ON THE DIFFERENTIABILITY OF RADEMACHER SERIES

J. E. Coury

In this paper, we present a detailed investigation of the differentiability properties of Rademacher series. A number of authors, among them L. A. Balašov [1], J. R. McLaughlin [13], [14], and A. I. Rubinstein [16], have considered related problems. (For a brief survey of the literature, see Balašov and Rubinstein [2, pp. 748, 749].) A principal theme in the present paper concerns the category- and measure-theoretic properties of the points of differentiability of Rademacher series, and we draw on the general theory of derivatives as developed in [20] by Z. Zahorski.

In Section 2, a "zero-one" law is proved for the set of points of differentiability. Using a result of M. K. Fort [7], we show that this set is of the second category on $[0, 1)$ if and only if the series is piecewise linear. It is also shown that if a Rademacher series possesses a nonzero derivative at even one point, then the sequence of coefficients is eventually strictly monotone.

We show in Section 3 that differentiability of a Rademacher series at at least one point is sufficient to guarantee that the series is of bounded pth variation for every $p > 1$. In Section 4, a necessary and sufficient condition is obtained for a Rademacher series to be continuous in the Darboux sense (that is, for the series to carry connected sets into connected sets). It is shown that a Rademacher series that is Darboux-continuous cannot possess a derivative at any point of $[0, 1)$, except in the case where the series is piecewise linear.

Section 5 deals with series satisfying Lusin's condition (N), that is, series that map nullsets into nullsets. For Rademacher series, we show that this condition is equivalent to the preservation of measurable sets. It is also shown that differentiability at even one point is sufficient to imply condition (N).

The Dini derivates of Rademacher series are examined in detail in Section 6. We show that on a residual set in $[0, 1)$, the upper and lower derivates are infinite and of opposite sign. This set is further shown to be of full measure in $[0, 1)$ unless the series has a derivative almost everywhere.

In the last two sections, we treat the problem of determining necessary and sufficient conditions for a Rademacher series to be of bounded variation. We show, in particular, that a Rademacher series is differentiable almost everywhere on $[0, 1)$ if and only if it is of bounded variation. This result answers a question raised by McLaughlin in [13], and it is an analogue for the Walsh system of the corresponding result for lacunary trigonometric series (McLaughlin [12]). Explicit representations for the total variation of a Rademacher series are also obtained. Finally, we show that if a Rademacher series possesses a nonzero derivative almost everywhere in $[0, 1)$, there exists a perfect set of positive measure on which the series is strictly monotone.

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1. PRELIMINARIES

Let $x \in [0, 1)$ have the dyadic expansion $\sum_{k=1}^{\infty} x_k/2^k$, where x_k is 0 or 1; in the case of dyadic rationals, we take the finite development. (We shall refer to x_k as the kth coordinate of x.) For $k \geq 1$, define the $(k - 1)$st Rademacher function r_{k-1} evaluated at x to be $(-1)^{x_k}$. (It should be noted that this definition is different from that given, for example, in [1] and [13], where $r_k(x) = (-1)^{x_k}$; the present definition, however, is consistent with the usual Walsh-Paley ordering of the Walsh system.) Addition of points in [0, 1) is always assumed to be modulo 1.

R(x) will denote the sum of the Rademacher series $\sum_{k=0}^{\infty} c_k r_k(x)$. We suppose in the sequel that this series is absolutely convergent, for in the contrary case R(x) is not continuous, and thus certainly not differentiable, at any point of [0, 1).

For $k \geq 0$, define $R_k = c_{k+1} + c_{k+2} + \cdots$. (The size of R_k plays an important role in the monotonicity of a Rademacher series; see, for example, [1, pp. 5-7].)

By $D^+R(x)$ and $D_+R(x)$ we denote the upper and lower right Dini derivates of R(x), respectively; by $D^-R(x)$ and $D_-R(x)$ we denote the upper and lower left derivates. The function R(x) is differentiable at a point if all four derivates at that point are equal and finite.

Finally, if C is a subset of [0, 1), -C denotes the set of points 1 - x for which x belongs to C.

2. GENERAL DIFFERENTIABILITY PROPERTIES

In many cases, Rademacher series satisfy a "zero-one" law, that is, a particular condition holds either almost everywhere on [0, 1) or almost nowhere. This phenomenon is also observed with respect to the differentiability of Rademacher series.

LEMMA 1. *Let* A *be a subset of the real line, and let* C *be a countable dense set. Define* $B = \bigcup (A + c)$, *where the union extends over* $c \in C$. *Then either* B *or its complement has Lebesgue measure* 0.

Proof. We may suppose that $m(A) > 0$. If $m(B') \neq 0$, Steinhaus's theorem implies that the difference set $B' - A$ contains an interval and therefore a point c of C. Writing $c = b' - a$, where $b' \in B'$ and $a \in A$, we then have the relation $b' = a + c \in B$. This contradiction establishes the lemma.

THEOREM 1. *Define* $\Delta = \{x\colon R \text{ is differentiable at } x\}$. *Then*

(i) Δ *is dense if* Δ *is not empty;*

(ii) Δ *has measure* 0 *or* 1.

Proof. Suppose that $x_0 \in \Delta$, and let α be a dyadic rational in [0, 1). The addition of α to x_0 affects only finitely many of the dyadic digits of x_0, and therefore $\lim_{h\to 0} [R(x_0 + \alpha + h) - R(x_0 + \alpha)]/h$ exists and equals $R'(x_0)$. Since α is arbitrary, it follows that $\Delta + \alpha$ is contained in Δ for every dyadic rational α, and hence $\bigcup_\alpha (\Delta + \alpha) \subset \Delta$. Thus Δ is dense, and by the previous lemma, either Δ or its complement has full measure in [0, 1).

