Then there exists a normal operator N such that $\|N - S\| < 100\sqrt{\varepsilon}$.

The estimate in the conclusion is not sharp. We are interested primarily in the fact that if Condition 1 holds, then a shift with a small self-commutator is close to a normal operator. In this form, Theorem 2 is best possible, except for the lack of sharpness in the estimate.

Condition 1 is clearly necessary. For if, say,

$$\liminf_{i \to -\infty} |a_i| = \limsup_{i \to \infty} |a_i| + \eta \qquad \text{for some } \eta > 0,$$

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then routine calculations show that $S - \lambda$ is surjective but not one-to-one when $\lambda = \limsup_{i \to \infty} |a_i| + \eta/2$. Now, considering the right inverse of such an operator $S - \lambda$, we see that if T is any operator such that $\|T\| < \eta/2$, then $S + T - \lambda$ is still surjective but not one-to-one. Therefore $S + T$ is not normal; in other words, if N is normal, then $\|S - N\| \geq \eta/2$.

Similarly, Condition 2 is necessary. Indeed, in general, the self-commutator $T^*T - TT^*$ of T is clearly a norm-continuous function of T that is 0 if and only if T is normal. But for our shift we see that $\|S^*S - SS^*\| = \sup_i \| |a_{i+1}|^2 - |a_i|^2 \|$. Easier still, if $\| |a_{i+1}| - |a_i| \| = \eta$, then $\| |S\phi_{i+1}| - |S^*\phi_{i+1}| \| = \eta$; therefore if N is normal, $\|S - N\| \geq \eta/2$.

We note that our hypotheses say nothing about the essential spectrum of S. This spectrum, which we recall to be the set of complex λ such that for no T can both $T(S - \lambda) - I$ and $(S - \lambda)T - I$ be compact, although well known to be invariant under compact perturbations, is extremely unstable under perturbations of small norm. The essential spectrum of a weighted shift is clearly radially symmetric. It is easy to see that if we add the hypothesis that $S^*S - SS^*$ is compact, then Condition 1 implies that the essential spectrum of S is a single annulus. In this case we see in Theorem 3 that our hypotheses yield compactness of $N - S$ as well as the inequality $\|N - S\| < 100\sqrt{\varepsilon}$. (That there exists a normal N such that $N - S$ is compact is also a special case of the remarkable Theorem 11.1 of Brown, Douglas, and Fillmore [1]. However, that theorem yields no estimate on the norm of $N - S$.) In this case, of course, the essential spectrum of N is that of S.

All our results hinge on our Theorem 1, the special case of Theorems 2 and 3 where the shift is finite-dimensional. The construction of the desired normal in this case is the heart of this note, and we shall describe it in detail. Theorems 2 and 3 will follow easily, once this construction is clear, and we shall merely indicate the proofs of these two theorems.

THEOREM 1. *Let* S *be the weighted shift on* E^n *defined by the formulas*

$$S(\phi_i) = a_i \phi_{i+1} \qquad \text{for } i = 1, \cdots, n-1,$$

$$S\phi_n = 0.$$

Suppose that for some even integer $M \geq 16$, *we have the inequalities* $|a_1| < 1/M^2$, $|a_{n-1}| < 1/M^2$, *and* $\| |a_i| - |a_{i+1}| \| < 1/M^2$ *for* $i = 1, \cdots, n-2$. *Finally, suppose* $\|S\| = 1$.

Then there exists a normal operator N *on* E^n *such that* $\|N - S\| < 100/M$.

Proof. We may assume, by making a unitary transformation, that $a_i \geq 0$. At the expense of a perturbation of norm at most $1/M$, we may assume that the weights occur in constant blocks of various lengths, each block having length at least M; we may further assume that the first and last blocks are of weight $1/M$ and that any two adjacent blocks have weights differing by exactly $1/M$. It is this weighted shift with weights occurring in constant blocks that we shall consider. We shall denote this shift by T.

We introduced the condition that M is a large even integer for technical convenience in constructing blocks of even-integer length; the reader will not lose the import of the theorem if he assumes that M is so large that one element more or less in a block of length M is important. Similarly, the value of the constant 100 is unimportant; what matters is the existence of a constant.

Now, if all the blocks have weight $1/M$, we replace our shift by 0 and the conclusion follows.

If the blocks are not all of weight $1/M$, then there exist one or more blocks of largest weight b, say, flanked by blocks of weight $b - 1/M$. Choose one block of weight b and call it the central block.

Now we perturb our operator T on the M-dimensional space spanned by the $M/2$ basis vectors of the central block that are closest to its flanking blocks; we keep this perturbation of norm less than $80/M$, so as to change T on these three blocks into the direct sum of a normal operator with all eigenvalues of modulus b and a shift on one long block of weight $b - 1/M$. That is, T is not only unchanged off the central block, but it is also unchanged on all elements of the central block that are further than $M/2$ from its flanking blocks.

