BOUNDS FOR THE COEFFICIENTS OF CYCLOTOMIC POLYNOMIALS

R. C. Vaughan

1. INTRODUCTION

We define the nth cyclotomic polynomial $\Phi_n(z)$ by the equation

$$\Phi_n(z) = \prod_{\substack{r=1 \\ (r,n)=1}}^{n} (z - e(r/n)) \qquad (e(\alpha) = e^{2\pi i\alpha}), \tag{1}$$

and we write

$$\Phi_n(z) = \sum_{m=0}^{\phi(n)} a(m, n)\, z^m, \tag{2}$$

where ϕ is Euler's function. P. T. Bateman [1] has shown that

$$|a(m, n)| < \exp\left(\frac{1}{2}\, d(n) \log n\right), \tag{3}$$

where d is the divisor function. P. Erdős has given two proofs [2], [3] of the existence of a positive number c such that for infinitely many natural numbers n,

$$\log \max_m a(m, n) > \exp\left(\frac{c \log n}{\log \log n}\right). \tag{4}$$

Erdős has asked whether it is possible to take c arbitrarily close to log 2, which would imply that Bateman's result is best possible. In Theorem 1 we give an affirmative answer to this question. In fact, we even show that the choice $c = \log 2$ is permissible.

THEOREM 1. *There are infinitely many natural numbers* n *such that*

$$\log \max_m a(m, n) > \exp\left(\frac{(\log 2)(\log n)}{\log \log n}\right). \tag{5}$$

Erdős and R. C. Vaughan [4] have shown that

$$|a(m, n)| < \exp((\tau^{1/2} + o(1))\, m^{1/2}) \tag{6}$$

uniformly in n as $m \to \infty$, where

Received November 12, 1974.

Michigan Math. J. 21 (1974).

$$\tau = \prod_p \left(1 - \frac{2}{p(p+1)}\right),$$

and that for every natural number m,

$$\log \max_n |a(m, n)| \gg \left(\frac{m}{\log m}\right)^{1/2},$$

where $\gg$ is Vinogradov's notation. A modification of the method used to prove Theorem 1 gives a slightly sharper lower bound for infinitely many m.

THEOREM 2. *There are infinitely many natural numbers* m *such that*

$$\log \max_m |a(m, n)| \gg \frac{m^{1/2}}{(\log m)^{1/4}}. \tag{7}$$

2. PRELIMINARIES

Throughout, χ denotes the quadratic character modulo 5. Also, y is a large real number, and

$$n = \prod_{\substack{p \le y \\ \chi(p) = -1}}' p, \tag{8}$$

where the dash signifies that the prime number 2 is included or excluded in the product according as the number of primes $p \le y$ with $\chi(p) = -1$ is odd or even. Then

$$\mu(n) = -1 \tag{9}$$

where μ is the Möbius function. By (1), when $|z| < 1$,

$$\Phi_n(z) = \prod_{r|n} (z^r - 1)^{\mu(n/r)}.$$

Hence, by (8),

$$|\Phi_n(z)| = \exp\left(-\Re\, \mu(n) \sum_{r|n} \mu(r) \sum_{m=1}^{\infty} \frac{1}{m} z^{mr}\right).$$

Thus, by (9),

$$|\Phi_n(z)| = \exp\left(\Re \sum_{m=1}^{\infty} c_m z^m\right), \tag{10}$$

where

$$c_m = \frac{1}{m} \sum_{r|(m,n)} r\,\mu(r). \tag{11}$$

Since χ is a primitive character modulo 5 and $\chi(-1) = 1$, we have for every integer m the equation

$$5^{1/2}\chi(m) = \tau(\chi)\chi(m) = \sum_{r=1}^{4} \chi(r)\,e(mr/5),$$

where

$$\tau(\chi) = \sum_{r=1}^{4} \chi(r)\,e(r/5)$$

is the Gaussian sum. Hence, for every positive real number x,

$$\sum_{m=1}^{\infty} c_m e^{-m/x}\left(e\left(\frac{m}{5}\right) - e\left(\frac{2m}{5}\right) - e\left(\frac{3m}{5}\right) + e\left(\frac{4m}{5}\right)\right) = 5^{1/2}\sum_{m=1}^{\infty} c_m\chi(m)\,e^{-m/x}. \tag{12}$$

Clearly,

$$\sum_{r=1}^{4} e(rm/5) = \begin{cases} 4 & (5\,|\,m), \\ -1 & (5\nmid m). \end{cases}$$

