DIVISIBILITY PROPERTIES IN SEMIGROUP RINGS

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1. INTRODUCTION

It is well known that if D is an integral domain with identity, then the polynomial ring $J = D[\{X_\lambda\}_{\lambda \in \Lambda}]$ is a unique factorization domain (UFD) if and only if D is a UFD. Also, J is a GCD-domain if and only if D is a GCD-domain, and J is a principal ideal domain (PID) if and only if D is a field and the cardinality $|\Lambda|$ of Λ is 1. In this paper we consider the problem of determining necessary and sufficient conditions on D and on an additive abelian semigroup S with zero in order that the semigroup ring D[X; S] should be a GCD-domain, a UFD, or a PID. Our main results are contained in Theorems 3.1, 5.2, 6.1, 6.4, 7.17, and 8.4.

For an associative ring R and a semigroup S (written additively), N. Jacobson [21, Exercise 2, p. 95] defines the semigroup ring of S over R to be the set of functions f from S into R that are finitely nonzero, with addition and multiplication defined as follows:

$$(f + g)(s) = f(s) + g(s),$$

$$(fg)(s) = \sum_{t+u=s} f(t)\, g(u),$$

where the symbol $\sum_{t+u=s}$ indicates that the sum is taken over all ordered pairs (t, u) of elements of S such that $t + u = s$. We adopt the notation of D. G. Northcott [29, p. 128] and write R[X; S] for the semigroup ring of S over R. In this paper, we deal only with the case in which the ring R and the semigroup S are commutative. A polynomial ring over R is a semigroup ring over R; in fact $R[\{X_\lambda\}_{\lambda \in \Lambda}]$ is isomorphic to the semigroup ring R[X; S], where S is the weak direct sum of $|\Lambda|$ copies of the additive semigroup Z_0 of nonnegative integers. Hence our results on divisibility properties of the rings D[X; S] are extensions of the results concerning polynomial rings mentioned in the preceding paragraph.

Following I. Kaplansky [24, p. 32], we say that an integral domain D with identity is a GCD-*domain* if each pair of nonzero elements of D has a greatest common divisor in D; other terms that have been used in the literature for this concept are *pseudo-Bezoutian ring* [4, p. 86] and HCF-*ring* [7]. Moreover, since D is a GCD-domain if and only if each pair of nonzero elements of D has a least common multiple in D, another natural term for this class of domains would be LCM-*domain.* GCD-domains have proved to be of interest at several points in the literature, notably in the work of H. Prüfer [32] and P. Jaffard [22, Chapter 3]; see also [7], [15], and [38]. In Theorems 6.1 and 6.4 we prove that the semigroup ring D[X; S] is a GCD-domain if and only if D is a GCD-domain and S is a torsion-free, cancellative

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semigroup with the property that the set of principal ideals of S is closed under finite intersection. A special case of the preceding result is Theorem 5.2: If G is an abelian group, then D[X; G] is a GCD-domain if and only if D is a GCD-domain and G is torsion-free.

Unique factorization domains are precisely the GCD-domains in which the ascending chain condition for principal ideals is satisfied. Hence our results on GCD-domains in Sections 4-6 are useful to us in considering D[X; S] as a UFD. In Theorem 7.17, we prove that the semigroup ring D[X; S] is a UFD if and only if D is a UFD, the semigroup S is a unique factorization semigroup, and each element of the maximal subgroup H of S is of type (0, 0, 0, ⋯) (an equivalent condition on H is that each subgroup of H of rank 1 is cyclic). Special cases of Theorem 7.17 are Theorems 7.9 and 7.12: If F is a field and G is a torsion-free abelian group, then the group ring F[X; G] is a UFD if and only if each element of G is of type (0, 0, 0, ⋯).

Theorem 7.12 is useful in constructing examples of non-Noetherian unique factorization domains. It is well known that the group ring F[X; G] is Noetherian if and only if G is finitely generated (see, for example, [10]). Thus, if F is a field and G is a nonfinitely generated, torsion-free group in which each element is of type (0, 0, 0, ⋯), then the group ring F[X; G] is a non-Noetherian UFD. Nonfinitely generated torsion-free groups in which each element is of type (0, 0, 0, ⋯) are well known (see [12, p. 151]). In [18], the first author has used Theorem 7.12 to construct examples of non-Noetherian unique factorization domains of arbitrary characteristic and arbitrary Krull dimension $k \geq 2$; for $k = 2$, it seems that such examples have not appeared in the literature.

In Section 8 we determine necessary and sufficient conditions under which D[X; S] is a PID. These conditions are given in Theorem 8.4: D[X; S] is a PID if and only if D is a field and S is isomorphic either to the additive semigroup Z_0 of nonnegative integers or to the additive group Z of integers. Consequently, D[X; S] is a PID if and only if D is a field and D[X; S] is isomorphic to D[Y] or to $D[Y, Y^{-1}]$.

For emphasis, we repeat a statement already made: *all rings and semigroups considered in this paper are assumed to be commutative.*

2. DIVISIBILITY PROPERTIES IN CANCELLATIVE SEMIGROUPS

Before embarking on a study of semigroup rings, we need to develop some preliminary results concerning divisibility properties of semigroups. In order to avoid complications, we assume that each semigroup S considered in this section is *cancellative*—that is, $ax = ay$ implies $x = y$ for all $a, x, y \in S$. Cancellative semigroups are precisely those that can be imbedded in a group; we call the smallest such group G in which a cancellative semigroup S can be imbedded *the quotient group of* S.

Let S be a cancellative multiplicative semigroup with identity. The concepts usually considered in connection with unique element factorization in an integral domain D with identity are defined in terms of the multiplicative semigroup $D - \{0\}$, and hence, they are meaningful for S; Section 3.1 of [8] is a good reference for this point of view. In particular, if $\{a_1, a_2, \cdots, a_n\}$ is a finite subset of S, then an element s of S is called a *greatest common divisor* of $a_1, a_2, \cdots, a_n$ (written $s = \gcd\{a_1, \cdots, a_n\}$) if s divides each a_i and s is divisible by each common divisor of $a_1, a_2, \cdots, a_n$; if $a_1, a_2, \cdots, a_n$ have a greatest common divisor in S, then it is unique to within unit factors. Similarly, a least common multiple of

$a_1, \cdots, a_n$ (lcm $\{a_1, \cdots, a_n\}$) can be defined, and it is uniquely determined, if it exists, to within unit factors. In analogy with results that are well known in the case of an integral domain, we have the following:

2.1. LEMMA. *Assume that* $x, y \in S$ *and* $\{a_i\}_1^n$, $\{b_i\}_1^m$ *are finite nonempty subsets of* S.

(1) lcm $\{a_1, \cdots, a_n\}$ *exists if and only if* lcm $\{xa_1, \cdots, xa_n\}$ *exists; if both exist, then* lcm $\{xa_1, \cdots, xa_n\} = x \cdot$ lcm $\{a_1, \cdots, a_n\}$.

(2) *If* gcd $\{xa_1, \cdots, xa_n\}$ *exists, then* gcd $\{a_1, \cdots, a_n\}$ *exists and* gcd $\{xa_1, \cdots, xa_n\} = x \cdot$ gcd $\{a_1, \cdots, a_n\}$.

(3) *If* $a =$ gcd $\{a_1, \cdots, a_n\}$ *and* $b =$ gcd $\{b_1, \cdots, b_m\}$ *exist, then* gcd $\{a_1, \cdots, a_n, b_1, \cdots, b_m\}$ *exists if and only if* gcd $\{a, b\}$ *exists; if these two greatest common divisors exist, they are equal.*

(4) *If* $c =$ lcm $\{x, y\}$ *exists, then* $xy = cs$ *for some element* s *in* S. *Moreover,* $s =$ gcd $\{x, y\}$.