If $R(x)$ is a finite sum, then $R(x)$ is a step function whose intervals of constancy have dyadic rational endpoints. Thus the series is differentiable except on a finite set, and Δ in this instance is of the second category. More generally, if the coefficients of $R(x)$ are eventually of the form $c_k = A/2^{k+1}$, say for $k \geq N$, then $R(x)$ is piecewise linear: for in this case, $R(x)$ is equal to a translate of the function $A(1/2^N - 2x)$ on each dyadic subinterval $[i/2^N, (i+1)/2^N)$. (One shows this in the same way one proves that $\sum_{k=0}^{\infty} A\,2^{-(k+1)} r_k(x)$ converges pointwise to $A(1 - 2x)$; see [19].) Thus in this case also, $R'(x)$ exists except at a finite number of points, and Δ is again a set of second category.

We show next that these are the only instances in which Δ can be of the second category in $[0, 1)$.

LEMMA 2 ([7], [10]). *Let* f *be a finite, real-valued function on* $[0, 1]$. *Let* D *denote the set of points of discontinuity, and* Δ *the points of differentiability of* f. *If* D *is dense in* $[0, 1]$, *then* Δ *is of the first category.*

THEOREM 2. *Suppose that* $R(x)$ *is not piecewise linear (equivalently, that the coefficients of* $R(x)$ *are not eventually of the form* $A/2^k$). *Then* Δ *is of the first category in* $[0, 1)$.

Proof. In view of the preceding lemma, it is enough to show that the set D of points of discontinuity of $R(x)$ is dense. By [13, p. 373], D is a subset of the dyadic rationals in $[0, 1)$. If D is finite, then $R(x)$ is continuous at $p/2^k$ for all sufficiently large values of k and all odd p; hence $c_k = R_k$ eventually ([13, p. 374]). This implies that $c_k = A/2^k$ (with A possibly 0) for all sufficiently large k, contrary to hypothesis.

Thus D is infinite and therefore contains an infinite sequence of dyadic rationals $p_k/2^{q_k}$, where each p_k is odd. However, by [13, p. 374], it follows that $p/2^{q_k}$ belongs to D for each q_k and *all* odd values of p. The set of all such $p/2^{q_k}$, with p odd, is evidently dense in $[0, 1)$, and therefore the same is true of D.

COROLLARY 1. *The set* Δ *is of second category if and only if the equation* $c_k = R_k$ *is satisfied for all sufficiently large* k.

It follows from an example in [1] that Δ may be nonempty although only of measure 0. Theorem 1, however, guarantees that Δ is at least countably infinite. One may ask whether in such instances Δ can be uncountable. (A category argument provides no information, since Δ is of first category.)

Balašov has shown in [1] that if $R(x) = \sum c_n r_n(x)$ possesses a derivative at even a single point, then the sequence $\{2^n c_n\}$ tends to a limit and

$$R'(x) = -2 \lim 2^{n+1} c_n$$

wherever the derivative exists. Thus if $R'(x)$ is nonzero at a point, the coefficients are eventually either positive or negative. However, much more can be said about the coefficients in this case.

PROPOSITION 1. *Suppose that* $R'(x)$ *exists at at least one point, and set* $A = \lim 2^n c_n$. *Then the sequence* $\{c_n\}$ *is eventually either strictly decreasing or strictly increasing to* 0, *according as* $A > 0$ *or* $A < 0$.

Proof. It is enough to prove the result for $A > 0$. By [1, p. 2], $R(x)$ is differentiable at $x_0 = 1/3$. Since the nth dyadic digit of x_0 is 0 if n is odd and 1 if n is even, we see that

$$I(x_0, 2^{-2n-2}) = 2^{2n+3}(-c_{2n} + c_{2n+1}),$$

where I(x, h) denotes the incrementary ratio $[R(x+h) - R(x)]/h$. The right side of this expression is negative for all sufficiently large values of n, and thus the inequality $c_{2n} > c_{2n+1}$ holds eventually.

Similarly, R(x) is differentiable at $y_0 = 2/3$. The nth coordinate of y_0 is 0 or 1, according as n is even or odd; hence for all sufficiently large n,

$$I(y_0, 2^{-2n-1}) = 2^{2n+2}(-c_{2n-1} + c_{2n})$$

is negative. It follows that $c_{2n-1} > c_{2n}$ for all large n, which, together with the previous inequality, implies that $\{c_n\}$ is eventually strictly decreasing.

For the case where A = 0, the coefficients need satisfy no such monotonicity condition: consider, for example, the sequence $\{c_n\}$ defined by $c_n = (-1)^n/n2^n$.

3. BOUNDED pth VARIATION

McLaughlin [13] has given a sufficient condition for a Rademacher series to be of bounded pth variation. The following result shows that differentiability of the series at even a single point is enough to guarantee bounded pth variation.

PROPOSITION 2. *Suppose that* R'(x) *exists at at least one point. Then* R(x) *is of bounded* p*th variation for every* $p > 1$.

Proof. By [1], $A = \lim 2^n c_n$ exists and is finite. First suppose that $A \neq 0$; then $\lim |c_{n+1}/c_n| = (1/2) \lim |2^{n+1} c_{n+1}/2^n c_n| = 1/2$. Hence, by the ratio test, the series $\sum 2^{n/p} |c_n|$ converges for every $p > 1$, and it follows from [13, p. 376] that the pth variation of R(x) is finite.

Assume now that A = 0, and consider the series $R^* = R + H$, where $H(x) = \sum 2^{-(n+1)} r_n(x)$. Since R^* and R are differentiable at precisely the same set of points, it follows from the preceding paragraph that $R^*(x)$ is of bounded pth variation, for each $p > 1$. Thus R(x) itself is of bounded pth variation, since H(x) is linear.

COROLLARY 2 (see [4, p. 65]). *Suppose that* R(x) *possesses a derivative at at least one point. Then* $\sum |c_n|^\gamma < \infty$ *for every* $\gamma > 0$.

4. THE DARBOUX PROPERTY

A finite function f possesses the *Darboux property,* or is *continuous in the Darboux sense,* if in each interval [a, b] it assumes every value between f(a) and f(b).

PROPOSITION 3. *Suppose that* $\{c_n\}$ *contains infinitely many nonzero terms. Then*

(i) *a necessary and sufficient condition for a Rademacher series* $R(x) = \sum c_n r_n(x)$ *to be continuous in the Darboux sense is that* $c_n \leq R_n$ *for every* n;

(ii) *more generally, if* $c_n \leq R_n$ *for all* $n \geq N$, *then* R(x) *has the Darboux property on each dyadic interval* $[i/2^N, (i+1)/2^N)$.

