HOMOTOPY CLASSES OF MAPS BETWEEN PRO-SPACES

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1. INTRODUCTION

A pro-space is an inverse system $\{X_s\}$ of spaces, though maps between pro-spaces are not simply inverse systems of maps. The role of pro-spaces in questions about homology of nonsimply connected spaces and in shape theory is well established [1], [2], [3], [4]. The philosophy is that more can be learned about spaces by transferring the investigation to this larger, more general setting, where nicer results hold, and then specializing back to spaces.

In [6] we developed the fundamental notions for a "homotopy theory" for the category of pro-spaces indexed by the natural numbers. There was, however, no explicit description of the set of homotopy classes of maps between two (nice) pro-spaces. In the present paper we identify this set, $[\{X_s\}, \{Y_s\}]$, by showing that there is a short exact sequence $* \to K \to [\{X_s\}, \{Y_s\}] \to C \to *$, in which both K and C are limits of homotopy classes of maps involving the spaces comprising $\{X_s\}$ and $\{Y_s\}$. (In general this is an exact sequence merely of pointed sets.) We also interpret this exact sequence as showing the relationship between the homotopy category of pro-spaces and the pro-homotopy category studied by M. Artin and B. Mazur [1], in which the set of morphisms between $\{X_s\}$ and $\{Y_s\}$ is the set C of the sequence.

As a corollary we get a short exact sequence of A. K. Bousfield and D. M. Kan [2] involving $[X, \varprojlim Y_s]$. An analogous result holds for "pro-homological algebra"; we can identify $\mathrm{Ext}(\{M_s\}, \{N_s\})$, for pro-modules $\{M_s\}$ and $\{N_s\}$, in terms of Ext and Hom of modules. Finally, we can use this exact sequence to define the cohomology *groups* of a pro-space, with coefficients in a pro-abelian group.

In Section 2, we briefly review notions of pro-spaces and limits. In Section 3, the meaning of a homotopy between two maps is made more concrete, and we state the main result. Applications are treated in Section 4, and the proof is given in Section 5.

Note. D. A. Edwards and H. M. Hastings [5] have recently obtained many of the results of this paper in a different homotopy category of pro-spaces.

2. PRO-SPACES AND $\varprojlim^1$

In this section, we collect the facts about pro-spaces and $\varprojlim^1$ needed to state the main theorem. A full discussion of pro-spaces can be found in [6].

Let $\mathscr{C}$ be a pointed category. Then the category tow-$\mathscr{C}$ has as objects *towers* in $\mathscr{C}$,

$$\cdots \to X_{s+1} \to X_s \to \cdots \to X_1 \to X_0 = * ,$$

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written $\{X_s\}$, and has as maps

$$\mathrm{Hom}_{\text{tow-}\mathscr{C}}(\{X_s\}, \{Y_s\}) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j) .$$

Denote by $\{*\}$ the object of tow-$\mathscr{C}$ each level of which is $*$; it is clearly the initial and final object of tow-$\mathscr{C}$.

Sufficient reflection on the nature of $\mathrm{Hom}_{\text{tow-}\mathscr{C}}(\{X_s\}, \{Y_s\})$ will convince the reader that every map f can be *represented* by a *level map*, that is, by a coherent sequence of maps $\{f_s: X'_s \to Y'_s\}$, where $\{X'_s\}$ and $\{Y'_s\}$ are isomorphic in tow-$\mathscr{C}$ to $\{X_s\}$ and $\{Y_s\}$, respectively. This can be achieved, for example, with a cofinal subtower of $\{X_s\}$. Thus throughout this paper we shall deal only with such level representatives, and we leave it to the reader to see that we have really proved statements about tow-$\mathscr{C}$.

For the sake of notational simplification, we shall let p represent any map $X_{s+1} \to X_s$ within a tower. Thus the statement above that $\{f_s: X'_s \to Y'_s\}$ is coherent means that $pf_{s+1} = f_s p$ for each $s \geq 0$. Any map induced by such a map p will also be denoted by p.

We are interested mainly in the case $\mathscr{C} = \mathscr{S}_*$, the category of pointed spaces, that is, of pointed simplicial sets [2], [7]. Objects of tow-$\mathscr{S}_*$ are called *pro-spaces*.

It was shown in [6] that tow-$\mathscr{S}_*$ is a *model category* in the sense of D. G. Quillen [9], that is, that tow-$\mathscr{S}_*$ admits notions of *fibration, cofibration,* and *weak equivalence* with certain fundamental properties, notably the following factoring and lifting properties.