It is known that the converses of (2) and (4) of Lemma 2.1 may fail, even if S is the semigroup of nonzero elements of an integral domain (see, for example, [5, p. 108] or [17, pp. 76-77])—that is, if gcd $\{a_1, \cdots, a_n\}$ exists, then gcd $\{xa_1, \cdots, xa_n\}$ need not exist; and if x and y have a greatest common divisor, they need not have a least common multiple. On the other hand, if each pair of elements of S has a greatest common divisor, then each pair of elements has a least common multiple; this statement follows from the next result, Proposition 2.2.

Although the proof of Proposition 2.2 is easy, the result itself seems not to be well known, even for the multiplicative semigroup of an integral domain (see, however, [5, pp. 94-96], [24, p. 32], and [17, pp. 75-77]).

2.2. PROPOSITION. *Let* x *and* y *be elements of* S. *The following conditions are equivalent.*

(1) lcm $\{x, y\} = xy$.

(2) $xS \cap yS = xyS$.

(3) *For all* z *in* S, $x \mid z$ *and* $y \mid z$ *implies that* $xy \mid z$.

(4) *For all* z *in* S, gcd $\{zx, zy\} = z$.

(5) gcd $\{x, y\} = 1$ *and* gcd $\{zx, zy\}$ *exists for each* z *in* S.

(6) *For each* z *in* S, $x \mid yz$ *implies that* $x \mid z$.

Proof. It is clear that conditions (1)-(3) are equivalent, and it is also clear that (4) implies (5). If lcm $\{x, y\} = xy$, then lcm $\{xz, yz\} = xyz$ by part (1) of Lemma 2.1, and hence gcd $\{xz, yz\} = z$ by part (4) of Lemma 2.1; therefore (1) implies (5).

(5) $\Rightarrow$ (4): It is clear that z divides gcd $\{zx, zy\}$—say gcd $\{zx, zy\} = zt$. Since S is cancellative, it follows that t divides both x and y. Therefore t is a unit of S, and gcd $\{zx, zy\} = z$.

(2) $\Rightarrow$ (6): If $yz = xs$, then $yz \in xS \cap yS = xyS$, so that $yz = xyt$ for some t in S. Since S is cancellative, $z = xt$ and $x \mid z$.

(6) $\Rightarrow$ (2): If $xa = yb \in xS \cap yS$, then $x \mid yb$, and by (6), $x \mid b$ so that $b \in xS$ and $yb \in xyS$. Consequently, $xS \cap yS = xyS$.

2.3. COROLLARY. *The following conditions are equivalent.*

(1) *Each pair of elements of* S *has a greatest common divisor in* S.

(2) *Each finite set of elements of* S *has a greatest common divisor in* S.

(3) *Each pair of elements of* S *has a least common multiple in* S.

(4) *Each finite set of elements of* S *has a least common multiple in* S.

Proof. The implications (2) ⇒ (1) and (4) ⇒ (3) are obvious, and the reverse implications follow by induction from part (3) of Lemma 2.1. Part (4) of Lemma 2.1 shows that (3) implies (1); we prove that (1) implies (3). Pick $x, y \in S$, let $d = \gcd\{x, y\}$, and write $x = dx_1$, $y = dy_1$. Then $\gcd\{x_1, y_1\} = 1$, by part (2) of Lemma 2.1, and $\gcd\{zx_1, zy_1\}$ exists for each z in S. Therefore, $\operatorname{lcm}\{x_1, y_1\} = x_1 y_1$, by Proposition 2.2, and $\operatorname{lcm}\{x, y\} = \operatorname{lcm}\{dx_1, dy_1\} = dx_1 y_1$. This completes the proof of Corollary 2.3.

We call a semigroup satisfying the equivalent conditions of Corollary 2.3 a GCD-*semigroup* (see [6]). If S is a GCD-semigroup, then part (3) of Lemma 2.1 is a consequence of the fact that the binary operations "gcd" and "lcm" are associative [11, p. 34].

For the proof of Theorem 4.4, we need the following two results concerning GCD-semigroups.

2.4. LEMMA. *Assume that* S *is a* GCD-*semigroup and that* $a, b, c \in S$. *If* $\gcd\{a, b\} = d_1$, *then* $\gcd\{a, bc\} = \gcd\{a, d_1 c\}$.

Proof. Let $d = \gcd\{a, bc\}$, and let $t = \gcd\{a, d_1 c\}$. It is clear that t divides d. But d divides $\gcd\{ac, bc\} = c \cdot \gcd\{a, b\} = cd_1$, and hence d divides t. Therefore $d = \gcd\{a, d_1 c\}$.

2.5. PROPOSITION. *Suppose that* S *is a* GCD-*semigroup, that* $a_1, \cdots, a_n \in S$ *are such that* $\gcd\{a_1, \cdots, a_n\} = 1$, *and that* $k_1, k_2, \cdots, k_n$ *are positive integers. Then* $\gcd\{a_1^{k_1}, \cdots, a_n^{k_n}\} = 1$.

Proof. It is clear that we need only consider the case where

$$k_1 = k_2 = \cdots = k_{n-1} = 1 .$$

Moreover, since

$$\gcd\{a_1, \cdots, a_n\} = \gcd\{\gcd\{a_1, \cdots, a_{n-1}\}, a_n\} ,$$

it suffices to establish Proposition 2.5 for the case where $n = 2$. Thus, assume that $\gcd\{a, b\} = 1$ and that k is a positive integer such that $\gcd\{a, b^k\} = 1$. By Lemma 2.4, $\gcd\{a, b \cdot b^k\} = \gcd\{a, 1 \cdot b^k\} = 1$. This completes the proof of Proposition 2.5.

3. AN ANALOGUE OF NAGATA'S THEOREM

In this section we prove Theorem 3.1, a result that will subsequently allow us to reduce several questions concerning semigroup rings to the same questions for group rings. In [27], M. Nagata has proved the following theorem, which we label (NT).

(NT) *If* D *is a Noetherian domain with identity, if* S *is a multiplicative system in* D *generated by prime elements, and if* D_S *is a* UFD, *then* D *is a* UFD.

The preceding result is frequently referred to in the literature as Nagata's Theorem; the following generalization of (NT) appears in [8, p. 116].

$(NT)^*$ *Assume that* D *is an integral domain with identity such that each nonzero element of* D *is a finite product of irreducible elements of* D. *If* S *is a multiplicative system in* D *generated by prime elements of* D, *and if* D_S *is a* UFD, *then* D *is a* UFD.

We seek an analogue of Nagata's Theorem for GCD-domains. The result we obtain, Theorem 3.1, uses the following terminology. If S is a nonempty subset of the set of nonzero elements of an integral domain J with identity, and if x is a nonzero element of J, then x is LCM-*prime to* S if $xJ \cap sJ = xsJ$ for each s in S.

3.1. THEOREM. *Let* N *be a multiplicative system in an integral domain* D *with identity, and let* T *be the set of elements of* D *that are* LCM-*prime to* N. *Assume that the following two conditions are satisfied.*

(1) *Each pair of elements of* N *has a least common multiple in* D.

(2) *Each nonzero element of* D *can be expressed as the product of an element of* N *and an element of* T.

If D_N *is a* GCD-*domain, then* D *is a* GCD-*domain.*

Proof. We prove first that $tD_N \cap D = tD$ for each element t in T. The containment $tD \subseteq tD_N \cap D$ always holds, and if $d \in D$ is such that $dn = tm$ for some $n \in N$, $m \in D$, then, since $(n) \cap (t) = (nt)$, n divides m (Proposition 2.2) and $d \in tD$. Therefore $tD = tD_N \cap D$, as asserted.