We accomplish this perturbation by blending a shift-like section of a scalar multiple of a unitary operator with the shift of constant weight; thus we obtain our shift of nonconstant weight. The blending takes place on a set of at most M basis vectors that appear as the first $M/2$ and the last $M/2$ vectors of the central block of the original shift.

We now give the details of this blending operation.

We are confronted with a section of T which, appropriately reindexed, is of the form

$$\xrightarrow{a}\phi_{-1}\xrightarrow{a}\overbrace{\underbrace{\phi_0\xrightarrow{b}\phi_1\xrightarrow{b}\cdots\xrightarrow{b}\phi_k}_{\text{block to be perturbed}}\xrightarrow{b}\phi_{k+1}\xrightarrow{b}\cdots\xrightarrow{b}\underbrace{\phi_{r-k}\xrightarrow{b}\phi_{r-k+1}\xrightarrow{b}\cdots\xrightarrow{b}\phi_r}_{\text{block to be perturbed}}}^{\text{center block}}\xrightarrow{b}\phi_{r+1}\xrightarrow{a}$$

Since our block is of length at least M, we may let $k = M/2 - 1$.

We now let $\alpha_j = \cos j\pi/2k$ and $\beta_j = \sin j\pi/2k$, and we define

$$\xi_j = \alpha_j \phi_j + \beta_j \phi_{r-k+j} \qquad \text{for } 0 \leq j \leq k.$$

We then define

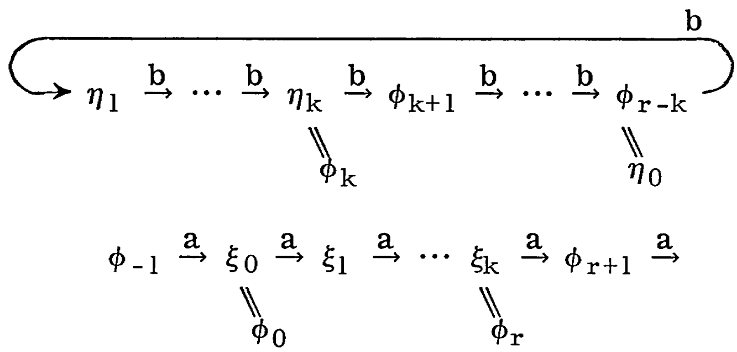
$$\eta_j = (-\beta_j \phi_j + \alpha_j \phi_{r-k+j}) e^{j\pi i/k} \qquad \text{for } 0 \leq j \leq k.$$

The idea is that $\{\xi_j\}$ slowly tapers from ϕ_0 to ϕ_r, and that $\{\eta_j\}$ tapers from ϕ_{r-k} to ϕ_k. Moreover, $\xi_0, \xi_1, \cdots, \xi_k, \eta_0, \eta_1, \cdots, \eta_k$ is a basis for the span of $\phi_0, \phi_1, \cdots, \phi_k, \phi_{r-k}, \phi_{r-k+1}, \cdots, \phi_r$, because the two elements ξ_j and η_j span the same space as ϕ_j and ϕ_{r-k+j}.

We now define our new operator T' by the equations

$$\begin{aligned}
T'(\xi_j) &= a\,\xi_{j+1} && \text{for } 0 \leq j \leq k-1, \\
T'(\xi_k) &= a\,\phi_{r+1}, \\
T'(\eta_j) &= b\,\eta_{j+1} && \text{for } 0 \leq j \leq k-1, \\
T'(\eta_k) &= T(\phi_k) = b\,\phi_{k+1}, \\
T' &= T \text{ elsewhere.}
\end{aligned}$$

We see that T' is of the desired form, namely the direct sum of a scalar multiple of a unitary operator and a weighted shift with smaller weights.



We reiterate that our perturbation is made only on the initial and terminal sections $\phi_0, \cdots, \phi_k$ and $\phi_{r-k}, \cdots, \phi_r$ of the block. We note that ϕ_{r+1}, the initial vector of the succeeding block, is in the range of the perturbation, but that this causes no difficulty, because ϕ_{r+1} is not in the range or domain of later perturbations.