Factoring Property. Each map $f: \{X_s\} \to \{Y_s\}$ has factorizations $f = gh$ and $f = g'h'$, where

$$h: \{X_s\} \to \{Z_s\}, \qquad g: \{Z_s\} \to \{Y_s\}, \qquad h': \{X_s\} \to \{Z'_s\}, \qquad g': \{Z'_s\} \to \{Y_s\},$$

and where h and h' are cofibrations, g and g' are fibrations, and g' and h are weak equivalences.

Lifting Property. If f is a fibration and g is a cofibration in the solid-arrow commutative diagram

$$\begin{array}{ccc} \{A_s\} & \longrightarrow & \{E_s\} \\ {\scriptstyle g}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \{X_s\} & \longrightarrow & \{B_s\} \end{array}$$

then the dotted arrow exists if either f or g is a weak equivalence.

The prototype model category is $\mathscr{S}_*$, in which fibrations are Kan fibrations, cofibrations are inclusions, and weak equivalences are maps that induce isomorphisms on homotopy groups in all dimensions, at every basepoint.

Quillen [9] showed that one can develop homotopy theory in a model category, much as one does in $\mathscr{S}_*$, or in the category of topological spaces. In particular, we define a *homotopy* between maps $f, g: \{X_s\} \to \{Y_s\}$ to be a map $h: \{\overline{X}_s\} \to \{Y_s\}$ for which the commutative diagram

$$\begin{array}{ccc} \{X_s \vee X_s\} & \xrightarrow{f \vee g} & \{Y_s\} \\ {\scriptstyle \{id_s \vee id_s\}} \downarrow & \searrow^{d} & \uparrow {\scriptstyle h} \\ \{X_s\} & \xleftarrow[e]{} & \{\overline{X}_s\} \end{array}$$

exists, where e is a weak equivalence and $\{X_s \vee X_s\} = \{X_s\} \vee \{X_s\}$ is the direct limit of the diagram $\{X_s\} \leftarrow \{*\} \rightarrow \{X_s\}$. If there is a homotopy between f and g, then f is said to be homotopic to g. Further, define a *fibrant* object $\{X_s\}$ to be one for which $\{X_s\} \rightarrow \{*\}$ is a fibration, and a *cofibrant* object $\{X_s\}$ to be one for which $\{*\} \rightarrow \{X_s\}$ is a cofibration. Then "is homotopic to" is an equivalence relation on maps from a cofibrant object to a fibrant object. We form the *homotopy category,* Ho(tow-$\mathscr{S}_*$), by taking as objects the objects of tow-$\mathscr{S}_*$ that are both fibrant and cofibrant, and as maps from $\{X_s\}$ to $\{Y_s\}$ the set of homotopy classes of maps, written $[\{X_s\}, \{Y_s\}]$. As usual, we write $[f]$ for the homotopy class of f. Similarly, Ho($\mathscr{S}_*$) is the usual homotopy theory of pointed Kan complexes.

Because of the way fibration and cofibration are defined in tow-$\mathscr{S}_*$, all objects are cofibrant, and the fibrant objects are the towers that are (up to isomorphism in tow-$\mathscr{S}_*$) towers of Kan fibrations $\cdots \rightarrow X_{s+1} \rightarrow X_s \rightarrow \cdots \rightarrow X_0 = *$, such that each X_s has nontrivial homotopy groups in only finitely many dimensions.

If $X \in \mathscr{S}_*$, we shall denote the *reduced suspension* [7, p. 124] of X by SX.

Finally, we recall [2, Definition IX.2.1] the definition of $\varprojlim^1$ for a tower of groups. Let $\cdots \rightarrow G_{s+1} \rightarrow G_s \rightarrow \cdots \rightarrow G_0 = *$ be a tower of groups and homomorphisms. Define an action of the group $\prod_{s=0}^{\infty} G_s$ on the set $\prod_{s=0}^{\infty} G_s$ by

$$(g_0, g_1, \cdots) \cdot (x_0, x_1, \cdots) = (g_0 x_0 (pg_1)^{-1}, g_1 x_1 (pg_2)^{-1}, \cdots) .$$

Then $\varprojlim^1 G_s = \prod_{s=0}^{\infty} G_s/\text{action}$. In general, $\varprojlim^1 G_s$ is only a pointed set; but if each G_s is abelian, then $\varprojlim^1 G_s$ inherits an abelian group structure [8]. We leave it to the reader to show that $\varprojlim^1$ (as well as $\varprojlim$) is in fact a functor on tow-(groups).

