COMMON FIXED POINTS OF COMMUTING HOLOMORPHIC MAPS OF THE HYPERBALL

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1. INTRODUCTION

Let f and g be continuous functions on the closed unit disk of the complex plane, and suppose

(i) f and g are holomorphic in the open unit disk,

(ii) f and g map the closed disk into itself, and

(iii) f and g commute under the operation of function composition.

In [10], A. L. Shields showed that under these conditions, f and g have a common fixed point. D. J. Eustice [5] has extended this result to the polydisk in complex 2-space. W. M. Boyce [1] and J. P. Huneke [8] have independently given counterexamples to show that two continuous functions that commute and map the closed unit interval into itself need not have a common fixed point. In this paper, we extend the result of Shields to finite-dimensional inner-product spaces.

For a characterization of commuting polynomials, see [9] and [2]. For a more complete discussion of the history of problems concerning commuting maps, see [1].

2. HOLOMORPHIC IDEMPOTENTS ON THE UNIT BALL OF A HILBERT SPACE

We shall use the following notation.

(i) H is a Hilbert space (either finite-dimensional or infinite-dimensional) with inner product $\langle \cdot, \cdot \rangle$;

(ii) B is the unit ball of H, that is, $B = \{z \in H: \langle z, z \rangle < 1\}$, and $\overline{B}$ is the closure of B;

(iii) $\mathscr{H}(B)$ is the set of functions $f: B \to B$ that are holomorphic on B; and

(iv) if $f \in \mathscr{H}(B)$, then f^k is defined inductively for $k = 1, 2, \cdots$ by $f^1 = f$ and $f^{k+1} = f \circ f^k$.

The unit ball B of H is known to be a homogeneous domain (see [3] and [7]). That is, corresponding to each pair of points $u, v \in B$ there exists a function $L \in \mathscr{H}(B)$ such that $L(u) = v$, L is a one-to-one map of B onto B, and L^{-1} is holomorphic. Such a map is called a biholomorphic map of B onto B. For example, if we write u in the form $u = \alpha b$, where $\|b\| = 1$ and α is a complex number such that $|\alpha| < 1$, then the map L_u defined by

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$$L_u(x) = \frac{1}{1 - \langle x, \alpha b \rangle}(\langle x - \alpha b, b \rangle b + \sqrt{1 - |\alpha|^2}(x - \langle x, b \rangle b))$$

is a biholomorphic map of B onto B such that $L_u(u) = 0$. The most general biholomorphic map of B onto B that has the value 0 at the point u is $T \circ L_u$, where T is a unitary map. For a discussion of maps of this type for other spaces, see [6, pp. 13-40].

A holomorphic idempotent on B is a holomorphic function $f: B \to B$ such that $f^2 = f$. The holomorphic idempotents on B are characterized in the following theorem.

THEOREM 1. *The holomorphic idempotents* f *mapping* B *into* B *are of the form*

(1) $$f = L \circ F \circ L^{-1},$$

where L *is a biholomorphic map of* B *onto* B, F(B) *is the intersection of* B *with a linear subspace of* H, *and* $F(0) = 0$.

Proof. Assume f is a holomorphic idempotent on B. If $f(0) = 0$, let L be the identity map, and let $F = f$. If $f(0) \neq 0$, let $b \in f(B)$ $(b \neq 0)$, and let L be a biholomorphic map of B onto B such that $L(0) = b$. Since f is an idempotent, the equality $f(b) = b$ holds, and setting $F = L^{-1} \circ f \circ L$, we conclude that $F(0) = 0$, $F^2 = F$, and f satisfies (1). It remains to show that F(B) is the intersection of B with a linear subspace of H.

