COMPACT, TOTALLY DISCONNECTED SETS THAT CONTAIN K-SETS

Frank B. Miles

It is well known that every infinite subset of a discrete abelian group contains an infinite Sidon set. In this paper, we present two analogous theorems on K-sets in nondiscrete, locally compact abelian groups. Each theorem says, roughly, that certain compact, metrizable, totally disconnected sets E contain K-sets homeomorphic to themselves and that each such set E is almost a K-set in the sense that the identity map from E to E can be uniformly approximated by homeomorphisms of E onto K-sets in E. More precisely, the two theorems are as follows:

THEOREM A. *Let* G *be a nondiscrete, locally compact, abelian* T_0*-group, and let* E *be an independent, nonvoid, compact, metrizable, totally disconnected subset of* G. *Then there exist a metric space* $C_\lambda(E, E)$ *of continuous functions from* E *to* E, *complete in the uniform topology and containing the identity map from* E *to* E, *and a subset* H *of the first category in* $C_\lambda(E, E)$ *with the property that each* $f \in C_\lambda(E, E) \setminus H$ *maps* E *homeomorphically onto a* K*-set.*

THEOREM B. *Let* G *be a nondiscrete, locally compact, abelian* T_0*-group, and suppose that the torsion subgroup of* G *is at most countable. Let* E *be a subset of* G *homeomorphic to Cantor's ternary set. Then the set* C(E, E) *of continuous functions from* E *to* E *with the uniform topology contains a set* H *of the first category with the property that each* $f \in C(E, E) \setminus H$ *maps* E *homeomorphically onto a* K*-set.*

Definitions and Notation. In all that follows, G denotes a locally compact abelian T_0-group with character group X. We write C(E, T) for the set of continuous functions from E to the unit circle T in the complex plane.

A nonvoid compact subset E of G is called a K-*set* if $X \mid E$, the set of restrictions to E of continuous characters of G, is uniformly dense in C(E, T). We remind the reader that a K-set consists solely of independent elements of infinite order and that a nonvoid finite independent set is necessarily a K-set. (A finite subset $\{x_1, \cdots, x_k\}$ of G is called *independent* if the relation $x_1^{n_1} \cdots x_k^{n_k} = e$, where e is the identity of G and the exponents n_j are integers, implies that all the exponents n_j are zero. An infinite subset of G is called *independent* if every finite subset of it is independent. The void set is independent.)

Remarks. (a) We prove both theorems by using an argument whose original form is due to R. Kaufman [2]. A modification of Kaufman's argument given by Y. Katznelson [1, pp. 184-185] has been adapted for use here and in the related paper [3].

(b) Suppose that E consists of a convergent sequence together with its limit point x. Then each homeomorphism of E into E must map x to itself. Thus, the set of homeomorphisms of E into E is not dense in C(E, E). This example and a little further thought show that, in order to obtain a conclusion of the form "all

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functions except those in a set of first category map E homeomorphically onto a K-set" we must restrict ourselves to the functions in C(E, E) that take limit points to limit points, limit points of limit points to limit points of limit points, and so forth. We are thus led to consider the subspace $C_\lambda(E, E)$ of C(E, E) defined below.

(c) The hypothesis in Theorem B that the torsion subgroup of G is countable is necessary, as is shown by the following example. Let G be the product of countably infinitely many copies of T. Let E be the subset of G consisting of those elements each coordinate of which is ± 1. Then E is homeomorphic to Cantor's ternary set; but no subset of E is a K-set, since elements of K-sets have infinite order. If we translate E by an element of infinite order, then no subset of E containing more than one element is a K-set, since K-sets are independent.

In the general case, we may drop the hypothesis that the torsion subgroup of G is countable if we add the hypothesis that E is independent; but then we have a special case of Theorem A, for $C_\lambda(E, E) = C(E, E)$ when E has no isolated points (see the definition of $C_\lambda(E, E)$ below).

(d) Let d_1 and d_2 be equivalent metrics on the compact metric space E. For f and g $\in$ C(E, E), let

$$D_j(f, g) = \sup\{d_j(f(x), g(x)) \colon x \in E\} \qquad (j = 1, 2).$$

The topologies on C(E, E) induced by D_1 and D_2 are the same, since each is the compact-open topology on C(E, E). It follows that a sequence in C(E, E) is a D_1-Cauchy sequence if and only if it is a D_2-Cauchy sequence. We may therefore speak of the uniform topology on C(E, E) and of Cauchy sequences in C(E, E) without specifying a particular metric on E.

