ON THE DERIVED ALGEBRA OF L_p OF A COMPACT GROUP

Alan L. Armstrong

INTRODUCTION

In [6], S. Helgason defined the derived algebra D_A of a semisimple commutative Banach algebra A. In [2], G. F. Bachelis showed that if $A = L_p(G)$, where G is a compact abelian group and $1 \leq p < \infty$, then D_A is the ideal of unconditionally convergent Fourier series. D. S. Browder [3] has extended the definition of the derived algebra of $L_p(G)$ to the more general case where G is a compact group, and he has extended Bachelis's result by showing that the derived algebra of the center of $L_p(G)$ is the same as the center of the ideal of unconditionally convergent Fourier series. He asks whether there are any necessary and sufficient conditions to ensure that the derived algebra of $L_p(G)$ is the ideal of unconditionally convergent Fourier series. This paper attempts to answer that question.

1. PRELIMINARIES

Throughout the paper, G will denote a compact group, and $\Gamma = \Gamma(G)$ will denote the set of equivalence classes of irreducible unitary representations. Corresponding to each $\alpha \in \Gamma$, we denote by U_α a fixed representative of α, by d_α its degree, and by χ_α the trace $\operatorname{Tr} U_\alpha$. If $f \in L_1(G)$, its Fourier series is given by

$$f(x) \sim \sum_\alpha d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x)),$$

where A_α is the matrix determined by the equation

$$A_\alpha = \int_G f(x)\overline{U_\alpha(x)}\,dx.$$

We shall also denote A_α by $\hat{f}(\alpha)$.

If A is a matrix over the complex numbers with absolute value $|A|$ [7, p. 691], we define

$$\|A\|_\infty = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } |A|\}$$

and

$$\|A\|_{\Phi_2} = \left(\sum |\lambda|^2\right)^{1/2},$$

the sum being over all eigenvalues of $|A|$.

For $f \in L_1(G)$, we define

Received July 18, 1974.

Michigan Math. J. 21 (1974).

$$\|\hat{f}\|_\infty = \sup_{\alpha \in \Gamma} \|\hat{f}(\alpha)\|_\infty .$$

Definition 1. *The derived algebra* D_p *of* $L_p(G)$ is given by

$$D_p = \left\{ f \in L_p : \sup_{g \in L_p} \frac{\|f * g\|_p}{\|\hat{g}\|_\infty} < \infty \right\}$$

(* denotes convolution).

The *norm* of $f \in D_p$ is given by

$$\|f\|_{D_p} = \sup_{g \in L_p} \frac{\|f * g\|_p}{\|\hat{g}\|_\infty},$$

and it is clearly equal to

$$\sup_{\substack{g \in T(G) \\ \|\hat{g}\|_\infty = 1}} \|f * g\|_p ,$$

where $T(G)$ is the space of trigonometric polynomials.

Let $\mathscr{F}$ denote the family of all nonempty finite subsets of Γ, and let $D(F) = \sum_F d_\alpha \chi_\alpha$, where $F \in \mathscr{F}$.

Definition 2. The space S_p of functions in $L_p(G)$ having unconditionally convergent Fourier series is defined by

$$S_p = \{ f \in L_p : \lim_{F \in \mathscr{F}} \|f - f * D(F)\|_p = 0 \} .$$

The norm of $f \in S_p$ is

(1)
$$\|f\|_{S_p} = \sup_{F \in \mathscr{F}} \|f * D(F)\|_p .$$

Browder [3] has shown that S_p is an ideal in $L_p(G)$, and also that $f \in S_p$ if and only if (1) is finite.

2. NECESSARY AND SUFFICIENT CONDITIONS THAT $S_p = D_p$

For $1 \leq p < 2$, we find a necessary and sufficient condition that $S_p = D_p$. For $p > 2$, we find a necessary as well as a slightly stronger sufficient condition. We begin by giving some equivalent conditions for a function to be in S_p or D_p.

LEMMA 1. *If* $f \sim \sum d_\alpha \operatorname{Tr}(A_\alpha U_\alpha) \in L_p$, *then* $f \in S_p$ *if and only if*

(2)
$$\sup \left\{ \left\| \sum_F \delta_\alpha d_\alpha \operatorname{Tr}(A_\alpha U_\alpha) \right\|_p : \delta_\alpha = \pm 1,\ F \in \mathscr{F} \right\} < \infty .$$

Proof. If $f \in S_p$, then the left side in (2) is less than $2 \sup_F \|f * D(F)\|_p < \infty$. The sufficiency of condition (2) is obvious.