We wish to show that if $u, v \in F(B)$, then $F(\alpha u + \beta v) = \alpha u + \beta v$ whenever $\|\alpha u + \beta v\| < 1$. We begin by assuming $u \in F(B)$, $u \neq 0$. Then, since F is an idempotent, $F(u) = u$. Let $\ell_u \in H^*$ be the linear functional $\ell_u(x) = \langle x, u/\|u\| \rangle$, so that $\ell_u(u) = \|u\|$ and $\|\ell_u\| = 1$. Define $g(\alpha)$ in $|\alpha| < 1$ by $g(\alpha) = \ell_u(F(\alpha u/\|u\|))$, so that g is holomorphic, $g(0) = 0$, and $|g(\alpha)| < 1$ when $|\alpha| < 1$. Applying Schwarz's lemma and using the fact that $g(\|u\|) = \ell_u(F(u)) = \ell_u(u) = \|u\|$, we conclude that $g(\alpha) = \alpha$ when $|\alpha| < 1$. Thus we have shown that

(2) $$|\alpha| = |\langle F(\alpha u/\|u\|), u/\|u\| \rangle| \leq \|F(\alpha u/\|u\|)\| \leq |\alpha|,$$

so that equality must hold in (2). This means that $F(\alpha u/\|u\|) = \alpha u/\|u\|$ when $|\alpha| < 1$, or equivalently, that $F(\alpha u) = \alpha u$ when $|\alpha| < 1/\|u\|$.

Now suppose $v \in F(B)$, and let α and β be complex numbers such that $\|\alpha u + \beta v\| \leq 1$. Then

$$\frac{1}{\lambda} F(\lambda(\alpha u + \beta v)) = \sum_{k=1}^{\infty} \frac{1}{k!} \lambda^{k-1} D^k F(0)(\alpha u + \beta v)^k$$

is a holomorphic function of λ in the disk $\{|\lambda| < 1\}$, where

$$D^k F(0)(x)^k = D^k F(0)(x, x, \cdots, x).$$

Since DF(0) is linear and $u, v \in F(B)$, we know

$$DF(0)(\alpha u + \beta v) = \alpha DF(0)(u) + \beta DF(0)(v) = \alpha u + \beta v.$$

By Schwarz's lemma, $\|F(\lambda(\alpha u + \beta v))\| \leq |\lambda| \, \|\alpha u + \beta v\|$, so that

$$\left\| \frac{1}{\lambda} F(\lambda(\alpha u + \beta v)) \right\| \leq \|\alpha u + \beta v\|,$$

with equality when $\lambda = 0$. We now apply the strong-maximum principle of E. Thorp and R. Whitley [11] to conclude $F(\lambda(\alpha u + \beta v)) = \lambda(\alpha u + \beta)$, where $|\lambda| < 1$, whenever $\|\alpha u + \beta v\| \leq 1$ and u, $v \in F(B)$. This completes the proof of the theorem.

THEOREM 2. *Let* f *be holomorphic in* B *and continuous in* $\overline{B}$, *with* $f(B) \subset \overline{B}$. *If* f *is an idempotent, then either* f *is given by* (1) *or* f *is a constant of norm* 1.

Proof. This follows easily from Theorem 1, the strong-maximum principle of Thorp and Whitley [11], and the observation that the functions L are continuous on $\overline{B}$.

3. COMMON FIXED POINTS

Suppose L is a biholomorphic map of B onto B. The following results were obtained in [7]. The map L takes affine subspaces into affine subspaces (actually, the intersection of affine subspaces with B), L extends to an automorphism of $\overline{B}$ onto $\overline{B}$ and has a fixed point in $\overline{B}$, and if L does not have a fixed point in B then it has a unique fixed boundary point or it has exactly two fixed boundary points. Concerning the latter case, we have the following result, which is related to a result of A. Denjoy [4] and J. Wolff [13], [14] for the complex plane.

THEOREM 3. *If* L *has two fixed points in* $\overline{B}$ *and no fixed points in* B, *then the iterates* L^n *of* L *converge to a fixed point of* L. *The convergence is uniform on the ball of radius* $r < 1$.