To prove that D is a GCD-domain, we show that if b and c are nonzero elements of D, then the ideal $bD \cap cD$ is principal. We write $b = n_1 t_1$ and $c = n_2 t_2$, where $n_1, n_2 \in N$ and $t_1, t_2 \in T$. Let $n = \operatorname{lcm}\{n_1, n_2\}$. Since the elements of N are units of D_N, we see that $bD_N = t_1 D_N$ and $cD_N = t_2 D_N$. Moreover, since D_N is a GCD-domain, $bD_N \cap cD_N = t_1 D_N \cap t_2 D_N$ is a principal ideal of D_N, and hence is the extension of a principal ideal of D. In fact, we can assume that $bD_N \cap cD_N = tD_N$, where $t \in T$. We prove that $bD \cap cD = ntD$. Since $t \in t_i D_N$ for i = 1, 2, we conclude that $t \in t_i D_N \cap D = t_i D$, and $nt \in n_i t_i D$. Therefore $ntD \subseteq bD \cap cD$. If $x \in bD \cap cD$, we write x as ms, where $m \in N$ and $s \in T$. Then $n_i \mid ms$ for i = 1, 2, and $(n_i) \cap (s) = (n_i s)$, so that $n_i \mid m$; consequently, $n \mid m$. On the other hand,

$$xD_N \cap D = sD \subseteq bD_N \cap cD_N \cap D = tD,$$

so that $t \mid s$. We conclude that $nt \mid ms = x$, and $bD \cap cD = ntD$, as asserted. This completes the proof of Theorem 3.1.

Our proof of Theorem 7.17 requires a slight generalization of $(NT)^*$; we establish this generalization in the next result.

3.2. THEOREM. *Let* D *be an integral domain with identity, and let* $\mathscr{P} = \{p_\alpha\}$ *be a nonempty set of prime elements of* D *satisfying the following condition* (Δ):

(Δ) *No nonzero element of* D *is divisible by infinitely many of the primes* p_α *or by infinitely many powers of a fixed prime* p_α.

Let N *be the multiplicative system generated by the set* $\mathscr{P}$.

(a) *If the ascending chain condition for principal ideals* (a.c.c.p.) *is satisfied in* D_N, *then* a.c.c.p. *is satisfied in* D.

(b) *If* D_N *is a* GCD-*domain, then* D *is a* GCD-*domain.*

(c) *If* D_N *is a* UFD, *then* D *is a* UFD.

Proof. Let T be the set of elements of D that are LCM-prime to N (in this case, T is the set of nonzero elements of D divisible by no p_α); T is a multiplicative system in D, and it is straightforward to show that condition (Δ) implies that (1) and (2) of Theorem 3.1 are satisfied. Hence (b) follows from Theorem 3.1. To prove (a), let $a_1 D \subseteq a_2 D \subseteq \cdots$ be an ascending sequence of nonzero principal ideals of D, and for each i, express a_i as $n_i t_i$, where n_i is in N and t_i is in T. The hypotheses on N and T imply that $n_i D \subseteq n_{i+1} D$ and $t_i D \subseteq t_{i+1} D$ for each i; moreover, $a_i D = a_{i+1} D$ if and only if $n_i D = n_{i+1} D$ and $t_i D = t_{i+1} D$. Therefore, the chain $a_1 D \subseteq a_2 D \subseteq \cdots$ stabilizes if and only if each of the chains $n_1 D \subseteq n_2 D \subseteq \cdots$ and $t_1 D \subseteq t_2 D \subseteq \cdots$ stabilizes.

The proof of Theorem 3.1 shows that $t_i D_N \cap D = t_i D$ and $n_i D_T \cap D = n_i D$ for each i. Hence, if a.c.c.p. is satisfied in D_N, then the chain $t_1 D_N \subseteq t_2 D_N \subseteq \cdots$ stabilizes, and consequently, the chain $t_1 D \subseteq t_2 D \subseteq \cdots$ also becomes stable. Since each principal ideal of D_T is the extension of a principal ideal of D, each principal ideal of D_T is of the form nD_T for some n in N. Therefore D_T is a UFD, the chains $n_1 D_T \subseteq n_2 D_T \subseteq \cdots$ and $n_1 D \subseteq n_2 D \subseteq \cdots$ becomes stable, and a.c.c.p. is satisfied in D, as asserted. This completes the proof of (a).

Part (c) of Theorem 3.2 follows from parts (a) and (b) and the fact that an integral domain J with identity is a UFD if and only if J is a GCD-domain in which a.c.c.p. is satisfied [2], [35, p. 16].

We remark that the analogue of Theorem 3.2 for cancellative semigroups with identity is valid. Part (b) of Theorem 3.2 has been obtained independently by M. Schexnayder [37].

4. A REDUCTION TO THE CASE OF A FIELD

We now have the results needed to determine conditions under which D[X; S] is a GCD-domain. The main result of this section is Theorem 4.4, which states that D[X; S] is a GCD-domain if and only if D and K[X; S] are GCD-domains, where K is the quotient field of D.

It is known that if R is a commutative ring and S is an additive abelian semigroup, then the semigroup ring R[X; S] is an integral domain if and only if R is an integral domain and S is torsion-free and cancellative [1], [14]. (To say that S is *torsion-free* means that there do not exist distinct elements x, y of S and a positive integer n such that nx = ny; S is torsion-free if and only if the quotient group of S is torsion-free.) Hence, in trying to determine conditions under which a semigroup ring is a GCD-domain, we restrict ourselves to the case where the coefficient ring is an integral domain with identity and the semigroup is torsion-free and cancellative.

4.1. LEMMA. *Let* D *be an integral domain, and let* S *be a torsion-free, cancellative, additive semigroup. If* f *and* g *are nonzero elements of* D[X; S] *and if* fg *is a monomial, then* f *and* g *are monomials.*

Proof. Since S is torsion-free and cancellative, it admits a total order $<$ compatible with the semigroup structure [29, p. 123]. We write

$$f = f_1 X^{s_1} + \cdots + f_n X^{s_n} \quad \text{and} \quad g = g_1 X^{t_1} + \cdots + g_m X^{t_m},$$

where f_1 and f_n are nonzero and $s_1 < \cdots < s_n$, and where g_1 and g_m are nonzero and $t_1 < \cdots < t_m$. Because D is an integral domain and S is cancellative, it is then clear that fg is not a monomial if either $n > 1$ or $m > 1$.

4.2. COROLLARY. *If* D *is an integral domain with identity and* S *is a torsion-free, cancellative semigroup, then the group of units of* D[X; S] *is*

$$\{uX^s \mid u \text{ is a unit of } D \text{ and } s \text{ has an additive inverse in } S\}.$$

In our next result, Theorem 4.4, we use the following terminology. If D is a GCD-domain and $f = \sum_{i=1}^{n} f_i X^{s_i}$ is a nonzero element of D[X; S], then we say that f is *primitive* if $\gcd\{f_1, f_2, \cdots, f_n\} = 1$. Our proof of one part of Theorem 4.4 will use a generalization of the so-called Dedekind-Mertens Lemma (see [17, Section 28]); the following result, which we label Theorem 4.3, is proved by D. G. Northcott in [28, pp. 286-287] (generalizations of Theorem 4.3 appear in [19, Theorem 3.7] and in [30]).

4.3. THEOREM. *Let* R *be a commutative ring, and let* S *be a torsion-free, cancellative semigroup. For* $h \in R[X; S]$, *let* B_h *be the additive group generated by the coefficients of* h. *If* f *and* g *are nonzero elements of* R[X; S], *then there exists a positive integer* K *such that* $B_f^k B_g = B_f^{k-1} B_{fg}$ *for each* $k \geq K$.

4.4. THEOREM. *Assume that* D *is an integral domain with identity having quotient field* K, *and* S *is a torsion-free, cancellative, additive semigroup with zero. The following conditions are equivalent.*

(1) D[X; S] *is a* GCD-*domain.*

(2) D *and* K[X; S] *are* GCD-*domains.*

Proof. If D[X; S] is a GCD-domain, then K[X; S] is a GCD-domain, for $K[X; S] = D[X; S]_N$, where N is the set of nonzero elements of D. Moreover, if $d_1, d_2 \in D$, then the ideal

$$d_1 D[X; S] \cap d_2 D[X; S] = (d_1 D \cap d_2 D)[X; S]$$

is principal. Since the ideal $(d_1 D \cap d_2 D)[X; S]$ contains monomials, it follows from Lemma 4.1 that $(d_1 D \cap d_2 D)[X; S]$ is generated by a monomial, say dX^s. It is then easy to show that $dD = d_1 D \cap d_2 D$ and that s has an additive inverse in S. At any rate, $d_1 D \cap d_2 D$ is principal, and D is a GCD-domain.