Proof. The proof of the sufficiency of the condition in (i) is essentially the same as the proof of Riemann's result on rearrangements of conditionally convergent series. To prove the necessity, suppose that $c_K > R_K$ for some K, and let $W = \{R(x): x \in [0, 1)\}$. One easily verifies in this case that W cannot contain the interval

$$(c_0 + \cdots + c_{K-1} - c_K + R_K, \; c_0 + \cdots + c_{K-1} + c_K - R_K)$$

and therefore cannot be continuous in the Darboux sense.

Part (ii) is obvious because the function $\sum_{n=0}^{N-1} c_n r_n(x)$ is constant on dyadic intervals of order N.

The proposition is not true if $\{c_n\}$ is eventually 0 (consider the series $R(x) = -r_0(x)$). However, if $c_n = 0$ for all $n \geq N$, the series evidently has the Darboux property on dyadic intervals of order N.

We show next that a Rademacher series with the Darboux property cannot be differentiable at even a single point unless it is piecewise linear.

LEMMA 3. *If* $\{c_n\}$ *is not eventually* 0 *and* $\lim 2^n c_n = 0$, *then* $c_n > R_n$ *holds for infinitely many* n.

Proof. Set $\alpha_k = 2^k |c_k|$; then $|R_n| \leq \sum_{k=n+1}^{\infty} \alpha_k 2^{-k} \leq 2^{-n} \max\{\alpha_k : k > n\}$ for each n, and thus $\{2^n R_n\}$ converges to 0. For $i = 0, 1, \cdots, 2^n - 1$, define $I_n^i = [i2^{-n}, (i+1)2^{-n})$, and let W denote the set of values assumed by R(x). For each $n \geq 1$, it is easily verified that $R(I_n^i)$ is contained in

$$[\varepsilon_0 c_0 + \cdots + \varepsilon_{n-1} c_{n-1} - R_{n-1}, \; \varepsilon_0 c_0 + \cdots + \varepsilon_{n-1} c_{n-1} + R_{n-1}],$$

where $\varepsilon_j = \pm 1$. It follows that W is contained in the union of 2^n intervals, each of length $2R_{n-1}$. Thus $m(W) \leq 2^{n+1} R_{n-1}$ for every $n \geq 1$, and hence the measure of W is 0.

If the inequality $c_n \leq R_n$ is satisfied eventually, the previous proposition implies that R(x) has the Darboux property on a (dyadic) subinterval of [0, 1). In particular, W is of positive measure. This contradiction establishes the lemma.

PROPOSITION 4. *Suppose that* $c_n \leq R_n$ *for all sufficiently large values of* n. *Then* R(x) *possesses a derivative at no point of* [0, 1) *unless it is piecewise linear (that is, unless* $c_n = R_n$ *eventually).*

Proof. If $c_n = 0$ for all large n, R(x) is evidently piecewise linear. Thus suppose that infinitely many c_n are nonzero. If R'(x) exists at some point, then $A = \lim 2^n c_n$ exists, and the previous lemma shows that A cannot be 0. Define $R^*(x) = \sum c_n^* r_n(x)$, where $c_n^* = c_n - A/2^n$; then $\{2^n c_n^*\}$ converges to 0, and $c_n^* \leq c_{n+1}^* + c_{n+2}^* + \cdots$ for all sufficiently large n. It follows from Lemma 3 that $c_n = A/2^n$ eventually, and hence that R(x) is piecewise linear.

COROLLARY 3. *Suppose that* R(x) *is continuous in the Darboux sense. Then either* Δ *is empty, or it has a finite complement in* [0, 1).

5. LUSIN'S CONDITION (N)

A finite function satisfies *condition* (N), or is an N-*function*, if it maps sets of measure zero into sets of measure zero. (For a discussion of condition (N), see [17, pp. 224-228].) The importance of this condition is seen in the following result.

PROPOSITION 5. *Let* R(x) *be an absolutely convergent Rademacher series. Then a necessary and sufficient condition for* R(x) *to preserve measurable sets is that* R(x) *satisfy condition* (N).

Proof. First suppose that R(x) is an N-function. Let E be measurable, and write $E = \left(\bigcup F_n\right) \cup C$, where each F_n is closed and C is a nullset. R(C) is measurable, since it has zero measure. Also, each $R(F_n)$ is a Borel set, since R(x) is continuous except possibly on a countable set. Thus R(E) is measurable.

Conversely, if R is not an N-function, there is a nullset E for which R(E) has positive outer Lebesgue measure. Thus R(E) contains a nonmeasurable subset H. If we set $C = R^{-1}(H) \cap E$, it follows that C is measurable but R(C) = H is not.

LEMMA 4. *If* $\{2^n R_n\}$ *is bounded, then* R(x) *is an* N-*function.*

Proof. By multiplying R(x) by a constant if necessary, we may assume that $2^n R_n \leq 1/4$ for each n, and hence that $2R_n \leq 1/2^{n+1}$. We then have the relations $m(R(I_n^i)) \leq 2R_{n-1} \leq 1/2^n = m(I_n^i)$, where I_n^i is defined as in the proof of Lemma 3. It follows that R(x) does not increase the measure of open sets, and this in turn implies condition (N).

COROLLARY 4. *Suppose that* $\{2^n c_n\}$ *is bounded. Then* R(x) *is an* N-*function.*

Proof. Write $2^n |c_n| = \alpha_n$, and let M be such that $\alpha_n \leq M$ for each n. Then $|R_n| \leq \sum_{k=n+1}^{\infty} \alpha_k/2^k \leq M/2^n$, and hence $\{2^n R_n\}$ is bounded.

PROPOSITION 6. *If* R(x) *possesses a derivative at at least one point, then* R(x) *satisfies condition* (N).

The hypothesis in the previous proposition, while sufficient to ensure condition (N), is not necessary. Consider, for example, the sequence defined by $c_{2k} = 1/2^{2k+2}$ and $c_{2k+1} = 1/2^{2k+1}$, for $k \geq 0$. Since $\lim 2^n c_n$ does not exist, the corresponding series possesses a derivative at no point.