LEMMA 2. *If* $f \sim \sum d_\alpha \operatorname{Tr}(A_\alpha U_\alpha) \in L_p$, *then* $f \in D_p$ *if and only if*

$$\sup \left\{ \|\sum_F d_\alpha \operatorname{Tr}(V_\alpha A_\alpha U_\alpha)\|_p : V_\alpha \text{ unitary, } F \in \mathscr{F} \right\} < \infty . \tag{3}$$

Proof. Suppose $f \in D_p$. If $g = \sum_F d_\alpha \operatorname{Tr}(V_\alpha U_\alpha)$, where V_α is unitary and $F \in \mathscr{F}$, then $\|\hat{g}\|_\infty = 1$ and the left side in (3) is bounded by $\|f\|_{D_p}$.

The sufficiency follows from the fact that if A is a finite-dimensional matrix with $\|A\|_\infty \leq 2$, then $A = V + W$, where V and W are unitary [7, p. 399].

Since $\|f\|_{S_p} \leq \|f\|_{D_p}$, it is clear that $D_p \subseteq S_p$. Browder [3] has shown that if G is a compact group and $1 \leq p \leq 2$, then $D_p = L_2(G)$. He has also given an example to show that the inclusion $D_p \subseteq S_p$ can be strict if $p \neq 2$. The example is the group $G = \prod_1^\infty \mathscr{S}_3$, where $\mathscr{S}_3$ denotes the nonabelian group of order 6.

Definition 3. A set $E \subseteq \Gamma(G)$ is said to be a local Λ_p-*set* if there exist a constant M and an $r < p$ such that for all $f(x) = \operatorname{Tr}(A_\alpha U_\alpha(x))$ and $\alpha \in E$,

$$\|f\|_p \leq M \|f\|_r .$$

If this inequality holds for some $r < p$, then with M replaced by $M(s)$, it holds for all $s < p$.

THEOREM 1. *The following conditions are equivalent for a compact group* G.

(1) $\Gamma(G)$ *is a local* Λ_2-set.

(2) $S_p = D_p$ *for all* $p (1 \leq p \leq 2)$.

(3) $S_p = D_p$ *for some* $p (1 \leq p < 2)$.

Proof. (3) $\Rightarrow$ (1). Suppose $S_p = D_p (= L_2)$ for some p $(1 \leq p < 2)$. If $f \in S_p$, then

$$\|f\|_{S_p} = \sup_{F \in \mathscr{F}} \|f * D(F)\|_p \leq \sup_{F \in \mathscr{F}} \|f * D(F)\|_2 = \|f\|_2 .$$

The open-mapping theorem implies the existence of a constant K such that $\|f\|_2 \leq K \|f\|_{S_p}$. Now, if $f = \operatorname{Tr}(A_\alpha U_\alpha)$, then $\|f\|_{S_p} = \|f\|_p$. Thus $\Gamma(G)$ is a local Λ_2-set.

(1) $\Rightarrow$ (2). Let $H = \prod_1^\infty Z_2$, where $Z_2 = \{1, -1\}$, and let r_n be the projection onto the nth coordinate. There exists a constant B such that if $h(t) = \sum_n a_n r_n(t)$, where $t \in H$ and the coefficients a_n are complex numbers, then $\|h\|_2 \leq B \|h\|_1$. The functions h are the well-known Rademacher functions.

Now let $f \in S_1$ and $f \sim \sum d_\alpha \operatorname{Tr}(A_\alpha U_\alpha)$. The relation $f \in S_1$ implies that there exists a constant $K < \infty$ such that

$$\sup_{F\in\mathscr{F}} \int_G \left|\sum_F \pm d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x))\right| dx \le K .$$

Now, for $x \in G$ and $F \in \mathscr{F}$, define a function h on H by

$$h(t) = \sum_F r_\alpha(t)\, d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x)) .$$

By the Plancherel theorem, we have the equality

$$\int_H \left|\sum_F r_\alpha(t)\, d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x))\right|^2 dt = \sum_F |d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x))|^2 .$$

Thus

$$\begin{aligned}
&\int_G \Big(\sum_F |d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x))|^2\Big)^{1/2} dx \\
&\quad = \int_G \Big(\int_H \left|\sum_F r_\alpha(t)\, d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x))\right|^2 dt\Big)^{1/2} dx \\
(4)\qquad &\quad \le \int_G B \int_H \left|\sum_F r_\alpha(t)\, d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x))\right| dt\, dx \\
&\quad = B \int_H \int_G \left|\sum_F r_\alpha(t)\, d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x))\right| dx\, dt \le BK .
\end{aligned}$$