Definition. Let E be a compact metric space. For each ordinal $\alpha \leq \Omega$ (the first uncountable ordinal), define E_α as follows. Let $E_0 = E$. Let

$$E_{\alpha+1} = \{x \in E_\alpha \colon x \text{ is a limit point of } E_\alpha\}.$$

When α is a limit ordinal, let $E_\alpha = \bigcap_{\beta<\alpha} E_\beta$.

LEMMA 1. *Let* E *be a compact metric space.*

(i) *For some* α *that strictly precedes* Ω, *we have the relation* $E_\alpha = E_{\alpha+1}$.

(ii) *If* F *is open and closed in* E, *then* $F_\alpha = F \cap E_\alpha$ *for all* $\alpha \leq \Omega$.

Proof. (i) Write $E = P \cup C$, where P is perfect, C is countable, and $P \cap C = \emptyset$. Clearly, $P \subset E_\Omega$. Hence, $\bigcup_{\alpha<\Omega} (E_\alpha \setminus E_{\alpha+1}) \subset C$. Since Ω has uncountably many predecessors and C is countable, $E_\alpha \setminus E_{\alpha+1} = \emptyset$ for some α that strictly precedes Ω.

(ii) The proof is an easy argument by means of transfinite induction.

Definitions. Suppose that E is a compact metric space.

(i) By α_E we denote the first ordinal α that satisfies the relation $E_\alpha = E_{\alpha+1}$ in part (i) of Lemma 1.

(ii) For $x \in E$, let $\lambda(x) = \alpha_E$ when $x \in E_{\alpha_E}$. When $x \in E \setminus E_{\alpha_E}$, let $\lambda(x)$ be the last ordinal α such that $x \in E_\alpha$. (Proof that such an α exists: Let β be the first ordinal such that $x \notin E_\beta$. Then $x \in E_\gamma$ for all $\gamma < \beta$; therefore, if β were a limit ordinal, x would be an element of $\bigcap_{\gamma<\beta} E_\gamma = E_\beta$, a contradiction.) Observe that $\lambda(x) \geq \alpha$ implies that $x \in E_\alpha$.

(iii) Let F be a nonvoid open and closed subset of E. If $F \cap E_{\alpha_E} \neq \emptyset$, define $\lambda(F) = \alpha_E$. If $F \cap E_{\alpha_E} = \emptyset$, let $\lambda(F)$ be the last α such that $F_\alpha \neq \emptyset$. (Proof that such an α exists: By Lemma 1, there is a first ordinal β such that $F_\beta = \emptyset$. Then $F_\gamma \neq \emptyset$ for all $\gamma < \beta$ and the F_γ are nested; therefore, if β were a limit ordinal, the set $\bigcap_{\gamma<\beta} F_\gamma = F_\beta$ would be empty, contradicting the fact that the F_γ are a family of closed sets with the finite-intersection property.) Observe that when $\lambda(F) < \alpha_E$, the set $F_{\lambda(F)}$ is finite and $\lambda(x) \leq \lambda(F)$ for all $x \in F$.

(iv) Let $C_\lambda(E, E)$ consist of all $f \in C(E, E)$ that satisfy the two conditions (a) for all $x \in E$, $\lambda(x) \leq \lambda(f(x))$ and (b) for all $y \in E \setminus E_{\alpha_E}$, $f^{-1}(y)$ contains at most one element of $E_{\lambda(y)}$.

THEOREM 1. *Let* E *be a nonvoid, compact metric space. Then* $C_\lambda(E, E)$ *is complete in the topology of uniform convergence.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $C_\lambda(E, E)$ that converges to $f \in C(E, E)$. We show that f satisfies the two conditions in the definition of $C_\lambda(E, E)$.

Let $x \in E$. Then $\lambda(x) \leq \lambda(f_n(x))$ for all n; therefore, every $f_n(x)$ is in $E_{\lambda(x)}$. Since $E_{\lambda(x)}$ is closed and $f_n(x) \to f(x)$, we see that $f(x) \in E_{\lambda(x)}$, and hence that $\lambda(x) \leq \lambda(f(x))$.

Now let $y \in E$ with $\lambda(y) = \alpha < \alpha_E$, and assume that there exist distinct x_1 and x_2 in E_α such that $f(x_1) = f(x_2) = y$. We show that this leads to a contradiction. Since y is an isolated point of E_α, there exists a neighborhood U of y such that $U \cap E_\alpha = \{y\}$. We have the relations $\alpha \leq \lambda(x_j) \leq \lambda(f_n(x_j))$ for all n and j. For sufficiently large n, we see that $f_n(x_j) \in U$, hence $\lambda(f_n(x_j)) \leq \alpha$, hence $\lambda(f_n(x_j)) = \alpha$, hence $f_n(x_j) = y$ $(j = 1, 2)$. Thus, for large n, the set $f_n^{-1}(y)$ contains at least two elements of E_α; this contradicts the hypothesis $f_n \in C_\lambda(E, E)$.