6. DINI DERIVATES

It is well known (see, for example, [9, pp. 25-34]) that a Rademacher series is almost everywhere convergent if and only if its sequence of coefficients is square-summable. However, if the series is nonabsolutely convergent, the set of points of divergence always forms a residual set (though possibly of measure zero). In fact, more can be said in this case about the points where the series diverges: except on a set of first category, the upper and lower limits of the sequence of partial sums are $+\infty$ and $-\infty$, respectively.

A similar phenomenon occurs at the points of nondifferentiability of a Rademacher series: at almost all such points, and on a residual set in [0, 1), the upper and lower derivates are infinite and of opposite sign. More precisely, we have the following result.

THEOREM 3. *Suppose that* R(x) *is not piecewise linear, and let*

$$K = \{x: D^+R(x) = D^-R(x) = +\infty,\ D_+R(x) = D_-R(x) = -\infty\}$$

denote the set of "knot-points" of R(x). *Then* K *is residual in* [0, 1). *Further, if* R(x) *is differentiable only on a set of measure* 0, *then* K *has measure* 1.

Proof. Let E represent the set of points where R(x) is continuous and at least one of the four derivates is infinite. By [18, p. 189], E is residual and hence uncountable; thus there exists a dyadic irrational $z \in E$ for which at least one of $D^+R(z)$, $D^-R(z)$, $D_+R(z)$, and $D_-R(z)$ is infinite. With no loss of generality, suppose that $D^+R(z) = +\infty$; then plainly $D^+R(z+\alpha) = +\infty$ for each dyadic rational α. Define $K^+ = \{x: D^+R(x) = +\infty\}$; then $K^+ = \bigcap_{n=1}^{\infty} K_n^+$, where K_n^+ denotes the set $\{x: D^+R(x) \geq n\}$. Since K^+ is dense, so also is each K_n^+.

By [20, p. 8], each K_n^+ can be written in the form $G \cup K_0$, where G is a G_δ-subset of the points of continuity of R(x) (thus G contains only dyadic irrationals) and K_0 is a subset of the points of discontinuity. Since each K_n^+ contains the set $T(z) = \{z + \alpha: \alpha$ is a dyadic rational$\}$ and since $z + \alpha$ is dyadically irrational for each such α, it follows that G contains T(z). Hence G is a dense G_δ-set and is therefore residual; thus K_n^+ is residual. It follows that K^+ is itself residual.

One easily verifies that if x is not dyadically rational, then $r_n(-x)$ equals $-r_n(x)$ for every n, where -x denotes the point 1 - x. By [15, pp. 194, 195], at each point x of continuity of R one may calculate each of the derivates by taking the appropriate limit over sequences of dyadic rationals tending to 0. If $\{h_n\}$ is such a sequence and x is dyadically irrational, then

$$\begin{aligned} D^+R(-x) &= \limsup_{h_n \to 0+} \frac{R(-x+h_n) - R(-x)}{h_n} = \limsup_{h_n \to 0+} \left(- \frac{R(x-h_n) - R(x)}{h_n} \right) \\ &= -\liminf_{h_n \to 0+} \frac{R(x-h_n) - R(x)}{h_n} = -\liminf_{h_n \to 0-} \left(- \frac{R(x+h_n) - R(x)}{h_n} \right) \\ &= \limsup_{h_n \to 0-} \frac{R(x+h_n) - R(x)}{h_n} = D^-R(x). \end{aligned}$$

Thus $D^+R(-x) = D^-R(x)$ for every dyadic irrational x, and hence

$$K^- = \{x: D^-R(x) = +\infty\}$$

is also residual.

Now suppose that R' exists only on a set of measure zero. By [17, p. 271], one of the following four relations holds for R(x) (or indeed for any finite function) at almost every x:

(1) $D^+R(x) = D^-R(x) = +\infty, \quad D_+R(x) = D_-R(x) = -\infty;$

(2) $D^+R(x) = D_-R(x) \neq \pm\infty, \quad D_+R(x) = -\infty, \quad D^-R(x) = +\infty;$

(3) $D_+R(x) = D^-R(x) \neq \pm\infty, \quad D^+R(x) = +\infty, \quad D_-R(x) = -\infty;$

(4) $D^+R(x) = D_+R(x) = D^-R(x) = D_-R(x) \neq \pm\infty.$

Let A_2, A_3, A_4 denote the set of points where (2), (3), and (4) hold, respectively. Each of these sets is evidently invariant under translation by dyadic rationals, and thus each has measure 0 or 1. Since A_4 is the set Δ, it follows by hypothesis that $m(A_4) = 0$. If A_2 has measure 1, the set $A_2 \cap (-A_2)$ also has full measure in $[0, 1)$; since $D^+R(-x) = D^-R(x)$ if x is dyadically irrational, we conclude that (2) cannot hold at any point of $A_2 \cap (-A_2)$, a contradiction. Thus A_2 has measure 0, and we show similarly that A_3 is a nullset. It follows that condition (1) holds at almost every point of $[0, 1)$.

In particular, the set $K_+ = \{x: D_+R(x) = -\infty\}$ is nonempty and therefore dense. Since the lower derivate has the value $\underline{D}R(x) = \inf\{D_+R(x), D_-R(x)\}$, it follows from [5, p. 74] that $\underline{K} = \{x: \underline{D}R(x) = -\infty\}$ is residual. Write $\underline{K} = K_+ \cup K_-$, where $K_- = \{x: D_-R(x) = -\infty\}$. Arguing as before, we conclude (since $D_+R(-x) = D_-R(x)$ for dyadically irrational values of x) that both K_+ and K_- are residual. It follows that K, being the intersection of K^+, K^-, K_+, and K_-, is itself a residual set in $[0, 1)$.