3. STATEMENT OF THE THEOREM

Let Ho(tow-$\mathscr{S}_*$) be the homotopy theory of pro-spaces described in Section 2.

THEOREM. *For* $\{X_s\}, \{Y_s\} \in$ Ho(tow-$\mathscr{S}_*$), *there is a natural short exact sequence*

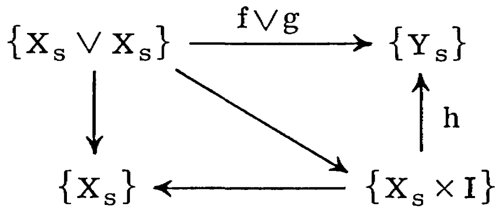
$$* \rightarrow \varprojlim_j{}^1 \varinjlim_i [SX_i, Y_j] \xrightarrow{\alpha} [\{X_s\}, \{Y_s\}] \xrightarrow{\beta} \varprojlim_j \varinjlim_i [X_i, Y_j] \rightarrow * .$$

Remark. It is not necessary to assume that $\{X_s\}$ is fibrant. Moreover, the proof does not use the fact that $\pi_n Y_s = 0$ for almost all n.

In this section we define α and β; the proof that the definitions are sound and the sequence is exact is relegated to Section 5. Before we proceed, however, we need a more explicit description of homotopies in tow-$\mathscr{S}_*$.

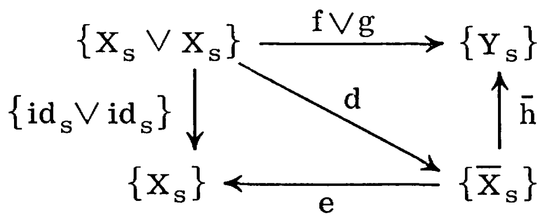
Let I be the standard 1-simplex (intuitively, the unit interval). Since we are working with pointed spaces, we shall abuse notation somewhat and write $X \times I$ for the *pointed* space $(X \times I)/(* \times I)$.

PROPOSITION 1. *Suppose that* $\{X_s\}$, $\{Y_s\} \in$ tow-$\mathscr{S}_*$, *that* $\{Y_s\}$ *is fibrant, and that* f *and* g *are homotopic maps from* $\{X_s\}$ *to* $\{Y_s\}$. *Then there is a commutative diagram*

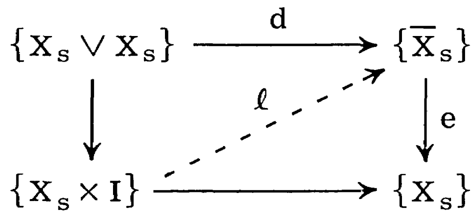


in tow-$\mathscr{S}_*$ *in which the unlabeled maps are the natural ones.*

Proof. Since f and g are homotopic, there is a commutative diagram



with e a weak equivalence. By factoring e into a cofibration followed by a fibration, both of which are also weak equivalences, we may assume that e is in fact already a fibration. Next consider the solid-arrow commutative diagram



where the unlabeled horizontal map is natural projection and the unlabeled vertical map is the natural imbedding $\{(\mathrm{id}_s \times 0) \vee (\mathrm{id}_s \times 1)\}$. By the Lifting Property, the dotted arrow ℓ exists. Let $h = \bar{h}\ell$, and the desired diagram commutes.

COROLLARY 1 (Representation Theorem for Homotopies). *Let* f *and* g *be maps in* tow-$\mathscr{S}_*$ *with the same domain and range, the range being fibrant. Then* f *and* g *are homotopic if and only if there exist level representatives* $\{f_s\colon X_s \to Y_s\}$ *and* $\{g_s\colon X_s \to Y_s\}$ *of* f *and* g, *respectively, and a level map* $\{h_s\colon X_s \times I \to Y_s\}$ *such that* $h_s(\ , 0) = f_s$ *and* $h_s(\ , 1) = g_s$.

Proof. Choose a level representative of the diagram in Proposition 1.