We prove that if D and K[X; S] are GCD-domains, then D[X; S] is a GCD-domain (by proving that D[X; S] and the multiplicative system $N = D - \{0\}$ of D[X; S] satisfy conditions (1) and (2) of Theorem 3.1). If $n_1, n_2 \in N$ and n is the least common multiple of n_1 and n_2 in D, then n is the least common multiple of n_1 and n_2 in D[X; S]; this is true since extension of ideals distributes over intersection, in passage from D to D[X; S]. We let T be the set of primitive elements of D[X; S]. It is clear that each nonzero element of D[X; S] is of the form nt, for some $n \in N$, $t \in T$. To prove that $(n) \cap (t) = (nt)$ for $n \in N$ and $t \in T$, we take f, $g \in D[X; S] - \{0\}$ such that $nf = tg$. By Theorem 4.3, there exists a positive integer k such that

$$B_t^k B_g = B_t^{k-1} B_{tg} = B_t^{k-1} B_{nf} = B_t^{k-1} n B_f.$$

It follows that if g_j is a nonzero coefficient of g and $\{t_1, \cdots, t_r\}$ is the set of nonzero coefficients of t, then n divides $t_i^k g_j$ for each i between 1 and r. Hence n divides

$$\gcd\{t_1^k g_j, t_2^k g_j, \cdots, t_r^k g_j\} = g_j \cdot \gcd\{t_1^k, \cdots, t_r^k\},$$

and by Proposition 2.5, $\gcd\{t_1^k, \cdots, t_r^k\} = 1$. Therefore n divides each g_j, so that $g \in nD[X; S]$ and $tg \in (nt)$, as we wished to show. We have proved that N and T satisfy the conditions of Theorem 3.1, and since $D[X; S]_N = K[X; S]$ is a GCD-domain, Theorem 3.1 shows that D[X; S] is a GCD-domain.

4.5. COROLLARY. (See [17, Theorem 34.10].) *The polynomial ring* $D[\{X_\lambda\}]$ *is a* GCD-*domain if and only if* D *is a* GCD-*domain.*

In the proof of the implication (2) ⇒ (1) in Theorem 4.4, it is possible to avoid the use of Theorem 4.3. Note that Theorem 4.3 was used in the proof of Theorem 4.4 to establish the containment $(n) \cap (t) \subseteq (nt)$, where $n \in N$ and t is primitive. A more elementary proof of this containment can be based on the fact that a product of primitive elements is primitive (this result is well known in the case of polynomial rings; see [17, p. 425], [24, Exercise 8, p. 42], [39, Lemma 4.2]). Thus, if $h = nf = tg \in (n) \cap (t)$, then we write $g = n_1 t_1$, where n_1 is the greatest common divisor of the coefficients of g and t_1 is primitive. Since tt_1 is primitive, it follows that n_1 is divisible by n. and $h = t n_1 g_1 \in (nt)$. The following proof that a product of primitive elements of D[X; S] is primitive is a modification of a proof given by Eduardo Bastida in the case of polynomial rings.

4.6. PROPOSITION. *Assume that* D *is a* GCD-*domain and that* S *is a torsion-free, cancellative semigroup. If* f *and* g *are primitive elements of* D[X; S], *then* fg *is also primitive.*

Proof. We assume that the relation $<$ on S is a total order compatible with the semigroup structure of S, and we write

$$f = \sum_{i=0}^{m} a_i X^{s_i} \quad \text{and} \quad g = \sum_{i=0}^{n} b_i X^{t_i},$$

where $s_0 < s_1 < \cdots < s_m$, $t_0 < t_1 < \cdots < t_n$, and each a_i and each b_j is nonzero. To prove Proposition 4.6, we prove that if d is a nonzero nonunit of D, then d fails to divide some coefficient of fg.

Without loss of generality, we may assume that $\gcd\{a_0, d\} \neq 1$, for if $\gcd\{a_0, d\} = \gcd\{b_0, d\} = 1$, then $\gcd\{a_0 b_0, d\} = 1$ and d does not divide $a_0 b_0$, a coefficient of fg. We consider the sequence $\{d_j\}_{j=0}^m$, where

$$d_j = \gcd\{a_0, a_1, \cdots, a_j, d\}.$$

Since f is primitive, there is a smallest integer i such that $d_i = 1$, and by assumption, $i \geq 1$. Thus d_{i-1} is a nonunit divisor of d, and to prove that d fails to divide some coefficient of fg, it suffices to prove this assertion for d_{i-1}; hence we assume that $d = d_{i-1}$. If k is chosen minimal with respect to the property that $\gcd\{b_0, b_1, \cdots, b_k, d\} = 1$, then by replacing d by $d' = \gcd\{b_0, b_1, \cdots, b_{k-1}, d\}$ (if k = 0, then d = d'), we can assume that d is a nonunit divisor of $a_0, a_1, \cdots, a_{i-1}, b_0, b_1, \cdots, b_{k-1}$ and that $\gcd\{a_i, d\} = \gcd\{b_k, d\} = 1$. If ϕ is the canonical homomorphism of D onto D/dD and if ϕ^* is the canonical extension of ϕ to a homomorphism from D[X; S] onto (D/dD)[X; S], then it follows that

$$\phi^*(f) = \sum_{0}^{m} \phi(a_j) X^{s_j} = \sum_{j=i}^{m} \phi(a_j) X^{s_j} \quad \text{and} \quad \phi^*(g) = \sum_{j=k}^{n} \phi(b_j) X^{t_j}.$$

Moreover, since $\gcd\{a_i, d\} = \gcd\{b_k, d\} = 1$, and consequently $\gcd\{a_i b_k, d\} = 1$, it follows that $\phi(a_i b_k) \neq 0$, $\phi^*(fg) \neq 0$, and d fails to divide some coefficient of fg. This completes the proof of Proposition 4.6.

5. THE CASE OF A GROUP

The next step in determining conditions under which $D[X; S]$ is a GCD-domain is the treatment of the case where S is a group; we prove in Theorem 5.2 that $D[X; G]$ is a GCD-domain if D is a GCD-domain and G is a torsion-free group. The proof of Theorem 5.2 uses a proposition that is of interest in itself.

5.1. PROPOSITION. *Assume that* S *is an additive abelian semigroup with zero and that* G *is a subsemigroup of* S *such that* G *contains* 0 *and* G *is a group. If* R *is a commutative ring with identity, then the semigroup ring* $R[X; S]$ *is a free* $R[X; G]$-*module.*

Proof. If $\{s_\alpha\}$ is a complete set of representatives of the cosets of G in S, then it is easy to verify that $\{X^{s_\alpha}\}$ is a free $R[X; G]$-module basis for $R[X; S]$.

We remark that the proof of Proposition 5.1, for groups, is essentially contained in the proof of Lemma 2.4 of [31, p. 6]; see also Theorem A of [4].