LEMMA 2. *Let* E *be a compact metric space with metric* d; *let* $x_1, \cdots, x_n$ *be distinct elements of* E; *let* $g \in C_\lambda(E, E)$, *and let* $\eta > 0$. *Then there are distinct elements* $y_1, \cdots, y_n$ *of* E *such that* $\lambda(x_j) \leq \lambda(y_j)$ *and* $d(y_j, g(x_j)) < \eta$ *for* $1 \leq j \leq n$.

Proof. Let $F = \{g(x_1), \cdots, g(x_n)\}$, and write $\alpha_j = \lambda(x_j)$ and $\beta_j = \lambda(g(x_j))$ $(1 \leq j \leq n)$. We may suppose that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. We choose the elements y_j of E by induction as follows.

Case 1: $\alpha_j < \beta_j$ or $\alpha_j = \beta_j = \alpha_E$. Here $g(x_j)$ is a limit point of E_{α_j}, and we choose $y_j \in E_{\alpha_j}$ so that $d(y_j, g(x_j)) < \eta$, $y_j \notin F$, and y_j is distinct from each previously chosen element y_k.

Case 2: $\alpha_j = \beta_j < \alpha_E$. In this case, choose $y_j = g(x_j)$. It is sufficient to show that y_j is distinct from each previously chosen y_k. Assume that $y_j = y_k$ for some $k < j$. We show that this leads to a contradiction. If $\alpha_k < \beta_k$, then y_k was chosen as described in Case 1, so that $y_k \notin F$, contrary to the relation $y_j \in F$. If $\alpha_k = \beta_k < \alpha_j = \beta_j$, then we chose y_k by taking $y_k = g(x_k)$. Hence,

$$\beta_k = \lambda(y_k) = \lambda(y_j) = \beta_j,$$

contrary to the relation $\beta_k < \beta_j$. If $\alpha_k = \beta_k = \alpha_j = \beta_j$, we again see that $y_k = g(x_k)$. Therefore, $g^{-1}(y_j)$ contains at least two points of $E_{\lambda(y_j)}$, and this contradicts the hypothesis $g \in C_\lambda(E, E)$.

Proof of Theorem A. If E is finite, take $H = \emptyset$. Since $f \in C_\lambda(E, E)$ if and only if f maps E onto E, and since a finite independent set is a K-set, the result holds.

Suppose now that E is infinite. For $h \in C(E, T)$, $f \in C(E, E)$, and $\varepsilon > 0$, let the statement "(*) holds for h, f, and ε" mean "there is a $\gamma \in X$ such that $|\gamma(f(y)) - h(y)| < \varepsilon$ for all $y \in E$." Let $f \in C(E, E)$. Clearly, f is a homeomorphism of E onto f(E) if and only if f is one-to-one. Also, if f is not one-to-one, it is clear that there exist $h \in C(E, T)$ and $\varepsilon > 0$ such that (*) fails for h, f, and ε. Hence, f is a homeomorphism of E onto f(E) and f(E) is a K-set if and only if for every $h \in C(E, T)$ and every $\varepsilon > 0$, (*) holds for h, f, and ε.

Let d be a metric on E compatible with the topology of E. For f and g in C(E, E), let $D(f, g) = \sup\{d(f(y), g(y)): y \in E\}$.

Let $h \in C(E, T)$, $g \in C_\lambda(E, E)$, $\varepsilon > 0$, and $\eta > 0$. We shall show that there is an $f \in C_\lambda(E, E)$ such that $D(f, g) < \eta$ and (*) holds for h, f, and ε. Write $E = \bigcup_{j=1}^n E_j$, where the sets E_j are pairwise disjoint, nonvoid, open and closed subsets of E, and where h varies less than $\varepsilon/2$ and g varies less than $\eta/2$ on each E_j. (The sets E_j exist, since E is totally disconnected.) Let $\lambda(E_j) = \alpha_j$ $(1 \leq j \leq n)$. If $\alpha_j < \alpha_E$, then $(E_j)_{\alpha_j}$ is finite, so that we may suppose without loss of generality that E_j contains exactly one point x_j such that $\lambda(x_j) = \alpha_j$. If $\alpha_j = \alpha_E$, let x_j be any point of $E_j \cap E_{\alpha_E}$. By Lemma 2, there are distinct points $y_1, \cdots, y_n$ in E such that

$$\lambda(y_j) \geq \lambda(x_j) \quad \text{and} \quad d(y_j, g(x_j)) < \eta/2 \qquad \text{for } 1 \leq j \leq n.$$