Finally, we see that

$$\|T - T'\| \leq \max_{0 \leq i \leq k} |T(\eta_i) - T'(\eta_i)| + |T(\xi_i) - T'(\xi_i)| .$$

But $T(\eta_j) = (-b\beta_j \phi_{j+1} + b\,\alpha_j \phi_{r-k+j+1})e^{j\pi i/k}$,

$$T'(\eta_j) = b(\eta_{j+1}) = (-b\beta_{j+1}\phi_{j+1} + b\,\alpha_{j+1}\phi_{r-k+j+1})e^{(j+1)\pi i/k}.$$

Recalling that $k = M/2 - 1$, we now see by a crude estimate that

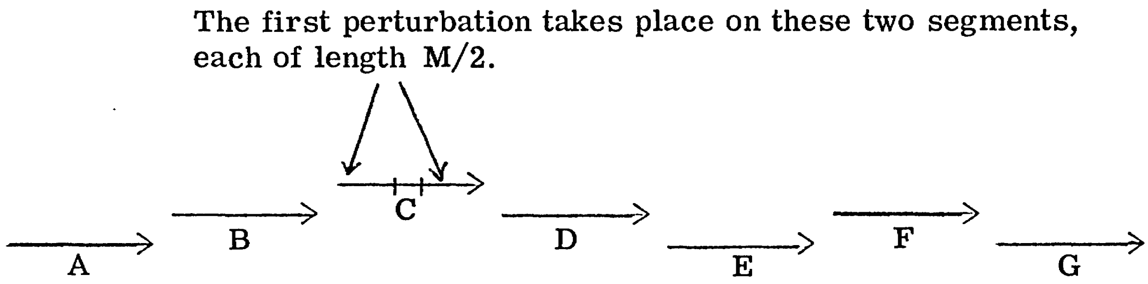
$$\|T(\eta_j) - T'(\eta_j)\| < \frac{4\pi}{k} + \frac{2\pi^2}{k^2} < \frac{40}{M}.$$

Recalling that $b - a = 1/M$, we get a similar estimate for ξ_j. Therefore we see that $\|T - T'\| < 80/M$. This completes our discussion of the perturbation required to lower one central block and produce a normal direct summand.

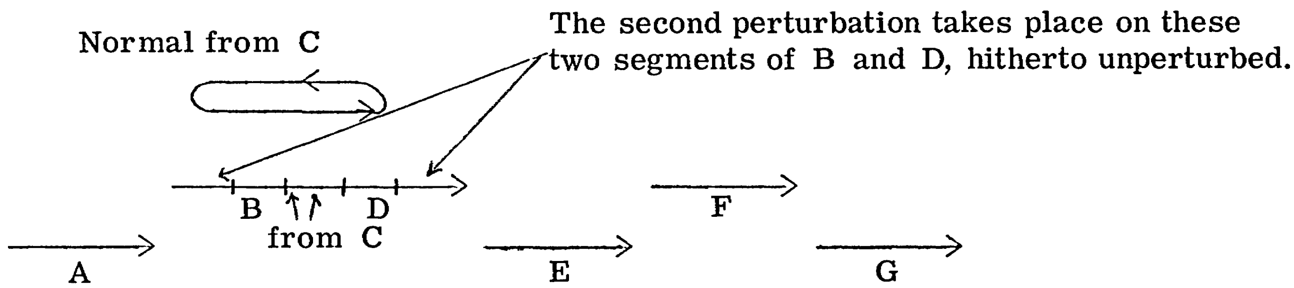
We have just decomposed T into the direct sum of a normal operator and a weighted shift. We now consider this new shift T', and we observe that it has one less block at weight b than T. We again choose a block of largest weight, and we perturb T, so as to reduce the weight of this block and produce another normal direct summand. We observe that the perturbation required at this step is necessarily on vectors unperturbed at the previous stage, because each perturbed block becomes a part of a larger central block with M unperturbed vectors at each end. Since the only vectors perturbed are those within M/2 of a flanking block, we never again perturb the once perturbed block. Similarly, the first vector in a flanking block immediately succeeding a perturbed central block never again appears either in the domain or the range of another perturbation.

We continue in this manner until we are left with a shift with constant weight 1/M, at which point we replace this shift by 0, and we are then left with a normal operator, at the expense of a perturbation of norm at most $\frac{1}{M} + \frac{80}{M} + \frac{1}{M} < 100/M$. This completes the proof. The following drawing may prove helpful.

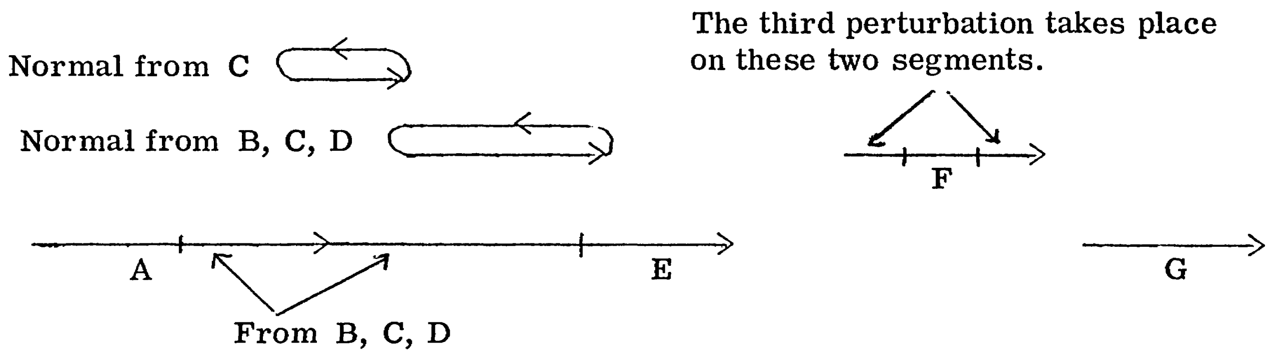
Decomposition of a weighted shift into the direct sum of
a normal operator and a shift.