5.2. THEOREM. *If* D *is a* GCD-*domain and* G *is a torsion-free abelian group, then* $D[X; G]$ *is a* GCD-*domain.*

Proof. Consider nonzero elements $f = \sum_{i=1}^{n} f_i X^{s_i}$ and $g = \sum_{i=1}^{m} g_i X^{t_i}$ of $D[X; G]$, and let H be the subgroup of G generated by the set

$$\{s_1, \cdots, s_n, t_1, \cdots, t_m\};$$

H is a direct sum of k copies of Z for some $k \leq n + m$, and hence $D[X; H]$ is a quotient ring of a polynomial ring in k indeterminates over D. Consequently, $D[X; H]$ is a GCD-domain. Let $J_H = D[X; H]$, let $J_G = D[X; G]$, and let $\{y_\lambda\}$ be a free module basis for J_G over J_H. If h generates the principal ideal $fJ_H \cap gJ_H$ of J_H (that is, if h is the least common multiple of f and g in J_H), then

$$fJ_G \cap gJ_G = f\Big(\sum_\lambda J_H y_\lambda\Big) \cap g\Big(\sum_\lambda J_H y_\lambda\Big) = \Big(\sum_\lambda fJ_H y_\lambda\Big) \cap \Big(\sum_\lambda gJ_H y_\lambda\Big)$$

$$= \sum_\lambda (fJ_H \cap gJ_H) y_\lambda = \sum_\lambda hJ_H y = h\Big(\sum_\lambda J_H y_\lambda\Big) = hJ_G.$$

We conclude that the set of principal ideals of $D[X; G]$ is closed under finite intersection--that is, $D[X; G]$ is a GCD-domain.

Our original proof of Theorem 5.2 was much more complicated than the proof given above. Through conversations with J. Brewer, D. Costa, and A. Grams, the first author became aware of Proposition 5.1, and that result simplified our previous proof.

6. A RESOLUTION OF THE PROBLEM FOR GCD-DOMAINS

In this section we solve the problem of determining necessary and sufficient conditions in order that a semigroup ring should be a GCD-domain (Theorems 6.1 and 6.4). We begin with the easier half of the problem.

6.1. THEOREM. *Assume that* D *is an integral domain with identity and that* S *is an additive semigroup with zero. If the semigroup ring* D[X; S] *is a* GCD-*domain, then* D *is a* GCD-*domain and* S *is a torsion-free* GCD-*semigroup.*

Proof. With S written additively, the condition that S is a GCD-semigroup translates, of course, to the condition that for all a, b $\in$ S, there is an element c in S such that $(a + S) \cap (b + S) = c + S$.

If D[X; S] is a GCD-domain, then clearly S is torsion-free and cancellative, and Theorem 4.4 implies that D is a GCD-domain. If a, b $\in$ S, then $(X^a) \cap (X^b)$ is a principal ideal of D[X; S]. Because $X^{a+b} \in (X^a) \cap (X^b)$, it follows from Lemma 4.1 that $(X^a) \cap (X^b)$ is generated by X^c for some element c in S. But (X^t), for $t \in S$, is simply the semigroup ring D[X; t + S]. Therefore, the equality $(X^a) \cap (X^b) = (X^c)$ implies that $(a + S) \cap (b + S) = c + S$, and S is a GCD-semigroup.

In order to establish the converse of Theorem 6.1, we must first establish an analogue of Theorem 4.3 for exponents of elements of a semigroup ring; we use the following notation. If R is a commutative ring and S is a semigroup, then for

$$f = \sum_{i=1}^{n} f_i X^{s_i} \in R[X; S],$$

E_f denotes the set of elements s_i of S such that $f_i \neq 0$ (that is, E_f is the *support of* f); if A and B are nonempty subsets of S, then A + B denotes the subset $\{a + b \mid a \in A, b \in B\}$ of S.

6.2. PROPOSITION. *Let* D *be an integral domain, and let* S *be a torsion-free, cancellative, additive semigroup. If* f *and* g *are nonzero elements of* D[X; S], *then* $kE_f + E_g = (k - 1)E_f + E_{fg}$, *where* k *is the number of nonzero monomials that appear in* g (*that is,* k *is the cardinality of* E_g).

Proof. We assume that < is a total order on S compatible with its semigroup structure, and we write

$$f = \sum_{i=1}^{r} f_i X^{a_i} \quad \text{and} \quad g = \sum_{i=1}^{k} g_i X^{b_i},$$

where $E_f = \{a_1 < a_2 < \cdots < a_r\}$ and $E_g = \{b_1 < b_2 < \cdots < b_k\}$.

Since $E_{fg} \subseteq E_f + E_g$, we see that $(k - 1)E_f + E_{fg} \subseteq kE_f + E_g$. To prove the reverse inclusion, we consider the set $\mathscr{E}$ of all (k + 1)-tuples α of the form $\alpha = (a_{i_1}, \cdots, a_{i_t}, b_t, a_{i_{t+1}}, \cdots, a_{i_k})$, where $a_{i_1} \leq \cdots \leq a_{i_k}$; note that b_t is the (t + 1)-st coordinate of α. Each such element α of $\mathscr{E}$ gives rise to an element $\alpha^* = a_{i_1} + \cdots + a_{i_t} + b_t + a_{i_{t+1}} + \cdots + a_{i_k}$ of $kE_f + E_g$. The mapping $\alpha \to \alpha^*$ of $\mathscr{E}$ into $kE_f + E_g$ is surjective, but it may not be one-to-one. We order the finite set $\mathscr{E}$ lexicographically by requiring that each element of E_g precedes each element of E_f (note that we are, in effect, considering the disjoint union of the sets E_g and E_f; if, for example, $a_1 = b_1$ and $a_2 = b_2$, then $(a_1, b_1, a_2, a_3, \cdots, a_k)$ and

$(a_1, a_1, b_2, a_3, \cdots, a_k)$ are distinct elements of $\mathscr{E}$, although, as $(k+1)$-tuples, they are identical) and that a_i precedes a_j if and only if $i < j$. As described, this is a total order on $\mathscr{E}$. We take an element m of $kE_f + E_g$, and we let $\{\mu_1, \cdots, \mu_e\}$ be the finite set of elements μ of $\mathscr{E}$ such that $m = \mu^*$; we assume that of these e elements of $\mathscr{E}$,

$$\mu_1 = (a_{i_1}, \cdots, a_{i_j}, b_j, a_{i_{j+1}}, \cdots, a_{i_k})$$

is first in the lexicographic order on $\mathscr{E}$. We show that this implies that the representation of the element $a_{i_j} + b_j$ of $E_f + E_g$ in the form $a_u + b_v$ is unique. Suppose not. Then $a_{i_j} + b_j = a_u + b_v$, where $u \neq i_j$ and $v \neq j$.

We claim that if α is the element of $\mathscr{E}$ with coordinates

$$a_{i_1}, \cdots, a_{i_{j-1}}, a_u, b_v, a_{i_{j+1}}, \cdots, a_{i_k},$$

then α precedes μ_1 in the lexicographic order.

Case I. If $b_j > b_v$, then b_v occurs as the $(v+1)$-st coordinate of α. Since $j + 1 > v + 1$ and b_v precedes each element of E_f (in particular, b_v precedes a_{i_v}), it follows that α precedes μ_1.

Case II. If $b_j < b_v$, then $a_{i_j} > a_u$. If $a_{i_{j-1}} \leq a_u$, then

$$\alpha = (a_{i_1}, \cdots, a_{i_{j-1}}, a_u, \cdots, b_v, \cdots),$$

and it is clear that α precedes μ_1. Otherwise, α can be written as $(\cdots, a_u, a_{i_t}, \cdots, a_{i_{j-1}}, \cdots, b_v, \cdots)$, where $u \leq i_t \leq i_{j-1}$, and again it is clear that α precedes μ_1.

The contradictions that follow in Cases I and II substantiate our claim concerning the representation of $a_{i_j} + b_j$ as an element of $E_f + E_g$. Since D is an integral domain, it follows that $a_{i_j} + b_j \in E_{fg}$, and consequently,

$$m = a_{j_1} + \cdots + a_{i_{j-1}} + a_{i_{j+1}} + \cdots + a_{i_k} + a_{i_j} + b_j \in (k-1)E_f + E_{fg}.$$

This establishes the inclusion $kE_f + E_g \subseteq (k-1)E_f + E_{fg}$, and therefore Proposition 6.2.