Hence, by (11) and (8),

$$\sum_{m=1}^{\infty} c_m e^{-m/x}\left(e\left(\frac{m}{5}\right) + e\left(\frac{2m}{5}\right) + e\left(\frac{3m}{5}\right) + e\left(\frac{4m}{5}\right)\right) = \sum_{m=1}^{\infty} c_m(e^{-5m/x} - e^{-m/x}). \tag{13}$$

Here it is convenient to recall the formula

$$\int_0^{\infty} e^{-m/x}x^{-\sigma-1}\,dx = m^{-\sigma}\Gamma(\sigma) \qquad (\sigma > 0), \tag{14}$$

where Γ as usual denotes the Gamma function.

By (11), c_m is multiplicative. Hence, by (8),

$$\sum_{m=1}^{\infty} \chi(m)c_m m^{-\sigma} = L(1+\sigma, \chi)\prod_{p|n}(1+p^{-\sigma}) \qquad (\sigma > 0)$$

and

$$\sum_{m=1}^{\infty} c_m m^{-\sigma} = \zeta(1+\sigma)\prod_{p|n}(1-p^{-\sigma}) \qquad (\sigma > 0),$$

where $L(s, \chi)$ is the Dirichlet L-function formed from the character χ, and where ζ is the Riemann zeta function. Hence, by (12), (13), and (14),

$$\int_0^\infty \left(\Re \sum_{m=1}^{\infty} c_m e\left(\frac{m}{5}\right) e^{-m/x}\right) x^{-\sigma-1}\, dx$$

$$= \frac{1}{4}\Gamma(\sigma)\left(5^{1/2} L(1+\sigma, \chi) \prod_{p|n} (1+p^{-\sigma}) - (1-5^{-\sigma})\zeta(1+\sigma)\prod_{p|n}(1-p^{-\sigma})\right) \quad (\sigma > 0). \tag{15}$$

3. THE PROOF OF THEOREM 1

Suppose that $0 < x \le 1$. Then, by (11),

$$\left|\sum_{m=1}^{\infty} c_m e\left(\frac{m}{5}\right) e^{-m/x}\right| \le \sum_{m=1}^{\infty} e^{-m/x} < x.$$

Therefore, by (15), when $0 < \sigma < 1$,

$$\sup_{x \ge 1}\left(\Re \sum_{m=1}^{\infty} c_m e\left(\frac{m}{5}\right) e^{-m/x}\right)$$

$$> \frac{1}{4}\sigma\,\Gamma(\sigma)\left(5^{1/2} L(1+\sigma, \chi) \prod_{p|n} (1+p^{-\sigma}) - (1-5^{-\sigma})\zeta(1+\sigma)\prod_{p|n}(1-p^{-\sigma})\right) - \frac{\sigma}{1-\sigma}.$$

On taking the limit as σ tends to 0, we find that

$$\sup_{x \ge 1}\left(\Re \sum_{m=1}^{\infty} c_m e\left(\frac{m}{5}\right) e^{-m/x}\right) \ge \frac{1}{4} 5^{1/2} L(1, \chi)\, d(n).$$

Therefore, by (10) and (2),

$$(1+\phi(n)) \max_m |a(m, n)| \ge \exp\left(\frac{1}{4} 5^{1/2} L(1, \chi)\, d(n)\right). \tag{16}$$

It is easily verified that

$$\sum_{m=0}^{\phi(n)} a(m, n) = \Phi_n(1) = \prod_{p|n} r^{\mu(n/r)} = \exp(\Lambda(n)),$$

where Λ is von Mangoldt's function. Hence, by (16),

$$\max_m a(m, n) > \exp\left(\frac{1}{4} 5^{1/2} L(1, \chi) d(n) - \log(\phi(n)(\phi(n)+1))\right).$$

We complete the proof of Theorem 1 by observing that by (8) and the prime number theorem for arithmetic progressions

$$\sum_{p|n} 1 = \frac{\log n}{\log\log n}\left(1 + \frac{1 - \log 2}{\log\log n} + O((\log\log n)^{-2})\right).$$