Now, if $g(x) = d_\alpha \operatorname{Tr}(B_\alpha U_\alpha(x))$, then, by hypothesis, $\|g\|_2 \le M \|g\|_1$, so that

$$\sqrt{d_\alpha}\, \|B_\alpha\|_{\Phi_2} \le M \int_G |g|\, dx . \tag{5}$$

If $\{y_\alpha\}$ is a square-summable sequence of positive numbers, then (4) and (5) yield the inequalities

$$\begin{aligned}
\sum_F \sqrt{d_\alpha}\, \|A_\alpha\|_{\Phi_2} y_\alpha &\le M \sum_F \int_G |d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x))|\, dx\, y_\alpha \\
&\le M\Big(\sum_F y_\alpha^2\Big)^{1/2} \int_G \Big(\sum_F |d_\alpha \operatorname{Tr}(A_\alpha U_\alpha(x))|^2\Big)^{1/2} \le MBK\Big(\sum_F y_\alpha^2\Big)^{1/2}
\end{aligned}$$

Therefore, $\left(\sum_\alpha d_\alpha \|A_\alpha\|^2_{\Phi_2}\right)^{1/2} < \infty$. The Peter-Weyl theorem implies that $f \in L_2(G)$. Thus $S_1 \subseteq L_2$, and therefore $L_2 = D_p \subseteq S_p \subseteq S_1 = L_2$. Therefore $D_p = S_p = L_2$ for $1 \le p \le 2$.

The implication (2) $\Rightarrow$ (3) is obvious.

For $p > 2$, we obtain a less satisfactory result.

THEOREM 2. *Let* $p > 2$. *If* $S_p = D_p$, *then* $\Gamma(G)$ *is a local* Λ_p-set.

Proof. Let $f(x) = \mathrm{Tr}(A_\alpha U_\alpha(x))$, and let W be the group of all d_α-by-d_α unitary matrices. The hypothesis $S_p = D_p$ implies that there exists a constant $K < \infty$ such that

$$\sup_{V \in W} \|\mathrm{Tr}(V A_\alpha U_\alpha)\|_p \leq \|f\|_{D_p} \leq K \|f\|_{S_p} = K \|f\|_p .$$

Thus, since K is independent of A_α, we have the inequality

$$\|\mathrm{Tr}(A_\alpha U_\alpha)\|_p \leq K \|\mathrm{Tr}(V A_\alpha U_\alpha)\|_p \quad \text{for all } V \in W.$$

Hence

$$\begin{aligned} \|\mathrm{Tr}(A_\alpha U_\alpha)\|_p^p &= \int_W \|\mathrm{Tr}(A_\alpha U_\alpha)\|_p^p \, dV \leq K^p \int_W \|\mathrm{Tr}(V A_\alpha U_\alpha)\|_p^p \, dV \\ &= K^p \int_G \int_W |\mathrm{Tr}(V A_\alpha U_\alpha)|^p \, dV \, dx = K^p \int_W |\mathrm{Tr}(V A_\alpha)|^p \, dV . \end{aligned} \tag{6}$$

Now, if $G^* = \prod_\alpha W_\alpha$, where each W_α is a unitary group, then the projection π_α onto the αth coordinate is an irreducible representation of G^*. A. Figà-Talamanca and D. Rider [5] have shown that for each p the π_α form a Λ_p-set in $\Gamma(G^*)$. In particular, the π_α form a local Λ_p-set. Thus there exists a constant K_0 such that for each W_α, each matrix A_α, and each $p > 2$.

$$\int_{W_\alpha} |\mathrm{Tr}(A_\alpha V)|^p \, dV \leq K_0 \|\mathrm{Tr}(A_\alpha V)\|_2^p . \tag{7}$$

Now, by the Peter-Weyl theorem, both the right side in (7) and $K_0 \|f\|_2^p$ are equal to $K_0 [d_\alpha^{-1} \|A_\alpha\|_{\Phi_2}^2]^{p/2}$. Thus (6) and (7) imply that Γ is a local Λ_p-set.

THEOREM 3. *If* $\sup_{\alpha \in \Gamma} d_\alpha < \infty$ *and* $1 \leq p < \infty$, *then* $S_p = D_p$.

Note: For $1 \leq p < 2$, this result is implied by Theorem 1. For $p > 2$, it is a partial converse to Theorem 2. Groups for which $\sup_{\alpha \in \Gamma} d_\alpha < \infty$ have been characterized by C. C. Moore [8].