Thus two maps are homotopic precisely when there is a coherent level homotopy between level representatives of the maps. In contrast, two level maps $\{f_s\}$ and $\{g_s\}$ represent the same element of $\varprojlim_j \varinjlim_i [X_i, Y_j]$ if there exist level homotopies

$h_s\colon f_s \simeq g_s$, but not necessarily with $ph_{s+1} = h_s p$ for all s. The theorem essentially tells when such homotopies can be made coherent. Since $\varprojlim_j \varinjlim_i [X_i, Y_j]$ is simply $\mathrm{Hom}_{\text{tow-Ho}(\mathscr{S}_*)}(\{X_s\}, \{Y_s\})$, where $\mathrm{Ho}(\mathscr{S}_*)$ is the ordinary homotopy category of pointed Kan complexes, the theorem shows that the projection $\mathscr{S}_* \to \mathrm{Ho}(\mathscr{S}_*)$ induces an epimorphism of sets of morphisms

$$\mathrm{Ho}(\text{tow-}\mathscr{S}_*) \to \text{tow-Ho}(\mathscr{S}_*)$$

and identifies the kernel. Artin and Mazur [1] have studied tow-Ho$(\mathscr{S}_*)$ in some detail.

It is easy to define β. Let $[f] \in [\{X_s\}, \{Y_s\}]$. Then $\beta([f])$ is the element of $\varprojlim_j \varinjlim_i [X_i, Y_j]$ represented by the homotopy classes of a level representative of f. To define α, we first need the following obvious fact [9, p. 2.9].

PROPOSITION 2. *Let* $X, Y \in \mathscr{S}_*$. *There is a natural one-to-one correspondence between* $[SX, Y]$ *and the set of homotopy classes of homotopies from the constant map* $*\colon X \to Y$ *to itself.*

For clarity in what follows, rather than using the simplicial definition of homotopy, we shall compose homotopies h and j by letting h act on $? \times [0, 1/2]$ and letting j act on $? \times [1/2, 1]$, and we write the result as $j \cdot h$. This composition is well-defined and associative up to homotopy.

We now define α. Since $\{Y_s\}$ is fibrant, we can assume $\{Y_s\}$ is a tower of fibrations. A class h in $\varprojlim^1_j \varinjlim_i [SX_i, Y_j]$ can be represented by a sequence of elements, the s^{th} element in $\varinjlim_i [SX_i, Y_s]$. Without loss of generality we can assume the existence of representatives $[h_s] \in [SX_{s+1}, Y_s]$; therefore h_s is a homotopy from $*\colon X_{s+1} \to Y_s$ to itself, $h_s\colon X_{s+1} \times I \to Y_s$. We define a sequence of homotopies $h'_s\colon X_s \times I \to Y_s$ by induction on s. Let $h'_0 = *$. If h'_s has been defined, then h'_{s+1} is a lifting of $(h'_s p) \cdot h_s\colon X_{s+1} \times I \to Y_{s+1}$ that restricts to $*$ on $X_{s+1} \times 0$. Such a lifting exists because $p\colon Y_{s+1} \to Y_s$ is a fibration. Set $f_s\colon X_s \to Y_s$ equal to $h'_s(\ , 1)$. Then $\alpha(h) = [\{f_s\}]$.

A comment about the exactness of the sequence in the theorem is in order. In general, we have only pointed sets, and exactness means that α is injective, β is surjective, and the image of α equals $\beta^{-1}(*)$. Of special interest is the case in which $\{X_s\}$ is a suspension $S\{W_s\} = \{SW_s\}$. Here $[\{X_s\}, \{Y_s\}]$ has a group structure, $\varprojlim_j \varinjlim_i [X_i, Y_j]$ is also a group, and $\varprojlim^1_j \varinjlim_i [SX_i, Y_j]$ is an abelian group. One can verify that in this case α and β are homomorphisms and the extension is central.

4. APPLICATIONS

We first specialize to the case of a constant pro-space X, that is, a tower where $X_s = X$ for each s, and each p is the identity. The following proposition can be loosely interpreted as stating that the functor $\varprojlim$ from $\mathrm{Ho}(\text{tow-}\mathscr{S}_*)$ to $\mathrm{Ho}(\mathscr{S}_*)$ is a right adjoint of the natural inclusion of $\mathrm{Ho}(\mathscr{S}_*)$ in $\mathrm{Ho}(\text{tow-}\mathscr{S}_*)$; see [2, Section XI.8] for a discussion of a similar situation with respect to the "homotopy inverse limit" functor of Bousfield and Kan.