Finally, suppose that R' exists almost everywhere in $[0, 1)$; with no loss of generality, we may assume that $\lim 2^n c_n > 0$. To conclude the proof, it is enough to show, as above, that the set K_+ is nonempty. We first note that if a point x has a 0 for its $(n + 1)$st coordinate and has its kth dyadic digit equal to 1 for $n + 1 < k \leq m$, then the incrementary ratio $I(x, 2^{-m})$ is equal to

$$-2^{m+1}(c_n - (c_{n+1} + \cdots + c_{m-1})).$$

We may assume that $c_n > R_n$ for infinitely many n, since in the contrary case, Proposition 4 implies that the derivative exists at no point (R is by hypothesis not piecewise linear).

Let N be such that $c_n > 0$ for every $n \geq N$. To construct a point ω in K_+, we first choose n_1 to be the first integer exceeding N for which $c_{n_1} > R_{n_1}$. Let ω have first coordinate 0 and kth coordinate 1, for $2 \leq k \leq n_1$. Next choose n_2 so that $c_{n_2} > R_{n_2}$, and so large that $2^{n_2}(c_{n_1} - R_{n_1}) > 2$; set $\omega_{n_1+1} = 0$ and $\omega_k = 1$ for $n_1 + 1 < k \leq n_2$. Proceeding by induction, if n_m has been determined, we choose n_{m+1} sufficiently large so that

$$c_{n_{m+1}} > R_{n_{m+1}} \quad \text{and} \quad 2^{n_{m+1}}(c_{n_m} - R_{n_m}) > m + 1;$$

set $\omega_{n_m+1} = 0$ and $\omega_k = 1$ for each k $(n_m + 1 < k \leq n_{m+1})$. For each m, we then have the relations

$$I(\omega, 2^{-n_m}) = -2^{n_m+1}(c_{n_{m-1}} - (c_{n_{m-1}+1} + \cdots + c_{n_m-1}))$$

$$< -2^{n_m+1}(c_{n_{m-1}} - R_{n_{m-1}}) < -2m.$$

It follows that $D_+R(\omega) = -\infty$, and therefore K_+ is nonempty. This completes the proof of the theorem.

7. FUNCTIONS OF BOUNDED VARIATION

In this section, we answer a question raised by McLaughlin in [13, p. 377]. We prove that a Rademacher series that is differentiable a.e. on $[0, 1)$ is necessarily of bounded variation. (This is an analogue for the Walsh system of the corresponding result for lacunary trigonometric series, proved by McLaughlin [12].) We begin with a number of preliminary results.

LEMMA 5. *Suppose that* $f(x)$ *is right-continuous on* $[0, 1)$ *and that* $f(x)$ *has total variation* M *on a dense subset* E. *Then* $f(x)$ *is of bounded variation on* $[0, 1)$ *and* $V_0^1 f = M$.

Proof. Let $\delta > 0$, and consider a partition $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$ of $[0, 1)$. For $i = 0, 1, \cdots, n$, choose $h_i \in E$ so that $x_i < h_i < x_{i+1}$ and $|f(h_i) - f(x_i)| < \delta/2^i$. Then, for each i,

$$|f(x_{i+1}) - f(x_i)| \leq |f(x_{i+1}) - f(h_{i+1})| + |f(h_{i+1}) - f(h_i)| + |f(h_i) - f(x_i)|$$
$$< |f(h_{i+1}) - f(h_i)| + 3\delta/2^{i+1},$$

and thus

$$\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| < \sum_{i=0}^{n-1} |f(h_{i+1}) - f(h_i)| + 3\delta \leq M + 3\delta.$$

Because δ is arbitrary, we conclude that $V_0^1 f \leq M$. Since the variation of f over $[0, 1)$ is at least M, the result follows.

PROPOSITION 7. *For a Rademacher series* $R(x) = \sum_{k=0}^{\infty} c_k r_k(x)$ *and each positive integer* N,

$$\frac{1}{2} \sum_{i=0}^{2^N-2} |R((i+1)/2^N) - R(i/2^N)| = \sum_{n=0}^{N-1} 2^n |c_n - (c_{n+1} + \cdots + c_{N-1})|$$

(*where, for* $n = N - 1$, *we set* $c_n - (c_{n+1} + \cdots + c_{N-1}) \equiv c_{N-1}$).

Proof. First note that in the left member of the expression above, R may be replaced by R_N, where $R_N(x) = \sum_{k=0}^{N-1} c_k r_k(x)$. For $0 \leq n \leq N - 1$, let D_n denote the set of dyadic rationals of the form $p/2^{n+1}$, with p odd. If $\alpha \in D_n$, then $r_k(\alpha) - r_k(\alpha - 1/2^N)$ is 0, -2, or 2, according as $0 \leq k \leq n - 1$, $k = n$, or $n + 1 \leq k \leq N - 1$. Thus

$$\frac{1}{2} \sum_{i=0}^{2^N-2} |R((i+1)/2^N) - R(i/2^N)| = \frac{1}{2} \sum_{n=0}^{N-1} \sum_{\alpha \in D_n} |R_N(\alpha) - R_N(\alpha - 1/2^N)|$$
$$= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{\alpha \in D_n} 2 |c_n - (c_{n+1} + \cdots + c_{N-1})|$$
$$= \sum_{n=0}^{N-1} 2^n |c_n - (c_{n+1} + \cdots + c_{N-1})|.$$

In view of this result, the sequence $\left\{\sum_{n=0}^{N} 2^n \left|c_n - (c_{n+1} + \cdots + c_N)\right|\right\}_{N=0}^{\infty}$ is always nondecreasing: successive values of N represent the variation of R(x) over larger sets of dyadic rationals. Appealing to Lemma 5, with E taken to be the set of dyadic rationals in [0, 1), we have the following result.