Proof. Let $f \sim \sum d_\alpha \mathrm{Tr}(A_\alpha U_\alpha) \in S_p$. We want to show that

$$\sup_{V_\alpha, F} \Big\| \sum_F d_\alpha \mathrm{Tr}(V_\alpha A_\alpha U_\alpha) \Big\|_p < \infty,$$

where the V_α are d_α-by-d_α unitary matrices. We know that

$$\sup_{F \in \mathscr{F}, \pm 1} \Big\| \sum_F \pm d_\alpha \mathrm{Tr}(A_\alpha U_\alpha) \Big\|_p \leq M < \infty. \tag{8}$$

Therefore we fix $F \in \mathscr{F}$ and consider $g = \sum_F \left(N^{-1} \sum_1^N r_i(\alpha)\, d_\alpha \operatorname{Tr}(A_\alpha U_\alpha) \right)$, where $r_i(\alpha) = \pm 1$ or 0. It follows from (8) that $\|g\|_p \leq M$. This implies that

$$\Big\| \sum_F b_\alpha d_\alpha \operatorname{Tr}(A_\alpha U_\alpha) \Big\|_p \leq M, \tag{9}$$

where b_α is rational and $-1 \leq b_\alpha \leq 1$. It follows easily that (9) remains valid, with M replaced by 2KM, for all b_α with $|b_\alpha| \leq K$.

Now let $y \in G$, and suppose $|b_\alpha| \leq K$ for $\alpha \in F$. Then

$$\Big\| \sum_F d_\alpha \operatorname{Tr}(b_\alpha U_\alpha(y) A_\alpha U_\alpha(x)) \Big\|_p = \Big\| \sum_F d_\alpha \operatorname{Tr}(b_\alpha A_\alpha U_\alpha(xy)) \Big\|_p$$

$$= \Big\| \sum_F d_\alpha \operatorname{Tr}(b_\alpha A_\alpha U_\alpha(x)) \Big\|_p \leq 2KM.$$

Put $b_\alpha(y) = d_\alpha (\overline{U_\alpha(y)})_{ij}$, and let B_{ij} be the matrix with a 1 in the ith row and jth column and zeros elsewhere. (The condition $\sup_\alpha d_\alpha < \infty$ allows us to assume that $d_\alpha = d_\beta$ for all $\alpha, \beta \in F$). Then

$$\Big\| \sum_F d_\alpha \operatorname{Tr}(B_{ij}(\alpha) A_\alpha U_\alpha(x)) \Big\|_p^p = \Big\| \sum_F d_\alpha \operatorname{Tr}\Big(\Big(\int_G b_\alpha(y) U_\alpha(y)\, dy \Big) A_\alpha U_\alpha(x) \Big) \Big\|_p^p$$

$$= \Big\| \int_G \Big(\sum_F d_\alpha \operatorname{Tr}(b_\alpha(y) U_\alpha(y) A_\alpha(x)) \Big) dy \Big\|_p^p$$

$$\leq \int_G dy \int_G \Big| \sum_F d_\alpha \operatorname{Tr}(b_\alpha(y) U_\alpha(y) A_\alpha U_\alpha(x)) \Big|^p dx \leq \int_G (2KM)^p\, dy = (2KM)^p.$$

Thus, for all $F \in \mathscr{F}$ and for all index pairs i, j,

$$\Big\| \sum_F d_\alpha \operatorname{Tr}(B_{ij}(\alpha) A_\alpha U_\alpha(x)) \Big\|_p \leq 2KM. \tag{10}$$

This follows since the hypothesis implies the existence of a $K < \infty$ such that $|b_\alpha(y)| \leq K$ for all $y \in G$ and $\alpha \in \Gamma$.

An argument similar to that at the beginning of the proof shows that (10) is valid, with possibly a new constant in place of 2KM, if the $B_{ij}(\alpha)$ are replaced by complex matrices all of whose entries are bounded in absolute value by some constant K. In particular, (10) holds if the $B_{ij}(\alpha)$ are replaced by unitary matrices.

For $p > 2$, I have not been able to show that $S_p = D_p$ if Γ is a local Λ_p-set. A more interesting question is whether $\sup_\alpha d_\alpha < \infty$ is equivalent to Γ being a local Λ_p-set for some p and hence for all p.

3. GROUPS FOR WHICH $S_p \neq D_p$

In this section, we identify some classes of groups for which Γ is not a local Λ_p-set. Our first result generalizes the example given by Browder.

THEOREM 4. *If* $G = \prod_1^\infty G_i$, *where each* G_i *is nonabelian, then* $S_p \neq D_p$ *for* $p \neq 2$.

The proof of the theorem rests on the following two lemmas.