PROPOSITION 3. $[X, \{Y_s\}] \cong [X, \varprojlim Y_s]$.

Proof. The correspondence is given by $\beta([\{f_s\}]) = [\varprojlim f_s]$. To see that β is well-defined, represent a homotopy between $\{f_s\}$ and $\{g_s\}$ by a level representative of the diagram in Proposition 1, where $X_s = X$ for each s, and take the inverse limit to produce a homotopy between $\beta([\{f_s\}])$ and $\beta([\{g_s\}])$. Clearly, β is bijective, by the universal property of $\varprojlim$.

COROLLARY 2 (Bousfield and Kan [2, Corollary IX.3.2]). *If $\{Y_s\}$ is a tower of fibrations in $\mathscr{S}_*$, then there exists a natural exact sequence*

$$* \to \varprojlim{}^1 [SX, Y_s] \to [X, \varprojlim Y_s] \to \varprojlim [X, Y_s] \to * .$$

Note that we have proved Corollary 2 only with the restriction that for each s, the homotopy groups $\pi_n(Y_s)$ are trivial when n is sufficiently large. This is no real restriction, however, because a tower $\{Y_s\}$ can be replaced by its Postnikov tower $\{P_s Y_s\}$ (made into a tower of fibrations), where $P_s Y_s$ is the s^{th} Postnikov stage [7, p. 33] of Y_s.

Next we consider cohomology of pro-spaces, dealing here only with nontwisted coefficients. Let $\{G_s\} \in$ tow-(abelian groups). For $n \geq 0$, we can form the canonical fibrant Eilenberg-MacLane pro-space $K(\{G_s\}, n)$, as $\{K(G_s, n)\}$. Define the cohomology groups of a pro-space $\{X_s\}$, with coefficients in $\{G_s\}$, as

$$H^n(\{X_s\}, \{G_s\}) = [\{X_s\} \cup \{*\}, K(\{G_s\}, n)].$$

COROLLARY 3. *There is a natural short exact sequence*

$$* \to \varprojlim_j{}^1 \varinjlim_i H^{n-1}(X_i, G_j) \to H^n(\{X_s\}, \{G_s\}) \to \varprojlim_j \varinjlim_i H^n(X_i, G_j) \to * .$$

In case $\{G_s\}$ is constant, we get Artin and Mazur's definition [1]

$$H^n(\{X_s\}, G) = \varinjlim H^n(X_s, G).$$

In case $\{X_s\}$ is constant, we get a generalized cohomology theory $[\ , \varprojlim K(G_s, n)]$ in which $H^0(*, \{G_s\}) = \varprojlim G_s$, $H^1(*, \{G_s\}) = \varprojlim{}^1 G_s$, and $H^n(*, \{G_s\}) = *$ $(n \geq 2)$.

As a final application of the ideas involved here, we state an abelianized analogue of the theorem (actually a special case of Corollary 3). The category tow-(abelian groups) is, in a natural way, an abelian category [morphism addition, for example, is induced by morphism addition in (abelian groups) at each level]. We can try to relate the resulting constructs of pro-homological algebra to limits of constructs in homological algebra. In the following proposition, we consider the group of extensions of one pro-abelian group by another.

PROPOSITION 4. *Let* $\{M_s\}$, $\{N_s\}$ $\in$ tow-(abelian groups). *Then there exists a natural exact sequence of abelian groups*

$$* \to \varprojlim_j{}^1 \varinjlim_i \mathrm{Hom}(M_i, N_j) \to \mathrm{Ext}(\{M_s\}, \{N_s\}) \to \varprojlim_j \varinjlim_i \mathrm{Ext}(M_i, N_j) \to * .$$

5. PROOF OF THE THEOREM

We must prove that a) β is well-defined and $\beta\alpha = *$; b) β is surjective; c) α is well-defined; d) α is injective; and e) $\ker \beta \subseteq \operatorname{im} \alpha$.

a) The mapping β is well-defined, by Corollary 1, and $\beta\alpha = *$, since by construction $(\alpha(h))_s \simeq *$ for each s.