COROLLARY 5. *The total variation over* [0, 1) *of a Rademacher series* R(x) *is given by*

$$V_0^1 R(x) = 2 \limsup_{N \to \infty} \sum_{n=0}^{N} 2^n \left|c_n - (c_{n+1} + \cdots + c_N)\right| .$$

Thus R(x) *is of bounded variation on* [0, 1) *if and only if*

$$\limsup_{N \to \infty} \sum_{n=0}^{N} 2^n \left|c_n - (c_{n+1} + \cdots + c_N)\right| < \infty .$$

For the case where each c_n is nonnegative and $c_n \geq R_n$ for every n, the sum $\sum_{n=0}^{N} 2^n \left|c_n - (c_{n+1} + \cdots + c_N)\right|$ reduces to $\sum_{n=0}^{N} c_n$. Thus the total variation of R(x) in this case cannot exceed 2M, where $M = \sum_{n=0}^{\infty} c_n$. Since R(0) = M and $\lim_{\varepsilon \downarrow 0} R(1 - \varepsilon) = -M$, it follows that the variation exceeds 2M if there exist a pair of points $x < y$ for which $R(x) < R(y)$. We have thus shown the following.

COROLLARY 6 (compare [1, p. 5]). *Let* $\{c_n\}$ *be a nonnegative sequence such that* $c_n \geq R_n$ *for every* n. *Then* R(x) *is decreasing on* [0, 1).

The next lemma, which describes the set of points where the incrementary ratio satisfies a "favorable" estimate, is the main tool required to show that a Rademacher series possessing a derivative almost everywhere is of bounded variation.

LEMMA 6. *Suppose that* R'(x) *exists* a.e. *on* [0, 1). *For* $n \geq 0$, *define sets*

$$E_n = \left\{x \in \Delta\colon 0 < h < 2^{-n} \text{ implies } \left|\frac{R(x+h) - R(x)}{h}\right| \leq |D| + 1\right\},$$

where $D = \lim 2^{n+2} c_n$. *Let* γ_{n+1} *denote the number of odd integers* j *for which the interval* $[j/2^{n+1}, (j+1)/2^{n+1})$ *contains neither points of* E_n *nor points* $x^* = x + 1/2^{n+1}$, *with* $x \in E_n$, *such that*

$$\left|\frac{R(x^* + 1/2^{n+1}) - R(x^*)}{1/2^{n+1}}\right| \leq 3(|D| + 1) .$$

Then the sequence $\{\gamma_n\}_{n=1}^{\infty}$ *is bounded.*

Proof. The value of the derivative, wherever it exists, is equal to -D; thus Δ is the union of the sets E_n. Since $E_0 \subset E_1 \subset \cdots$, we see that $\lim m(E_n) = 1$. Choose N so that $m(E_N) > 3/4$.

(I) For $i = 0, 1, \cdots, 2^n - 1$, define $I_n^i = [i/2^n, (i+1)/2^n)$ and set $E_n^i = E_n \cap I_n^i$. Then there exists an *even* integer i such that $m(E_N^i) > 1/2^{N+1}$: for otherwise, writing

$$E_N = \left(\bigcup_{i \text{ even}} E_N^i\right) \cup \left(\bigcup_{i \text{ odd}} E_N^i\right),$$

we would obtain the inequality $m(E_N) \leq 2^{N-1} \cdot \frac{1}{2^{N+1}} + \frac{1}{2} = \frac{3}{4}$, contrary to the choice of E_N.

We next show that when i and k are even, E_N^i and E_N^k have identical measure. First note that if i is even, each point of I_N^i has its Nth dyadic digit equal to 0. Also, if $0 < h < 1/2^N$, the Nth coordinate of h is 0. Therefore

$$R(x + h) - R(x) = R(y + h) - R(y),$$

where $x \in I_N^i$ and $y \equiv x + (k - i)/2^N \in I_N^k$. (This is clear, since x and y have identical developments from the Nth digit on, and since the first N - 1 terms cancel in each of the differences $R(x + h) - R(x)$ and $R(y + h) - R(y)$.) It follows that x belongs to E_N^i if and only if $x + (k - i)/2^N$ belongs to E_N^k. Thus, if i and k are even, E_N^i and E_N^k are translates of each other by a dyadic rational, and thus they have equal measure.

Consider now the following cases, where j is assumed to be odd.

(II) I_{N+1}^j is the right half of I_N^i, with i even. From the previous remarks, $m(E_N^i) > 1/2^{N+1}$ for every even value of i, and therefore E_N^i has nonvoid intersection with I_{N+1}^j. Thus I_{N+1}^j contains points of E_N.

(III) I_{N+1}^j is the right half of I_N^i, where i is odd and such that E_N^i is nonempty. If $E_N \cap I_{N+1}^j$ is not void, then I_{N+1}^j contains points of E_N. If $E_N \cap I_{N+1}^j = \emptyset$, fix $x \in E_N^i$ and set $x^* = x + 1/2^{N+1}$; thus x^* belongs to I_{N+1}^j. We therefore have the relation

$$\left|\frac{R(x^* + 1/2^{N+1}) - R(x^*)}{1/2^{N+1}}\right| = \left|\frac{R(x + 1/2^N) - R(x + 1/2^{N+1})}{1/2^{N+1}}\right|$$

$$\leq 2\left|\frac{R(x + 1/2^N) - R(x)}{1/2^N}\right| + \left|\frac{R(x) - R(x + 1/2^{N+1})}{1/2^{N+1}}\right|.$$

Since $x \in E_N$, the second term in the sum is less than or equal to $|D| + 1$. Also, since R is continuous at x, and since $|I(x, h)| \leq |D| + 1$ for every h $(0 < h < 1/2^N)$, the first term in the sum is not greater than $2(|D| + 1)$. Hence x^* satisfies the inequality in the statement of the lemma.

In view of (I) and (II), we conclude that γ_{N+1} is no greater than the number of intervals I_N^i where i is odd; that is, $\gamma_{N+1} \leq 2^{N-1}$. (We have in fact shown, using (III), that γ_{N+1} cannot exceed the number of intervals I_N^i such that i is odd and E_N^i is void. However, for our purposes the weaker estimate is sufficient.)

We show next that γ_{N+2} cannot exceed 2^{N-1}. In the remainder of the proof, j denotes an odd positive integer; thus the interval I_{N+2}^j is either the second or fourth quarter of I_N^i (where i may be even or odd).

(IV) I_{N+2}^j is the second quarter of I_N^i (i even or odd). Note that I_{N+2}^j is the right half of I_{N+1}^{2i}. Arguing as in (I), we see that $m(E_{N+1}^k) > 1/2^{N+2}$ for all even k, and thus I_{N+2}^j must meet E_{N+1}.