LEMMA 3. *Let* $W(n)$ *be the unitary matrices of degree* $n > 1$, *and let* $\chi_n(V) = \mathrm{Tr}\, V$ *for* $V \in W(n)$. *Then there exists a positive number* $A < 1$ *such that* $\|\chi_n\|_1 \leq A$ *for all* $n > 1$.

Proof. Suppose not. Then there exists a sequence $\{n_i\}$ with $\|\chi_{n_i}\|_1 \to 1$. Let $\delta > 0$, and let

$$E_n = \{x \in W(n): (|\chi_n(x)| - 1)^2 < \delta\}.$$

Now $\int_{W(n_i)} (|\chi_{n_i}| - 1)^2 \to 0$, since $\|\chi_{n_i}\|_1 \to 1$. Thus we can choose an integer $m = n_k$ such that $\int_{W(m)} (|\chi_m| - 1)^2 \leq \delta^2$, and therefore

$$\lambda_m(E_m^c) \leq \delta, \tag{11}$$

where λ_m denotes Haar measure on $W(m)$. Now there exists a constant B such that $\|\chi_n\|_8^4 \leq B$ for all n [7, p. 123]. Thus Hölder's inequality and (11) yield the inequalities

$$\int_{E_m^c} |\chi_m|^4 \, dV_m \leq \|\chi_m\|_8^4 \sqrt{\lambda_m(E_m^c)} \leq B\sqrt{\delta}.$$

On E_m, we have the inequality $\||\chi_m|^4 - 1| \leq g(\delta)$, where $g(\delta) \to 0$ as $\delta \to 0$ and g is independent of m. Therefore, if δ is chosen small enough, we can find an m such that $\|\chi_m\|_4^4$ is a close to 1 as we like. But this is a contradiction, since $\|\chi_m\|_4^4 = 2$ for $m \neq 1$ [7, p. 148].

LEMMA 4. *Let* G *be a compact group, and let* $\alpha \in \Gamma$ with $d_\alpha > 1$. *There exists a positive number* $A < 1$, *independent of* G *and* α, *such that* $\|\mathrm{Tr}\,(V\, U_\alpha(x))\|_1 \leq A$ *for some* d_α-by-d_α *unitary matrix* V.

Proof. The translation invariance of the Haar measure of $W(d_\alpha)$ implies that for $V \in W(d_\alpha)$,

$$\begin{aligned} \int_{W(d_\alpha)} \int_G |\mathrm{Tr}\,(V\, U_\alpha(x))| \, dx\, dV &= \int_G \int_{W(d_\alpha)} |\mathrm{Tr}\,(V\, U_\alpha(x))| \, dV\, dx \\ &= \int_G \int_{W(d_\alpha)} |\mathrm{Tr}\, V| \, dV\, dx = \int_{W(d_\alpha)} |\mathrm{Tr}\, V| \, dV. \end{aligned} \tag{12}$$

Thus, by (12) and Lemma 3, there exists a $V \in W(d_\alpha)$ such that $\int_G |\mathrm{Tr}\,(V\, U_\alpha(x))| \, dx \leq A$, where A can be taken as in Lemma 3.

Proof of Theorem 4. *Case* 1. $1 \leq p < 2$. By Theorem 1, we need only show that $\Gamma(G)$ is not a local Λ_2-set. This amounts to showing that corresponding to each $\delta > 0$, there exist an $\alpha \in \Gamma(G)$ and a unitary matrix V such that $\|\mathrm{Tr}(V U_\alpha(x))\|_1 \leq \delta$. This suffices, since $\|\mathrm{Tr}(V U_\alpha(x))\|_2 = 1$.

Since each G is nonabelian, there exist $\alpha_i \in \Gamma(G)$ with $d_{\alpha_i} > 1$. By Lemma 4, there exist unitary matrices V_i such that if $h_i(x) = \mathrm{Tr}(V_i U_{\alpha_i}(x))$, then $\|h_i\|_1 \leq A < 1$, where A is independent of i. (Notation: $\|h_i\|_1 = \int_{G_i} |h_i(x)|\,dx$.) Putting

$$f_n = \mathrm{Tr}((V_1 \otimes \cdots \otimes V_n)(U_{\alpha_1} \otimes \cdots \otimes U_{\alpha_n})),$$

we see that $\|f_n\|_1 = \prod_1^n \|h_i\|_1 \to 0$ as $n \to \infty$, since $A < 1$. This completes the proof, since $\alpha_1 \otimes \cdots \otimes \alpha_n \in \Gamma(G)$.