It is interesting to note that while the Dedekind-Mertens Lemma (Theorem 4.3) is valid for arbitrary semigroup rings, Proposition 6.2 fails without the hypothesis that D is an integral domain. In fact, if R is a commutative ring with identity and containing a nonzero element r such that $r^2 = 0$, then in the semigroup ring $R[Y] = R[X; Z_0]$, the elements $f = r + Y$ and $g = r - Y$ are such that $0 \in kE_f + E_g$ for each positive integer k, while $0 \in (k-1)E_f + E_{fg}$ for no k; this problem is not alleviated if E_f, E_g, and E_{fg} are replaced by the subsemigroups of Z_0 that they generate or if they are replaced by the subgroups of Z, the quotient group of Z_0, that they generate.

Our next result, Proposition 6.3, is the analogue of Proposition 4.6. We use the following terminology. Let D be an integral domain, S a GCD-semigroup, and

$f = \sum_{i=1}^{n} d_i X^{s_i}$ a nonzero element of $D[X; S]$, where no d_i is 0. We say that f is e-*primitive* (exponent-primitive) if $\gcd\{s_1, s_2, \cdots, s_n\} = 0$.

6.3. PROPOSITION. *Assume that* D *is an integral domain and that* S *is a* GCD-*semigroup with zero. If* f *and* g *are* e-*primitive elements of* $D[X; S]$, *then* fg *is also* e-*primitive.*

Proof. Let $X^a h = fg$, where $a \in S$ and h is e-primitive. By Proposition 4.6, there exists a positive integer k such that

$$kE_f + E_g = (k-1)E_f + E_{fg} = (k-1)E_f + E_{X^a h} = (k-1)E_f + \{a\} + E_h.$$

If $s_j \in E_g$ and if $\{t_1, t_2, \cdots, t_r\} = E_f$, then it follows that a divides $kt_i + s_j$ for each i between 1 and r (since the notation is addition, the statement that x divides y in S means there exists z such that $x + z = y$). Hence a divides

$$\gcd\{kt_1 + s_j, kt_2 + s_j, \cdots, kt_r + s_j\} = s_j + \gcd\{kt_1, \cdots, kt_r\};$$

moreover, by Proposition 2.5, $\gcd\{kt_1, \cdots, kt_r\} = 0$, since f is e-primitive. Therefore a divides each s_j. Since g is e-primitive, the only divisor of each s_j is 0. Consequently, $a = 0$ and $h = fg$ is primitive.

We have all the tools necessary to establish the converse of Theorem 6.1.

6.4. THEOREM. *If* D *is a* GCD-*domain and* S *is a torsion-free* GCD-*semigroup with zero, then the semigroup ring* $D[X; S]$ *is a* GCD-*domain.*

Proof. To prove that $D[X; S]$ is a GCD-domain, we intend to apply Theorem 3.1. Let N be the multiplicative system $\{X^a \mid a \in S\}$. As observed in the proof of Theorem 6.1, the hypothesis that S is a GCD-semigroup implies that each pair of elements of N has a least common multiple in N. We let T be the set of e-primitive elements of $D[X; S]$; clearly, each nonzero element of $D[X; S]$ is expressible as a product nt, where $n \in N$ and $t \in T$. We prove that $(X^a) \cap (t) = (X^a t)$ for each a in S and each t in T. Let $h = X^a f = tg$ be a nonzero element of $(X^a) \cap (t)$, and write $g = X^b g_1$, where g_1 is e-primitive. Then $X^a f = X^b t g_1$, where tg_1 is e-primitive. It follows that X^a divides X^b, and therefore divides g. Thus h belongs to $(X^a t)$. Since $(X^a t)$ is contained in $(X^a) \cap (t)$, it follows that $(X^a) \cap (t) = (X^a t)$. We have shown that conditions (1) and (2) of Theorem 3.1 are satisfied, and since $D[X; S]_N = D[X; G]$ is a GCD-domain (Theorem 5.2), where G is the quotient group of S, the domain $D[X; S]$ is a GCD-domain.

Except for the assertion "$D[X; S]$ a GCD-domain implies that D is a GCD-domain", Theorem 4.4 is included in Theorems 6.1 and 6.4, whereas Theorem 4.4 has not been used in an essential way to prove these two results. On the other hand, the proof of Theorem 4.4 is of interest in itself, because of the insight it yields concerning divisibility properties of $D[X; S]$, for a GCD-domain D.

7. SEMIGROUP RINGS AS UNIQUE FACTORIZATION DOMAINS

In this section, we determine necessary and sufficient conditions in order that a semigroup ring should be a UFD (Theorem 7.17). Since the semigroup ring is to be an integral domain, *we assume throughout the section that all coefficient rings considered are integral domains with identity and that all semigroups are torsion-free and cancellative, and that they contain a zero element.*

In the preceding section, we determined necessary and sufficient conditions in order that a semigroup ring should be a GCD-domain: D[X; S] *is a* GCD-*domain if and only if* D *is a* GCD-*domain and* S *is a* GCD-*semigroup*. Since the unique factorization domains are precisely the GCD-domains satisfying a. c. c. p. [9], [20], it is natural to consider the question of determining what semigroup rings satisfy a. c. c. p. To begin, we establish some preliminary results concerning the a. c. c. p. Using these results, we are able to establish analogues of Theorems 6.1 and 6.4; moreover, we are able to reduce the problem to the case where the ring of coefficients is an algebraically closed field. The next result is elementary and its proof will be omitted.

(7.1) *Assume that* D *and* J *are integral domains with identity, that* D *is a subring of* J, *and that* K *is the quotient field of* D.

(a) *If nonunits of* D *are nonunits of* J *and if* a. c. c. p. *is satisfied in* J, *it is also satisfied in* D.

(b) *If* $J \cap K = D$ *and if* a. c. c. p. *is satisfied in* J, *it is also satisfied in* D.

7.2. COROLLARY. *Assume that* D *and* J *are integral domains with identity and that* D *is a subring of* J. *Assume that* S_1 *is a subsemigroup of the semigroup* S_2. *Moreover, assume that nonunits of* D *are nonunits of* J *and that noninvertible elements of* S_1 *are noninvertible in* S_2. *If* a. c. c. p. *is satisfied in* $J[X; S_2]$, *it is also satisfied in* $D[X; S_1]$.

Proof. Apply part (a) of (7.1) and Corollary 4.2.

7.3. COROLLARY. *Assume that* F *is a subfield of the field* K *and that* H *is a subgroup of the group* G. *If* K[X; G] *is a* UFD, *then* F[X; H] *is a* UFD.

Proof. Since F[X; H] is a GCD-domain, it suffices to prove that a. c. c. p. is satisfied in F[X; H]; this follows from Corollary 7.2—a. c. c. p. is satisfied in the UFD K[X; G].

7.4. LEMMA. *Assume that* D *and* J *are integral domains with a common identity, that* D *is a subring of* J, *and that* S_1 *is a subsemigroup of the semigroup* S_2. *If* a. c. c. p. *is satisfied in* D *and in* $J[X; S_2]$ *and if noninvertible elements of* S_1 *are noninvertible in* S_2, *then* a. c. c. p. *is satisfied in* $D[X; S_1]$.

Proof. Assume that $(f_1) \subset (f_2) \subset \cdots$ is an infinite, strictly ascending sequence of principal ideals of $D[X; S_1]$. Since a. c. c. p. is satisfied in $J[X; S_2]$, we assume without loss of generality that $(0) \neq f_1 J[X; S_2] = f_k J[X; S_2]$ for each k. Hence, if $f_i = g_{i+1} f_{i+1}$ for each i, where g_{i+1} is in $D[X; S_1]$, then each g_{i+1} is a unit of $J[X; S_2]$—say $g_{i+1} = u_{i+1} X^{s_{i+1}}$, where $u_{i+1} \in D$, $s_{i+1} \in S_1$, u_{i+1} is a unit of J, and s_{i+1} is invertible in S_2. Thus s_{i+1} is invertible in S_1 and each u_{i+1} is a nonunit of D. Moreover, if a_1 is a nonzero coefficient of f_1, then $a_1 = u_2 a_2$ for some nonzero coefficient of f_2, and $a_2 = u_3 a_3$ for some nonzero coefficient of f_3, and so forth. It follows that $a_1 D \subset a_2 D \subset a_3 D \subset \cdots$, contrary to the assumption that a. c. c. p. is satisfied in D. Consequently, a. c. c. p. is satisfied in $D[X; S_1]$.