Define $f(y) = y_j$ when $y \in E_j$. Then $D(f, g) < \eta$ and $f \in C_\lambda(E, E)$. (The second condition in the definition of $C_\lambda(E, E)$ is satisfied, because when $\alpha_j < \alpha_E$, then E_j contains only one point x_j such that $\lambda(x_j) = \alpha_j$.) Since $\{y_1, \cdots, y_n\}$ is a finite independent set, it is a K-set, and therefore there exists a $\gamma \in X$ such that $|\gamma(y_j) - h(x_j)| < \varepsilon/2$ $(1 \leq j \leq n)$. For $y \in E_j$, we see that

$$|\gamma(f(y)) - h(y)| \leq |\gamma(y_j) - h(x_j)| + |h(x_j) - h(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, (*) holds for h, f, and ε.

For $h \in C(E, T)$ and $\varepsilon > 0$, let

$$H(h, \varepsilon) = \{f \in C_\lambda(E, E): (*) \text{ fails for } h, f, \text{ and } \varepsilon\}.$$

It is easy to show that $H(h, \varepsilon)$ is closed. By the preceding paragraph, $H(h, \varepsilon)$ is nowhere dense in $C_\lambda(E, E)$. Let $\{h_n\}_{n=1}^\infty$ be dense in C(E, T). Let $H = \bigcup_{n,k=1}^\infty H(h_n, 1/k)$. Then H is a first-category set in the complete metric space $C_\lambda(E, E)$. Also, if $f \in C_\lambda(E, E) \setminus H$, then every $h \in C(E, T)$ can be uniformly approximated by functions $\gamma \circ f$ $(\gamma \in X)$; therefore, by the second paragraph of this proof, f is a homeomorphism and f(E) is a K-set.

LEMMA 3. *Let* G *and* E *be as in Theorem* B, *and let* F *be a finite independent subset of* G. *Then there exists an* $x \in E \setminus F$ *such that* $\{x\} \cup F$ *is independent.*

Proof. If $F = \emptyset$, let x be any element of E of infinite order. Suppose now that $F \neq \emptyset$. Let F' be the subgroup of G generated (algebraically) by F, and let

$$\tilde{F} = \{x \in G: x^n \in F' \text{ for some nonzero integer } n\}.$$

Since the torsion subgroup of G is at most countable, it follows that $\tilde{F}$ is at most countable. Let x be any element of $E \setminus \tilde{F}$. Then $\{x\} \cup F$ is independent.

LEMMA 4. *Let* G *and* E *be as in Theorem* B. *Let* $\eta > 0$. *Let* d *be a metric on* E *compatible with the topology of* E. *Suppose that* $E = \bigcup_{j=1}^{n} E_j$, *where the sets* E_j *are pairwise disjoint, nonvoid, open and closed subsets of* E. *Let* $g \in C(E, E)$. *Then there exist distinct elements* $x_1, \cdots, x_n$ *of* E *such that* $\{x_1, \cdots, x_n\}$ *is a* K-*set and* $d(x_j, g(E_j)) < \eta$ *for* $1 \leq j \leq n$.

Proof. We use induction on n. For $1 \leq j \leq n$, let $F_j = \{x \in E: d(x, g(E_j)) < \eta\}$. Each F_j is nonvoid and hence contains a homeomorph D_j of E. Applying Lemma 3 to the case $E = D_1$ and $F = \emptyset$, we obtain an element $x_1 \in D_1$ such that $\{x_1\}$ is independent. Suppose now that $1 \leq k \leq n - 1$ and that distinct $x_j \in D_j$ have been chosen such that $\{x_1, \cdots, x_k\}$ is independent. Apply Lemma 3 to the case $E = D_{k+1}$ and $F = \{x_1, \cdots, x_k\}$ to obtain an element $x_{k+1} \in D_{k+1}$ distinct from $x_1, \cdots, x_k$ such that $\{x_1, \cdots, x_{k+1}\}$ is independent. The result now follows from the fact that a nonvoid finite independent subset of G is a K-set.

Proof of Theorem B. The proof is essentially the same as that given above for Theorem A, except that we use Lemma 4 instead of Lemma 2 and $C(E, E)$ in place of $C_\lambda(E, E)$.

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California State College
Dominguez Hills, California 90747