b) To show that β is surjective, let $\{[g_s]\}$ represent an element of $\varprojlim_j \varinjlim_i [X_i, Y_j]$; that is, let $g_s\colon X_s \to Y_s$ be a sequence of maps that are coherent up to homotopy. We shall construct a level map $\{f_s\}$ of tow-$\mathscr{S}_*$ such that $f_s \simeq g_s$; then clearly $\beta([\{f_s\}]) = \{[g_s]\}$. Let $f_0 = g_0$. Assume, by induction, that f_s has been defined, and that $f_s \simeq g_s$ and $pf_s = f_{s-1}p$. To define f_{s+1}, let h_{s+1} be a homotopy from pg_{s+1} to $f_s p$; since $Y_{s+1} \to Y_s$ is a fibration, we can lift h_{s+1} to a homotopy from g_{s+1} to f_{s+1}. Clearly, $pf_{s+1} = f_s p$.

c) Next we show that α is well-defined. Note that α was actually defined on $\prod_s \varinjlim_i [SX_i, Y_s]$. We show that each of the choices is immaterial. We observe that every ambiguity arising from $\varinjlim$ is absorbed in the definition of $\mathrm{Hom}_{\text{tow-}\mathscr{S}_*}[\{X_s\}, \{Y_s\}]$. Moreover, the choice of homotopy h_s or the choice of lifting h'_{s+1}, introduces no ambiguity, because at each stage we can lift every homotopy relating different choices to produce a coherent homotopy in $[\{X_s\}, \{Y_s\}]$.

It remains to show that if sequences $\{h_s\}$ and $\{j_s\}$ representing elements h and j in $\prod_s \varinjlim_i [SX_i, Y_s]$ are related by the action that defines $\varprojlim{}^1$, then $\alpha(h) = \alpha(j)$. To this end, suppose $h = k \cdot j$, where we represent k as a sequence of homotopies $\{k_s\colon X_s \times I \to Y_s\}$. Thus we assume that $h_s = (k_s p) \cdot j_s \cdot (pk_{s+1})^{-1}$. Let $k'_0 = k_0$. First make the choice of h'_1 in such a way that the restriction to $[0, 1/3]$ is $(k_1)^{-1}$. Let j'_1 (respectively, k'_1) be the restriction of h'_1 to $[1/3, 2/3]$ (respectively, to $[2/3, 1]$). Clearly, this is a suitable choice of j'_1 in the determination of $\alpha(j)$. Therefore $h'_1 = k'_1 \cdot j'_1 \cdot (k_1)^{-1}$. Clearly, k'_1 is a homotopy between $j'_1(\ , 1)$ and $h'_1(\ , 1)$, and $k'_0 p = pk'_1$. By induction, then, to get h'_{s+1}, we must lift

$$(h'_s p) \cdot h_s = (k'_s p) \cdot (j'_s p) \cdot (k_s p)^{-1} \cdot (k_s p) \cdot j_s \cdot (pk_{s+1})^{-1}$$
$$= (k'_s p) \cdot (j'_s p) \cdot j_s \cdot (pk_{s+1})^{-1} .$$

We lift in such a way that we get $(k_{s+1})^{-1}$ on $[0, 1/4]$; let j'_{s+1} be the restriction of h'_{s+1} to $[1/4, 3/4]$, and let k'_{s+1} be the restriction of h'_{s+1} to $[3/4, 1]$. Then j'_{s+1}

is a suitable lifting of $(j'_s p) \cdot j_s$; also, k'_{s+1} is a homotopy between $j'_{s+1}(\ , 1)$ and $h'_{s+1}(\ , 1)$, and $k'_s p = pk'_{s+1}$. Thus $\{k'_s\}$ is a (coherent) homotopy between $\alpha(h)$ and $\alpha(j)$. This completes the proof that α is well-defined.

d) By reversing the argument above we can show that α is one-to-one. Indeed, if $\{H_s\}$ is a (coherent) homotopy between $\alpha(j) = \{j'_s(\ , 1)\}$ and $\alpha(h) = \{h'_s(\ , 1)\}$, let $k_s = (h'_s)^{-1} \cdot H_s \cdot j'_s$; it is easy to show that $\{h_s\} = \{k_s\} \cdot \{j_s\}$.

e) Finally, we show that $\ker \beta \subseteq \operatorname{im} \alpha$. If $\{H_s\}$ is a (noncoherent) sequence of homotopies from $*$ to $f_s\colon X_s \times I \to \overline{Y}_s$, let $h_s = (H_s p)^{-1} \cdot (pH_{s+1})$. Then $[\{f_s\}] = \alpha(\{[h_s]\})$, because we can choose $h'_s = H_s$ at each stage.

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