(V) I_{N+2}^j is the fourth quarter of I_N^i (i even). Since $m(E_N^i)$ exceeds $1/2^{N+1}$, there are points of E_N, and therefore points of E_{N+1}, in either the third or fourth quarter of I_N^i. If some points lie in the fourth quarter, then I_{N+2}^j contains points of E_{N+1}. If they belong to the third quarter of I_N^i, we argue as in (III) to conclude that I_{N+2}^j contains points of the form $x^{**} = x + 1/2^{N+2}$, with $x \in E_{N+1}$, for which $|I(x^{**}, 1/2^{N+2})| \leq 3(|D| + 1)$.

It follows from (IV) and (V) that the only intervals I_{N+2}^j that can be counted among the γ_{N+2} "bad" intervals are the fourth quarters of I_N^i, where i is odd. Hence γ_{N+2} cannot exceed 2^{N-1}.

Proceeding in this fashion, we conclude that $\gamma_n \leq 2^{N-1}$ for every $n > N$. This completes the proof.

THEOREM 4. *If* $R'(x)$ *exists almost everywhere on* $[0, 1)$, *then* $R(x)$ *is of bounded variation.*

Proof. By adding to $R(x)$, if necessary, a suitable multiple of the series $\sum_{n=0}^{\infty} 2^{-(n+1)} r_n(x)$, we may assume that $D = \lim 2^{n+2} c_n > 0$. With no loss of generality, we shall suppose that $c_n > 0$ for every n. In view of Corollary 5, it suffices to show that

$$L \equiv \limsup_{n \to \infty} \sum_{k=0}^{n} 2^k |c_k - (c_{k+1} + \cdots + c_n)| < \infty .$$

By definition, the term in this sum for $k = n$ is $2^n c_n$; thus, since $\lim 2^n c_n$ exists and is finite, we may ignore this term in estimating L. Next, for $0 \leq k \leq n - 1$, observe that $2^k |c_k - (c_{k+1} + \cdots + c_n)|$ can be expressed as $2^{k-(n+2)} |I(x, 1/2^{n+1})|$, where x (which depends on k) is any point having a 0 in coordinates $1, 2, \cdots, k+1$ and a 1 in coordinates $k+2, \cdots, n, n+1$. Note that each such x belongs to a dyadic interval $I_{n+1}^{j_k}$ with *odd* index j_k, and that any point in $I_{n+1}^{j_k}$ may be chosen for x. It is clear that each k corresponds to a unique j_k and that different values of k give rise to different values of j_k.

For all but at most γ_{n+1} values of j_k (and therefore of k) we can, by the preceding lemma, find an $x \in I_{n+1}^{j_k}$ for which $|I(x, 1/2^{n+1})| \leq K$, where $K = 3(|D| + 1)$. Let I denote the set of indices k for which this inequality holds, and let M be a bound for the sequence $\{\gamma_n\}$.

For each k in the complement I' of I, we have the inequality

$$2^k |c_k - (c_{k+1} + \cdots + c_n)| \leq 2^k R_{k-1};$$

because the sequence $\{2^k R_{k-1}\}$ is bounded (by the proof of Corollary 4), say by C, we obtain the inequality

$$2^k |c_k - (c_{k+1} + \cdots + c_n)| \leq C$$

for each $k \in I'$. It follows that

$$\sum_{k=0}^{n-1} 2^k |c_k - (c_{k+1} + \cdots + c_n)| = \sum_{I} + \sum_{I'} \leq \sum_{k=0}^{n-1} 2^{k-(n+2)} K + \sum_{I'} < \frac{1}{4} K + MC .$$

Since the constants K, M, and C are independent of n, we conclude that L is finite and therefore that R(x) is of bounded variation on [0, 1).

If R(x) is differentiable almost everywhere on [0, 1), Proposition 6 and [8, p. 188] imply that the series assumes each of almost all of its values at most countably many times. However, the previous theorem and [17, p. 279] yield a stronger result in this case.

COROLLARY 7. *If* R'(x) *exists* a.e. *on* [0, 1), *then almost all of the level sets of* R(x) *are finite.*

8. FURTHER RESULTS ON BOUNDED VARIATION

McLaughlin [13, p. 376] has shown that $\sum 2^n |c_n| < \infty$ is a sufficient condition for R(x) to be of bounded variation. The following result improves on this.

THEOREM 5. *Suppose that* R(x) *is differentiable at at least one point. Then a necessary and sufficient condition for* R(x) *to have finite total variation on* [0, 1) *is that*

$$\sum_{n=0}^{\infty} 2^n |c_n - R_n| < \infty .$$

Proof. By assumption, $\lim 2^n c_n = A$ exists and is finite; it follows that $\lim 2^n R_n$ exists and equals A. Write

$$\sum_{n=0}^{N} 2^n |c_n - (c_{n+1} + \cdots + c_N)| = \sum_{n=0}^{N} 2^n |(c_n - R_n) + R_N|$$

$$< \sum_{n=0}^{N} 2^n |c_n - R_n| + 2^{N+1} |R_N| .$$

An application of Corollary 5 now yields the sufficiency of the condition.

To prove the necessity, we write

$$\sum_{n=0}^{N} 2^n |c_n - R_n| = \sum_{n=0}^{N} 2^n |c_n - (c_{n+1} + \cdots + c_N) - R_N|$$

$$< \sum_{n=0}^{N} 2^n |c_n - (c_{n+1} + \cdots + c_N)| + 2^{N+1} |R_N|$$

and appeal once again to Corollary 5.

In particular, it follows that the series $\sum a^n r_n(x)$ is of bounded variation if and only if $0 \leq a \leq 1/2$.

For the case where the derivative is 0 wherever it exists, we obtain an explicit representation for $V_0^1 R(x)$.

PROPOSITION 8. *Suppose that* $\lim 2^n c_n = 0$. *Then the total variation of* R(x) *on* [0, 1) *is given by*

$$V_0^1 R(x) = 2 \sum_{n=0}^{\infty} 2^n |c_n - R_n|.$$

This equation is valid even if the total variation is infinite.