Proof. Since L leaves two distinct points fixed and L takes affine subspaces into affine subspaces, the affine subspace joining the two fixed points is invariant under L. If K is a biholomorphic map of B onto B, then L^n converges if and only if $(K^{-1} \circ L \circ K)^n$ converges. Hence we may assume that the affine subspace joining the two fixed points contains the origin and that it is the set $\{\alpha x_0\colon |\alpha| < 1\}$, where $\|x_0\| = 1$. Then $L = U \circ h$, where

$$h(x) = \frac{1}{1 - \bar{\beta}\langle x, x_0\rangle} \left((\langle x, x_0\rangle - \beta)x_0 + \sqrt{1 - |\beta|^2}(x - \langle x, x_0\rangle x_0)\right).$$

U is unitary and $U(x_0) = \gamma x_0$ for some γ ($|\gamma| = 1$), and β is chosen so that $L(\beta x_0) = 0$. This representation follows if we apply Schwarz's lemma to $L \circ h^{-1}$ and use the invariance of $\{\alpha x_0\colon |\alpha| < 1\}$.

Writing $L_0(z) = \gamma \frac{z - \beta}{1 - \bar{\beta} z}$ (z complex, $|z| < 1$), we see that

$$L(x) = L_0(\langle x, x_0\rangle)x_0 + L_1(x),$$

where

$$L_1(x) = U\left(\frac{\sqrt{1 - |\beta|^2}(x - \langle x, x_0\rangle x_0)}{1 - \bar{\beta}\langle x, x_0\rangle}\right)$$

is orthogonal to x_0 for every $x \in B$.

Hence $L^n(x) = L_0^n(\langle x, x_0\rangle)x_0 + L_n(x)$, where $L_n(x)$ is orthogonal to x_0 for every $x \in B$. If L_0 has a fixed point z ($|z| < 1$), then zx_0 is a fixed point of L. Hence L_0 does not have such a fixed point and L_0^n converges to a fixed point on the boundary $|z| = 1$ (uniformly on subdisks) [4] and [13] (see also [10]). This implies $L_n(x) \to 0$ uniformly on the ball of radius $r < 1$. This concludes the proof of Theorem 3.

We assume from this point on that H is finite-dimensional. In this case, if f is holomorphic in B and continuous on $\overline{B}$ with values in $\overline{B}$, then the family of all functions that are holomorphic on B, map B into $\overline{B}$, and commute with f forms a compact topological semigroup. The proof is the same as that of Shields [10] for $n = 1$.

We require the following two results concerning compact topological semigroups; they were used also by Shields [10] and Eustice [5], and they can be found in [12].

(i) *If* S *is a compact topological semigroup and* $x \in S$, *then the closure of the iterates of* x (*denoted by* $\Gamma(x)$) *is an Abelian subsemigroup that contains exactly one idempotent* e.

(ii) *If the idempotent* $e \in \Gamma(x)$ *is an identity, then* $\Gamma(x)$ *is a group and* $x^{-1} \in \Gamma(x)$.

THEOREM 4. *If* f *and* g *are holomorphic on* B (*the unit ball in a finite-dimensional complex inner-product space* H) *and continuous on the closure* $\overline{B}$, *and if* $f(\overline{B}) \subset \overline{B}$, $g(\overline{B}) \subset \overline{B}$, *and* $f \circ g = g \circ f$, *then* f *and* g *have a common fixed point in* $\overline{B}$.

Proof. We proceed by induction on the dimension n of H. For $n = 1$, the result is that of Shields [10]. Assume that the theorem holds when $n < k$, where k is an integer ($k \geq 2$), and consider the case $n = k$. If $\Gamma(f)$ contains an element F such that $F(B) \not\subset B$, then $\|F(x)\| = 1$ for some $x \in B$, and the maximum principle of Thorp and Whitley [11] implies that $F(x) \equiv c$ for some c ($\|c\| = 1$). In this case,

$$g(c) = g(F(c)) = F(g(c)) = c = F(f(c)) = f(F(c)) = f(c),$$

so that f and g have a common fixed point in $\overline{B}$. Notice that according to Theorem 3, this case occurs when f is a biholomorphic map of B onto B that has exactly two fixed boundary points.