Section of the weighted shift T *before the first perturbation.*



Section of T *after the first perturbation.*



Section of T *after the second perturbation.*

THEOREM 2. *Let* H *be a separable Hilbert space with orthonormal basis* $\{\phi_i\}$ $(-\infty < i < \infty)$. *Let* $\{a_i\}$ *be a set of complex numbers, and let* S *be the weighted bilateral right shift defined by*

$$S(\phi_i) = a_i \phi_{i+1} .$$

Let $\sup |a_i| = 1$, *and let* $\{a_i\}$ *satisfy the following two conditions:*

Condition 1.

$$\liminf_{i\to-\infty} |a_i| \le \limsup_{i\to\infty} |a_i|, \qquad \liminf_{i\to\infty} |a_i| \le \limsup_{i\to-\infty} |a_i|.$$

Condition 2. *For some* $\varepsilon < 1/256$, *we have the inequality* $\sup \| |a_i| - |a_{i+1}| \| < \varepsilon$.
Then there exists a normal operator N *such that*

$$\|N - S\| < 100\sqrt{\varepsilon}.$$

Proof. Choose a_∞ to be a number such that

$$\liminf_{i\to-\infty} |a_i| \le |a_\infty| \le \limsup_{i\to-\infty} |a_i| \quad \text{and} \quad \liminf_{i\to+\infty} |a_i| \le |a_\infty| \le \limsup_{i\to+\infty} |a_i|.$$

Then the procedure in the proof of Theorem 1, modified to lower blocks at weight greater than $|a_\infty|$ to weight $|a_\infty|$, and similarly, to raise blocks with weight less than $|a_\infty|$ up to weight $|a_\infty|$, will serve. (Note that although at each stage we may have to deal with infinitely many blocks at a fixed weight, all the perturbations required to split off a normal at this weight take place on distinct blocks and therefore they form one perturbation of small norm.) However, at the last stage we are left with a shift of constant weight $|a_\infty|$, once again a normal operator, as our last direct summand. This completes the proof of Theorem 2.

THEOREM 3. *Let the hypotheses of Theorem* 2 *hold. In addition, let*

$$\lim_{\substack{i\to\infty \\ i\to-\infty}} (|a_i| - |a_{i+1}|) = 0.$$

That is, let $S^*S - SS^*$ *be compact. Then there exists a normal operator* N *such that* $\|N - S\| < 100\sqrt{\varepsilon}$ *and such that* N - S *is compact.*

Proof. Just as in the proof of Theorem 2, choose a_∞ so that $|a_\infty|$ lies between $\limsup |a_i|$ and $\liminf |a_i|$ at both ∞ and $-\infty$. Now, by the compactness of $S^*S - SS^*$, we may choose our blocks so as to allow the ε of Condition 2 to approach 0 as our blocks approach $\pm\infty$. That is, as our indices approach infinity, we get longer and longer blocks at more and more closely spaced weights. This yields a sequence of perturbations on orthogonal subspaces; the sequence goes to 0 in norm, and hence it defines a compact perturbation. Once again, as in the proof of Theorem 2, we are left with a shift of constant weight $|a_\infty|$ as a last normal direct summand. This completes the proof of Theorem 3.

The reader may be suspicious of the apparently wide choice of possible normal operators we produce in Theorem 3. After all, we may have several choices for $|a_\infty|$, which appears to contradict the fact that any two such normal operators must differ by a compact operator. However, a close look at the construction shows us that all these possible normal operators have the same essential spectrum, namely the annulus with inner radius $\liminf_{|i|\to\infty} |a_i|$ and outer radius $\limsup_{|i|\to\infty} |a_i|$, and therefore this implausibility is removed.

REFERENCE

1. L. G. Brown, R. G. Douglas, and P. A. Fillmore, Unitary equivalence modulo the compact operators and extensions of C^* algebras. Proceedings of a Conference on Operator Theory. Lecture Notes in Mathematics, Vol. 345. Springer-Verlag, Berlin 1973, pp. 58-128.

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