Case 2. $p > 2$. By Theorem 2, we need only show that $\Gamma(G)$ is not a local Λ_p-set, in other words, that for each M, there exist an $\alpha \in \Gamma(G)$ and a unitary matrix V such that $\|\mathrm{Tr}(V U_\alpha(x))\|_p \geq M$. This easily reduces to showing that there exists a $\delta > 0$ such that $\int_{W(n)} |\chi_n|^p \geq 1 + \delta$ for all $n > 1$. Since $\int_{W(n)} |\chi_n|^4 = 2$, we need only worry about $2 < p < 4$. Let $q = 2(p-2)/(p-1)$. Then, by Hölder's inequality, we have the inequality

$$1 = \int_{W(n)} |\chi_n|^2 \leq \left(\int |\chi_n|\right)^{q/2} \left(\int |\chi_n|^p\right)^{(2-q)/2}.$$

It follows from Lemma 3 that $\int_{W(n)} |\chi_n|^p$ must be bounded away from 1.

Let G be an infinite, compact Lie group. C. Cecchini [4] has shown that $\|\chi_\alpha\|_4 \to \infty$ as $d_\alpha \to \infty$, for $\alpha \in \Gamma(G)$. Thus for these groups $S_p \neq D_p$ for $p \geq 4$. For compact, connected Lie groups we can say more; but first we give a definition.

Definition 4. A set $E \subseteq \Gamma(G)$ is said to be a *local central* Λ_p-*set* if there exist a constant M and an $r < p$ such that for all $\alpha \in E$,

$$\|\chi_\alpha\|_p \leq M \|\chi_\alpha\|_r.$$

THEOREM 5. *If* G *is a compact, connected group, then the following are equivalent.*

(1) G *is abelian.*

(2) $D_p = S_p$, *for all* p $(1 \leq p < \infty)$.

(3) $D_p = S_p$ *for some* $p \geq 4$.

(4) $\Gamma(G)$ *is a local* Λ_p-set *for some* $p \geq 4$.

(5) $\Gamma(G)$ *is a local central* Λ_p-set *for some* $p \geq 4$.

Proof.

(1) $\Rightarrow$ (2) is Bachelis's theorem.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4) is Theorem 2.

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (1). A theorem of A. Weil [11, p. 91] says that G is a factor group of $G^* = \prod_\alpha G_\alpha \times A$, where each G_α is a compact, connected, simple Lie group and A is abelian. If G is nonabelian, then there exists a normal subgroup K of G such that G/K is a compact, connected, simple Lie group. If (5) holds, then this implies that $\Gamma(G/K)$ is a local central Λ_p-set; but this contradicts Cecchini's theorem.

Finally, we note that Rider [10] has shown the existence of a sequence of characters χ_m on the unitary group W(n) of degree n such that $\|\chi_m\|_{2+2/n} \to \infty$ as $m \to \infty$. Thus $S_p(W(n)) \neq D_p(W(n))$ for $p \geq 2 + 2/n$. J. F. Price [9] has shown that $\Gamma(SU(2))$ is not a local Λ_p-set for any $p > 1$. Therefore, $S_p(SU(2)) \neq D_p(SU(2))$ for $p \neq 2$.

4. CENTRAL DERIVED ALGEBRAS

In this section we investigate relationships among D_p^Z and S_p^Z, the centers of D_p and S_p, and the derived algebra $\mathscr{D}_p$ of the center of L_p.

Definition 5. The *derived algebra* $\mathscr{D}_p$ *of* L_p^Z is defined by

$$\mathscr{D}_p = \left\{ f \in L_p^Z\colon \sup_{g \in L_p^Z} \frac{\|f * g\|_p}{\|\hat{g}\|_\infty} < \infty \right\}.$$

The *norm* is given by

$$\|f\|_{\mathscr{D}_p} = \sup_{\substack{g \in T^Z(G) \\ \|\hat{g}\|_\infty = 1}} \|f * g\|_p .$$

Browder's principal result [3] is that $S_p^Z = \mathscr{D}_p$ for every compact group G. For the group $G = \prod_1^\infty \mathscr{S}_3$, Browder shows that if $1 \leq p < 4$, then $D_p^Z = \mathscr{D}_p$ if and only if p = 2. Our next theorem shows that this cannot be extended to p = 4.

THEOREM 6. *If* G *is a compact group, then* $D_p^Z = \mathscr{D}_p$ *for* p = 2s (s = 1, 2, 3, ...).