Proof. The hypothesis evidently implies that $\lim 2^n R_n = 0$. In view of Corollary 5 and the proof of sufficiency in the preceding theorem, we see that $V_0^1 R(x) \leq 2 \sum_{n=0}^{\infty} 2^n |c_n - R_n|$. The reverse inequality follows from Corollary 5 and the proof of necessity in Theorem 5.

The following representation for $V_0^1 R(x)$ is based on the work of V. A. Matveev [11].

THEOREM 6. (i) *If* R(x) *is of bounded variation on* [0, 1), *then*

$$V_0^1 R(x) = \lim_{h \to 0} \int_0^1 \left| \frac{R(x+h) - R(x)}{h} \right| dx.$$

(ii) *If* $\limsup_{h \to 0} \int_0^1 \left| \frac{R(x+h) - R(x)}{h} \right| dx$ *exists and is finite, then* R(x) *has finite total variation on* [0, 1).

Proof. Part (i) follows from [11, p. 136]. (The required hypotheses are met, since R(x) is right-continuous.) To prove part (ii), let

$$D_\beta = \limsup_{h \to 0} \int_0^\beta |I(x, h)|\, dx;$$

then $D_\beta \leq D_1 < \infty$ for $0 < \beta < 1$. If β is not dyadically rational, we may apply [11, Theorem 3, p. 136] to conclude that R(x) is of bounded variation on $[0, \beta]$ and $V_0^\beta R(x) = D_\beta$. It follows that R(x) has finite total variation on [0, 1).

The following sufficient condition for $V_0^1 R(x)$ to be finite is often more easily verified than the condition given in Theorem 5 (for example, if $c_n = 1/n^\alpha 2^n$, where $\alpha > 0$).

PROPOSITION 9. *Suppose that* $c_n \downarrow 0$ *and that the sequence* $\{2^n c_n\}$ *is eventually nonincreasing. Then* R(x) *is of bounded variation.*

Proof. Write $c_n = \varepsilon_n / 2^n$, where the sequence $\{\varepsilon_n\}$ is eventually nonincreasing. Then, for all sufficiently large n,

$$R_n = \sum_{k=n+1}^{\infty} c_k \leq \varepsilon_n (1/2^{n+1} + 1/2^{n+2} + \cdots) = \varepsilon_n / 2^n = c_n.$$

Therefore R(x) is piecewise monotone and thus of bounded variation.

If R'(x) exists almost everywhere, Theorem 4 shows that R(x) is the difference of two monotone functions, though not necessarily monotone itself. F. M. Filipczak

[6] and A. M. Bruckner, J. G. Ceder, and M. L. Weiss [3] have shown that every function continuous on a perfect set must be monotone on some perfect subset. For a certain class of Rademacher series, we can prove a stronger result.

THEOREM 7. *Suppose that* $R(x)$ *is of bounded variation on* $[0, 1)$ *and that* $A = \lim 2^n c_n \neq 0$. *Then there exists a perfect subset* P *of* $[0, 1)$, *of positive measure, on which* $R(x)$ *is continuous and strictly monotone. More generally, for each* α $(0 < \alpha < 1)$, *there exists a perfect set* P_α *of measure exceeding* α *such that the restriction of* $R(x)$ *to* P_α *is continuous, and* P_α *is the finite union of sets on each of which* $R(x)$ *is strictly monotone.*

Proof. We prove the result for $A > 0$. For $n \geq 1$, define sets

$$A_n = \{x \in \Delta: 0 < h < 1/2^n \text{ implies } I(x, h) < 0\};$$

then $\{A_n\}$ is an expanding sequence of sets whose union is Δ. Given $0 < \alpha < 1$, choose $N \equiv N(\alpha)$ so that $m(A_N) > \alpha$, and let P_α be a perfect subset of A_N of measure greater than α. Since P_α consists of dyadic irrationals, $R(x)$ is continuous on P_α. Let A_N^i be the part of A_N that meets the dyadic interval I_N^i, for $i = 0, 1, \cdots, 2^N - 1$. It is clear from the definition of A_N that $R(x)$ is strictly decreasing on each nonvoid A_N^i, and therefore P_α is the union of finitely many sets on each of which $R(x)$ is strictly monotone. Since A_N has positive measure, we see that $m(A_N^k) > 0$ for some value of k. Thus take P to be any perfect subset of A_N^k of positive measure.

Finally, while this theorem shows that a Rademacher series of bounded variation is monotone on certain sets of positive measure, the series need not be monotone on any subinterval of $[0, 1)$. Consider, for example, the series $R(x)$ with coefficients given by

$$c_{2n+1} = 1/2^{2n+1} \quad \text{and} \quad c_{2n} = 1/2^{3n+2} + 1/2^{2n},$$

for $n \geq 0$. Here, the function $R(x)$ is of bounded variation. (To see this, write

$$R(x) = \sum (c_n - 2^{-n}) r_n(x) + \sum 2^{-n} r_n(x) \equiv R^*(x) + H(x),$$

and observe that $R^*(x)$ is of bounded variation since its coefficients c_n^* satisfy the condition $\sum 2^n c_n^* < \infty$.) Evidently, $R(x)$ satisfies the hypotheses of Theorem 7.

However, since the condition $c_n \geq R_n$ does not hold eventually in the present example, $R(x)$ cannot be monotone on any dyadic subinterval of $[0, 1)$. This follows from the next result, whose proof is a straightforward modification of the proof of Theorem 2 in [1].

THEOREM 8. *Let* $R(x) = \sum c_n r_n(x)$, *and let* N *be a positive integer. For* $i = 0, 1, \cdots, 2^N - 1$, *define* $I_N^i = [i/2^N, (i+1)/2^N)$. *Then the following statements are equivalent:*

(i) $R(x)$ *is decreasing on* I_N^i *for some value of* i;

(ii) $R(x)$ *is decreasing on* I_N^i *for every value of* i;

(iii) $c_n \geq 0$ *and* $c_n \geq R_n$ *for every* $n \geq N$.

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University of British Columbia, Vancouver, Canada V6T 1W5