7.5. THEOREM. *Assume that* D *is an integral domain with identity and* S *is a semigroup. The semigroup ring* D[X; S] *is a* UFD *if and only if* D *is a* UFD *and* K[X; S] *is a* UFD, *where* K *is the quotient field of* D.

Proof. First assume that D[X; S] is a UFD. By Theorem 4.4, D is a GCD-domain. Moreover, since nonunits of D are nonunits of D[X; S] by Corollary 4.2, it follows from (7.1) that a. c. c. p. is satisfied in D. Consequently, D is a UFD, and since K[X; S] is a quotient ring of D[X; S], the domain K[X; S] is also a UFD.

Conversely, if D and K[X; S] are unique factorization domains, then Theorem 4.4 implies that D[X; S] is a GCD-domain, while Lemma 7.4 implies that a.c.c.p. is satisfied in D[X; S]. Therefore, D[X; S] is a UFD, as asserted.

In view of Theorem 7.5, we restrict our considerations in the rest of this section to the case in which the coefficient ring is a field. We first determine conditions under which a group ring over a field is a UFD.

7.6. LEMMA. *Let* F *be a field, and let* G *be an abelian group* (not necessarily torsion-free). *If* A *is a nonempty subset of* G, *then the ideal* I *of* F[X; G] *generated by the set* $\{1 - X^a \mid a \in A\}$ *is the kernel of the homomorphism* ϕ^*: F[X; G] → F[X; G/H], *where* H *is the subgroup of* G *generated by* A *and* ϕ^* *is the canonical extension to* F[X; G] *of the canonical homomorphism* ϕ: G → G/H (see [25, Lemma 1, p. 153] or [10]).

7.7. COROLLARY. *Let* F *be a field, and let* G *be a torsion-free abelian group. If* $g \in G$, *then* $1 - X^g$ *is prime in* F[X; G] *if and only if the subgroup* $\langle g \rangle$ *of* G *generated by* g *is pure in* G.

Proof. For a subgroup H of a torsion-free group G, H is pure in G if and only if G/H is torsion-free. Thus $F[X; G]/(1 - X^g) \simeq F[X; G/\langle g \rangle]$ is an integral domain if and only if $G/\langle g \rangle$ is torsion-free; that is, if and only if $1 - X^g$ is prime in F[X; G].

7.8. LEMMA. *Let* D *be an integral domain with identity, and let* G *be a torsion-free group. If* g = nh, *where* g *and* h *are nonzero elements of* G *and* n *is an integer greater than* 1, *then the principal ideal of* D[X; G] *generated by* $1 - X^g$ *is properly contained in the principal ideal generated by* $1 - X^h$.

Proof. The result follows from the equality

$$1 - X^g = (1 - X^h)(1 + X^h + X^{2h} + \cdots + X^{(n-1)h})$$

and the fact that $1 + X^h + \cdots + X^{(n-1)h}$ is not a unit of D[X; G].

Our next result refers to the *type* of an element of an abelian group G (for the definition, see [12, p. 147], [13, p. 109], or [34, p. 203]). To say that each element of G has type (0, 0, 0, ⋯) is equivalent to the condition that for each nonzero element g of G, there is a largest positive integer n_g such that the equation $n_g x = g$ is solvable in G, or to the condition that each subgroup of G of rank 1 is cyclic. We have resisted the temptation to attach a title to this class of groups; instead, we continue to describe them as "groups, each of whose elements is of type (0, 0, 0, ⋯)".

7.9. THEOREM. *If the group ring* F[X; G] *of* G *over the field* F *is a* UFD, *then* G *is torsion-free and each element of* G *is of type* (0, 0, 0, ⋯).

Proof. Clearly, G is torsion-free. If some element g of G is not of type (0, 0, 0, ⋯), then there is a sequence $g = g_1, g_2, g_3, \cdots$ of elements of G and a sequence $k_1, k_2, \cdots$ of integers greater than 1 such that $g_i = k_i g_{i+1}$ for each i. It then follows from Lemma 7.8 that $(1 - X^{g_1}) \subset (1 - X^{g_2}) \subset \cdots$ is a strictly increasing sequence of principal ideals of F[X; G]. Therefore F[X; G] is not a UFD since a.c.c.p. is not satisfied in F[X; G].

In Theorem 7.12, we establish the converse of Theorem 7.9—that is, we prove that if F is a field and G is a torsion-free group in which every element is of type (0, 0, 0, ⋯), then F[X; G] is a UFD. By Corollary 7.3, it will suffice to prove the preceding statement under the additional hypothesis that F is algebraically closed.

7.10. LEMMA. *Let* H *be a pure subgroup of the torsion-free group* G. *If* F *is a field, then prime elements of* F [X; H] *are prime in* F [X; G].

Proof. Let p be a prime element of F [X; H], and assume that p divides a product fg in F [X; G]—say fg = ph, where $h \in$ F [X; G]. Let K be the subgroup of G generated by H and the exponents that occur in the canonical forms of f, g, or h. Then H is pure in the torsion-free group K; moreover, K/H is torsion-free and finitely generated, so that K/H is a finite direct sum of infinite cyclic groups. Consequently, H is a direct summand of K [23, p. 15]—say $K = H \oplus K_1$. The element p is prime in F [X; H], and hence in F [X; K] $\simeq$ (F [X; H]) [Y; K_1]. Moreover, p divides fg in F [X; K], and consequently, p divides f or g in F [X; K], and in F [X; G]. Therefore p is prime in F [X; G], and our proof of Lemma 7.10 is complete.

Most of the work required for the proof of Theorem 7.12 is contained in the next result.

7.11. PROPOSITION. *Let* F *be an algebraically closed field, let* G *be a torsion-free group such that each element of* G *has type* (0, 0, 0, $\cdots$), *and let* H *be a finitely generated subgroup of* G *such that* G/H *is a torsion group. Then each prime element of* F [X; H] *can be expressed as a finite product of prime elements of* F [X; G].

Proof. Let p be a prime element of F [X; H], and write p as $r_1 X^{h_1} + r_2 X^{h_2} + \cdots + r_n X^{h_n}$, where each r_i is nonzero and $h_1 < h_2 < \cdots < h_n$, the symbol $<$ denoting a total order on G compatible with its group structure. Since X^{h_1} is a unit of F [X; H], we assume without loss of generality that $h_1 = 0$. Let k be the largest integral divisor of h_n in G. We prove that p is a finite product of prime elements of F [X; G] by showing that if $f_1, \cdots, f_t$ are nonunits of F [X; G] such that $f_1 f_2 \cdots f_t = p$, then $t \leq k$. Once this is proved, it will follow that p is a finite product of irreducible elements of F [X; G], and since F [X; G] is a GCD-domain, irreducible elements of F [X; G] are prime.

Thus, assume that there exist nonunits $f_1, \cdots, f_t$ of F [X; G], with $t > k$, such that $p = f_1 f_2 \cdots f_t$. Let K be the subgroup of G generated by H and the set of exponents that occur in the canonical form of some f_i; then K is a finitely generated, torsion-free group, and hence F [X; K] is a UFD. Consider the diagram

$$\begin{array}{ccc} F[X; K] & \text{——} & F(X; K) \\ | & & | \\ F[X; H] & \text{——} & F(X; H) \end{array} .$$

In the diagram, F(X; H) and F(X; K) denote the quotient fields of F [X; H] and F [X; K], respectively. We observe that F(X; K) is finite and normal over F(X; H) and that F [X; K] is the integral closure of F [X; H] in F(X; K). Thus, if $\{g_i\}_1^m$ is a finite set of generators of K, then F(X; K) = F(X; H) $(\{X^{g_i}\}_{i=1}^m)$. Moreover, for each i between 1 and m, there is a positive integer n_i such that $n_i g_i \in H$. It follows that X^{g_i} is a root of the pure equation $Y^{n_i} - X^{n_i g_i}$ over F(X; H). Since F is algebraically closed, F contains the n_ith roots of unity, and $Y^{n_i} - X^{n_i g_i}$ splits into linear factors in F(X; K) [Y]. Therefore, F(X; K)/F(X; H) is finite and normal. We have shown that F [X; K] is integral over F [X; H], and since F [X; K] is integrally closed, $F[X; K]$ is the integral closure of $F[X; H]$ in $F(X; K)$. Moreover, we have also shown that if $g \in K$, then the conjugates of X^g over $F(X; H)$ are of the form αX^g, where α is a root of unity in F; in particular, if $f = \sum_1^r a_i X^{g_i} \in F[X; K]$ and σ is an element of the Galois group S of $F(X; K)$ over $F(X; H)$, then $\sigma(f) = \sum_1^r a_i \sigma(X^{g_i})$, and hence the same exponents occur in the canonical forms of f and $\sigma(f)$.