Now suppose every $F \in \Gamma(f)$ satisfies $F(B) \subset B$. Then $\Gamma(f)$ contains an idempotent F as characterized in Theorem 1. Since commutativity, existence of common fixed points, and idempotency are all preserved if we replace f, g, and F by

$$L^{-1} \circ f \circ L, \quad L^{-1} \circ g \circ L, \quad \text{and } L^{-1} \circ F \circ L,$$

respectively, where L is a biholomorphic map of B onto B, we may assume F(B) is the intersection of B with a linear subspace of H, and that $F(0) = 0$.

Now assume F is not the identity. Then $\dim F(B) < \dim H = n$. If $\dim F(B) = 0$, then 0 is clearly a common fixed point of f and g; therefore we assume $\dim F(B) \geq 1$. Since F is the identity map on a subspace of H (actually, the intersection of a subspace of H with B), we may write $H = H_1 \oplus H_2$, where $F(B) = B_1$ (the unit ball in H_1), and $\dim H_1 < n$. It is clear that $g(B_1) \subset B_1$ and $f(B_1) \subset B_1$, so that f and g have a common fixed point in $\overline{B}_1$, by the induction hypothesis.

Now assume F is the identity. In this case, f is a biholomorphic map of B onto B. If f has no fixed points in B, then f has a unique fixed point on the boundary, and this is also a fixed point of g.

The remaining case is the case in which f is a biholomorphic map of B onto B with a fixed point in B. As before, we may assume $f(0) = 0$, so that f is the restriction of a unitary map U. Let H_1 be the subspace $\{x \in H: U(x) = x\}$. If $H_1 = H$, then every fixed point of g in $\overline{B}$ is a common fixed point of f and g. If $H_1 \neq H$, we see that $f(g(x)) = g(f(x)) = g(x)$ for all $x \in B_1$ (the unit ball in H_1), and this shows that $x \in B_1$ implies $g(x) \in \underline{B_1}$. By the induction hypothesis, f and g have a common fixed point in $\overline{B}_1$.

4. EXAMPLES

Example 1. Let $H = C^2$ with the usual inner product $\langle z, w \rangle = z_1\bar{w}_1 + z_2\bar{w}_2$, where $z = (z_1, z_2)$ and $w = (w_1, w_2)$. Let

$$f(z) = \left(\frac{\sqrt{1-r^2}\,z_2}{1-rz_1},\ -\frac{z_1-r}{1-rz_1}\right) \quad (r < 1),$$

so that f is holomorphic in a neighborhood of B and maps B onto B. The point $\left(\frac{1-\sqrt{1-r^2}}{r}, \frac{1-\sqrt{1-r^2}}{r}\right) = z_0$ is always a fixed point of f. It lies in B when $r^2 < 8/9$, and then f has two fixed points outside $\overline{B}$. When $r^2 = 8/9$, z_0 lies on the boundary of B and is the unique fixed point of f. When $1 > r^2 > 8/9$, z_0 is outside of $\overline{B}$ and f has two fixed points on the boundary of B. The iterates of f then converge to one of these fixed boundary points (assuming the domain is restricted to B), by Theorem 3.

Example 2. Let $r < 1$, and set

$$f(z) = \frac{1}{2}\left(\frac{z_1-r}{1-rz_1}, \frac{z_2-r}{1-rz_2}\right) \quad \text{and} \quad g(z) = (z_2, z_1).$$

We are again taking $H = C^2$ with the usual inner product. Then f and g commute and have the common fixed point (a, a), where $a = (1 - \sqrt{1+8r^2})/4r$.

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