4. THE PROOF OF THEOREM 2

Let

$$\sigma = 1 + \frac{1}{\log y} \tag{17}$$

and

$$u = \exp((\log y)^{1/4}). \tag{18}$$

Then, by (11),

$$\int_0^1 \left|\sum_{m=1}^{\infty} c_m e\left(\frac{m}{5}\right) e^{-m/x}\right| x^{-\sigma-1}\,dx \leq \int_0^1 \sum_{m=1}^{\infty} e^{-m/x} x^{-\sigma-1}\,dx \leq \int_0^1 2x^{1-\sigma}\,dx \ll 1,$$

$$\int_1^u \left|\sum_{m=1}^{\infty} c_m e\left(\frac{m}{5}\right) e^{-m/x}\right| x^{-\sigma-1}\,dx \leq \int_1^u x^{-\sigma}\,dx \leq (\log y)^{1/4},$$

and

$$\int_u^{\infty} (\log x)^{-1/2} x^{-\sigma}\,dx \leq \int_1^y (\log x)^{-1/2}\frac{dx}{x} + (\sigma - 1)^{1/2}\int_1^{\infty} x^{-\sigma}\,dx = 3(\log y)^{1/2}.$$

Therefore, by (15),

$$\begin{aligned} &3(\log y)^{1/2} \sup_{x \geq u}\left(x^{-1}(\log x)^{1/2}\, \Re \sum_{m=1}^{\infty} c_m e\left(\frac{m}{5}\right) e^{-m/x}\right) \\ &> \frac{1}{4}\Gamma(\sigma)\left(5^{1/2} L(1+\sigma, \chi)\prod_{p|n}(1+p^{-\sigma})\right. \\ &\qquad \left. - (1 - 5^{-\sigma})\zeta(1+\sigma)\prod_{p|n}(1-p^{-\sigma})\right) - 2(\log y)^{1/4}. \end{aligned} \tag{19}$$

By (17 and (18),

$$\prod_{p|n} (1+p^{-\sigma}) \gg \exp\Bigg(\sum_{\substack{p\leq y \\ \chi(p)=-1}} p^{-\sigma} \Bigg) \geq \exp\Bigg(\sum_{\substack{p\leq y \\ \chi(p)=-1}} p^{-1}(1-(\sigma-1)\log p) \Bigg).$$

Hence, by (17) and the prime number theorem for arithmetic progressions,

$$\prod_{p|n} (1+p^{-\sigma}) \gg (\log y)^{1/2}.$$

Therefore, by (19), there is a real number x, no smaller than u, such that

$$\Re \sum_{m=1}^{\infty} c_m e\left(\frac{m}{5}\right) e^{-m/x} \gg x(\log x)^{-1/2}. \tag{20}$$

By (6), there is a positive constant c such that

$$\left| \sum_{0\leq m\leq x} a(m, n)\, e\left(\frac{m}{5}\right) e^{-m/x} \right| < e^{cx^{1/2}}$$

and

$$\left| \sum_{m>cx^2} a(m, n)\, e\left(\frac{m}{5}\right) e^{-m/x} \right| < \sum_{m>cx^2} e^{-m/2x} < 1.$$

Therefore, by (2), (10), and (20),

$$\log \left| \sum_{x<m\leq cx^2} a(m, n)\, e\left(\frac{m}{5}\right) e^{-m/x} \right| \gg x(\log x)^{-1/2}.$$

Hence, for an m with $x < m \leq cx^2$,

$$\log(|a(m, n)|\, e^{-m/x}) \gg x(\log x)^{-1/2}.$$

Therefore, if $x \geq m^{1/2}(\log m)^{1/4}$, then

$$\log|a(m, n)| \gg m^{1/2}(\log m)^{-1/4},$$

and if $x < m^{1/2}(\log m)^{1/4}$, then

$$\log|a(m, n)| > m/x > m^{1/2}(\log m)^{-1/4}.$$

Thus, in either case there is an arbitrarily large natural number m such that

$$\log \max_n |a(m, n)| \gg m^{1/2}(\log m)^{-1/4},$$

as required.

REFERENCES

1. P. T. Bateman, *Note on the coefficients of the cyclotomic polynomial.* Bull. Amer. Math. Soc. 55 (1949), 1180-1181.

2. P. Erdős, *On the coefficients of the cyclotomic polynomial.* Portugal. Math. 8 (1949), 63-71.

3. ———, *On the growth of the cyclotomic polynomial in the interval* (0, 1). Proc. Glasgow Math. Assoc. 3 (1957), 102-104.

4. P. Erdős and R. C. Vaughan, *Bounds for the* r-th *coefficients of cyclotomic polynomials.* J. London Math. Soc. (2) 8 (1974), 393-400.

Imperial College
London, S. W. 7.
and
The University of Michigan
Ann Arbor, Michigan 48104