Proof. It is clear that $D_p^Z \subseteq \mathscr{D}_p$ for every p. We must show that $\mathscr{D}_p \subseteq D_p^Z$ for p = 2s, where s is an integer. Now, if $f \in \mathscr{D}_p$ and $f \sim \sum d_\alpha a_\alpha \chi_\alpha$, then

$$\sup \left\{ \|\sum_F d_\alpha a_\alpha b_\alpha \chi_\alpha\|_p\colon |b_\alpha| \leq 1,\ F \in \mathscr{F} \right\} < \infty .$$

We need to show that

$$\sup \left\{ \|\sum_F d_\alpha a_\alpha \operatorname{Tr}(V_\alpha U_\alpha)\|_p\colon V_\alpha \text{ unitary},\ F \in \mathscr{F} \right\} < \infty . \tag{13}$$

Now

$$\sup_{\{b_\alpha\},F} \|\sum_F d_\alpha a_\alpha b_\alpha \chi_\alpha\|_p^p \tag{14}$$
$$= \sup_F \sum_{(\alpha_1,\cdots,\alpha_{2s})} d_{\alpha_1}|a_{\alpha_1}| \cdots d_{\alpha_{2s}}|a_{\alpha_{2s}}| \int_G \chi_{\alpha_1}\bar{\chi}_{\alpha_2}\cdots\bar{\chi}_{\alpha_{2s}}\,dx\,,$$

since $|b_\alpha| \leq 1$ and $\int_G \chi_{\alpha_1}\bar{\chi}_{\alpha_2}\cdots\bar{\chi}_{\alpha_{2s}} \geq 0$ [7, p. 20]. The second sum in (14) extends over all 2s-tuples $(\alpha_1, \cdots, \alpha_{2s})$ of elements of F. We shall show that (13) is bounded by (14). Now

$$\int_G |\sum_F d_\alpha a_\alpha \operatorname{Tr}(V_\alpha U_\alpha)|^p$$
$$= \sum_{(\alpha_1,\cdots,\alpha_{2s})} d_{\alpha_1}a_{\alpha_1}d_{\alpha_2}\overline{a_{\alpha_2}}\cdots d_{\alpha_{2s}}\overline{a_{\alpha_{2s}}}\int_G \operatorname{Tr}(V_{\alpha_1}U_{\alpha_1})\overline{\operatorname{Tr}(V_{\alpha_2}U_{\alpha_2})}\cdots\overline{\operatorname{Tr}(V_{\alpha_{2s}}U_{\alpha_{2s}})}$$
$$= \sum_{(\alpha_1,\cdots,\alpha_{2s})} d_{\alpha_1}a_{\alpha_1}d_{\alpha_2}\overline{a_{\alpha_2}}\cdots d_{\alpha_{2s}}\overline{a_{\alpha_{2s}}}\int_G \operatorname{Tr}\Big(V\Big(m_1 I \oplus \sum \oplus m_i U_{\tau_i}\Big)\Big),$$

where $m_1 I \oplus \sum \oplus m_i U_{\tau_i}$ is a decomposition of the tensor product $U_{\alpha_1} \otimes \bar{U}_{\alpha_2} \otimes \cdots \otimes U_{\alpha_{2s-1}} \otimes \bar{U}_{\alpha_{2s}}$ into irreducible components, and where $V = V_{\alpha_1} \otimes \bar{V}_{\alpha_2} \otimes \cdots \otimes V_{\alpha_{2s-1}} \otimes \bar{V}_{\alpha_{2s}}$. Now

$$\int_G (U_{\tau_i})_{k\ell}\,dx = 0 \qquad (1 \leq k,\ \ell \leq d_{\tau_i}),$$

if $U_{\tau_i} \neq I$. Therefore the left side of (13) is majorized by

$$\sum_{(\alpha_1,\cdots,\alpha_{2s})} |d_{\alpha_1}a_{\alpha_1}\cdots d_{\alpha_{2s}}\overline{a_{\alpha_{2s}}}|\, m_1$$
$$= \sum_{(\alpha_1,\cdots,\alpha_{2s})} d_{\alpha_1}|a_{\alpha_1}|\cdots d_{\alpha_{2s}}|a_{\alpha_{2s}}|\int_G \chi_{\alpha_1}\overline{\chi_{\alpha_2}}\cdots\overline{\chi_{\alpha_{2s}}},$$

since $\int_G \chi_{\alpha_1}\overline{\chi_{\alpha_2}}\cdots\overline{\chi_{\alpha_{2s}}}$ is the number of times the identity representation appears in $U_{\alpha_1} \otimes \bar{U}_{\alpha_2} \otimes \cdots \otimes U_{\alpha_{2s-1}} \otimes \bar{U}_{\alpha_{2s}}$.

For $1 \leq p < 2$, we have a result analogous to Theorem 1.

THEOREM 7. *The following are equivalent.*

(1) $\Gamma(G)$ *is a local central* Λ_2-set.

(2) $D_p^Z = \mathscr{D}_p = L_2^Z$ *for all* p $(1 \leq p \leq 2)$.

(3) $D_p^Z = \mathscr{D}_p$ *for some* p $(1 \leq p < 2)$.

Proof. Because the proof is almost identical with the proof of Theorem 1, we omit it.

For connected groups, more can be said. The following, however, is stated as a conjecture, since I was not able to establish one direction of the proof.

CONJECTURE. *If* G *is a compact, connected group, then the following conditions can be added to Theorem* 7.

(4) $G = \dfrac{\prod_I G_i \times A}{N}$, *where each* G_i *is a compact, connected, simple Lie group,* A *is abelian, and* I *is finite.*

(5) $\dfrac{G}{Z(G)}$ *is semisimple, where* $Z(G)$ *is the center of* G.

Every compact, connected group can be written as in (4), where N is a closed normal subgroup of $\prod_I G_i \times A$ and I is an index set of arbitrary cardinality.

It is easy to see that (4) and (5) are equivalent. To see that (4) implies (1) of Theorem 7, let G be a compact, connected, simple Lie group, and let T be a maximal torus of G. The Weyl integration formula [1] states that

$$\int_G f\,dx = \frac{1}{W}\int_T f(t)\,|D(t)|^2\,dt$$

for all central functions f on G, where W is the order of the Weyl group, and where D is a polynomial on T with integral coefficients. Thus $\chi_\alpha(t)\,D^2(t)$ is also a polynomial on T with integral coefficients. Therefore,

$$\int_G |\chi_\alpha(x)|\,dx = \frac{1}{W}\int_T |\chi_\alpha(t)|\,|D(t)|^2\,dt \geq \frac{1}{W}$$

for all $\alpha \in \Gamma(G)$. This gives (1) for the case where G is a compact, connected Lie group. Condition (1) clearly holds for a finite product of such groups G as well as for any abelian group A. Also, (1) holds for factor groups of groups that satisfy (1). This gives the sufficiency of (4).

Showing the necessity of (4) amounts to showing that if I is infinite, then (1) does not hold. It would suffice to show the existence of an $A < 1$ such that for each compact, connected, simple Lie group G there exists an $\alpha \in \Gamma(G)$ with $\|\chi_\alpha\|_1 \leq A$. Lemma 3 contains a proof of this for the unitary groups. I have not been able to show this for the general case, even though it seems very likely to be true.

Finally, we give an analogue of Theorem 3. The proof is similar to that of Theorem 3, and we omit it.

THEOREM 8. *If* $\sup_{\alpha\in\Gamma} d_\alpha < \infty$ *and* $1 \leq p < \infty$, *then* $D_p^Z = \mathscr{D}_p$.

REFERENCES

1. J. F. Adams, *Lectures on Lie groups.* Benjamin, New York, 1969.

2. G. F. Bachelis, *On the ideal of unconditionally convergent Fourier series in* $L_p(G)$. Proc. Amer Math. Soc. 27 (1971), 309-312.

3. D. S. Browder, *Derived algebras in* L_1 *of a compact group.* Pacific J. Math. 43 (1972), 39-49.

4. C. Cecchini, *Lacunary Fourier series on compact Lie groups.* J. Functional Analysis 11 (1972), 191-203.

5. A. Figà-Talamanca and D. Rider, *A theorem of Littlewood and lacunary series for compact groups.* Pacific J. Math. 16 (1966), 505-514.

6. S. Helgason, *Multipliers of Banach algebras.* Ann. of Math. (2) 64 (1956), 240-254.

7. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. II: *Structure and analysis for compact groups. Analysis on locally compact abelian groups.* Die Grundlehren der mathematischen Wissenschaften, Volume 152. Springer-Verlag, New York, 1970.

8. C. C. Moore, *Groups with finite dimensional irreducible representations.* Trans. Amer. Math. Soc. 166 (1972), 401-410.

9. J. F. Price, *Non ci sono insiemi infiniti di tipo* $\Lambda(p)$ *per* SU_2. Boll. Un. Mat. Ital. (4) 4 (1971), 879-881.

10. D. Rider, *Norms of characters and central* Λ_p *sets of* $U(n)$. Proceedings of the Conference on Harmonic Analysis. Springer Lecture Notes, No. 266, Springer-Verlag, New York, 1972; pp. 287-294.

11. A. Weil, *L'intégration dans les groupes topologiques et ses applications.* Actualités Sci. Ind. 869-1145. Hermann, Paris, 1938.

Kansas State University
Manhattan, Kansas 66506