Since p can be factored in $F[X; K]$ as a product of t nonunits, where $t > k$, it follows that the prime factorization of p in $F[X; K]$ is of the form $p = p_1^{e_1} \cdots p_r^{e_r}$, where $e_1 + \cdots + e_r \geq t$. We write p_1 as $X^{g_0} q_1$, where

$$q_1 = b_0 + b_1 X^{g_1} + \cdots + b_v X^{g_v} \quad \text{and} \quad 0 < g_1 < \cdots < g_v.$$

It is clear that the ideal (q_1) of $F[X; K]$ generated by q_1 is a minimal prime of (p). Moreover, $(q_1) \cap F[X; H] = pF[X; H]$ by the lying-over theorem, and because $F(X; K)/F(X; H)$ is normal and $F[X; H]$ is an integrally closed domain, $\{\sigma((q_1)) = (\sigma(q_1)) \mid \sigma \in S\}$ is the set of prime ideals of $F[X; K]$ lying over $pF[X; H]$ in $F[X; H]$ [17, p. 120]. But it is also clear that $\{(p_i)\}_{i=1}^r$ is the set of primes of $F[X; K]$ lying over $pF[X; H]$, so that each (p_i) is of the form $(\sigma_i(q_1))$ for some $\sigma_i \in S$. Therefore

$$(p) = (p_1^{e_1} \cdots p_r^{e_r}) = ([\sigma_1(q_1)]^{e_1} [\sigma_2(q_1)]^{e_2} \cdots [\sigma_r(q_1)]^{e_r}),$$

so that there is a unit uX^g of $F[X; K]$ such that $p = uX^g [\sigma_1(q_1)]^{e_1} \cdots [\sigma_r(q_1)]^{e_r}$. But our previous observations show that each $\sigma_i(q_i)$ has order 0 and degree g_v. Since p has order 0 and degree h_n, it follows that $g = 0$ and $h_n = (e_1 + \cdots + e_r) g_v$. Because $e_1 + \cdots + e_r \geq t > k$, this contradicts our choice of k as the largest positive integer dividing h_n in G. Hence $t \leq k$, as asserted, and p is a finite product of prime elements of $F[X; G]$.

7.12. THEOREM. *Let* F *be a field, and suppose that each element of the torsion-free group* G *is of type* (0, 0, 0, ⋯). *Then* $F[X; G]$ *is a* UFD.

Proof. Corollary 7.3 allows us to assume that F is algebraically closed. Let f be an element of $F[X; G]$, and write f as $r_1 X^{h_1} + \cdots + r_n X^{h_n}$, where each r_i is nonzero. Let H be the subgroup of G generated by the set $\{h_1, \cdots, h_n\}$. Since H is a finitely generated, torsion-free group, H is a direct sum of infinite cyclic groups. Thus $F[X; H]$ is a UFD and f has a prime factorization in $F[X; H]$. By Lemma 7.10, we need only show that f has a prime factorization in $F[X; H^*]$, where H^* is the pure subgroup of G generated by H (see [34, p. 195] for the definition). By Proposition 7.11, each prime factor of f has a prime factorization in $F[X; H^*]$. Thus f is a finite product of primes in $F[X; G]$, and $F[X; G]$ is a UFD.

The first author has used Theorem 7.12 and some results concerning the (Krull) dimension of a group ring to give examples of non-Noetherian unique factorization domains of arbitrary characteristic and arbitrary dimension $k \geq 2$; for the details, see [18].

7.13. THEOREM. *Let* D *be an integral domain with identity, and let* G *be a torsion-free group. Then* $D[X; G]$ *is a* UFD *if and only if* D *is a* UFD *and each element of* G *is of type* (0, 0, 0, ⋯).

Proof. Apply Theorems 7.5 and 7.12.

7.14. COROLLARY. *Let* D *be an integral domain with identity, and let* G *be a torsion-free group. The group ring* D[X; G] *satisfies* a.c.c.p. *if and only if* D *satisfies* a.c.c.p. *and each element of* G *is of type* (0, 0, 0, ⋯).

Proof. Suppose that D[X; G] satisfies a.c.c.p. Then (7.1) implies that a.c.c.p. is satisfied in D, and Lemma 7.8 implies that each element of G is of type (0, 0, 0, ⋯). For the converse, we note that Theorem 7.13 implies a.c.c.p. is satisfied in K[X; G], where K is the quotient field of D, and Lemma 7.4 implies that a.c.c.p. is satisfied in D[X; G].

7.15. LEMMA. *Let* S *be a* UFS *with maximal subgroup* H. *If each element of* H *is of type* (0, 0, 0, ⋯), *then each element of* G, *the quotient group of* S, *is of type* (0, 0, ⋯).

Proof. We write the semigroup operation on S as addition. The group H is merely the set of invertible elements of S. Moreover, the definition of a unique factorization semigroup implies that if $\{p_\alpha\}$ is a complete set of nonassociate prime elements of S, then S is the weak direct sum of H and the family $\{\langle p_\alpha \rangle\}$ of subsemigroups with zero generated by the prime elements p_α. Hence S is isomorphic to $H \oplus \sum_\alpha^w (Z_0)_\alpha$, where Z_0 is the additive semigroup of nonnegative integers, and $G \simeq H \oplus \sum_\alpha^w (Z)_\alpha$. Since each element of H and of Z is of type (0, 0, 0, ⋯), we conclude that each element of G is also of type (0, 0, 0, ⋯).

7.16. LEMMA. *Let* p *be an element of the unique factorization semigroup* S (again written additively). *Then* X^p *is a prime element of* D[X; S], *where* D *is an integral domain with identity, if and only if* p *is a prime element of* S.

Proof. Suppose that X^p divides fg in D[X; S]. Write $f = X^a f_1$ and $g = X^b f_2$, where f_i is e-primitive for i = 1, 2. By the proof of Theorem 6.4, $(X^c) \cap (h) = (X^c h)$ for each c in S and each e-primitive element h of D[X; S]. Thus X^p divides $X^{a+b} f_1 f_2$, and $f_1 f_2$ is e-primitive by Proposition 6.3. Since

$$(X^p) \cap (f_1 f_2) = (X^p f_1 f_2),$$

it follows that X^p divides X^{a+b} in D[X; S]. Therefore $p + c = a + b$ for some c in S—that is, p divides $a + b$ in S. Since p is prime in S, the element p divides a or divides b in S, and X^p divides either f or g in D[X; S].

Conversely, suppose that X^p is prime in D[X; S]. If $a, b \in S$ and p divides $a + b$ in S, then X^p divides X^{a+b} in D[X; S]. Thus X^p divides X^a or X^b in D[X; S]. By Lemma 4.1, p divides a or b in S.

7.17. THEOREM. *The semigroup ring* D[X; S] *is a* UFD *if and only if* D *is a* UFD, *the semigroup* S *is a* UFS, *and each element of the maximal subgroup* H *of* S *is of type* (0, 0, 0, ⋯).

Proof. To prove that D[X; S] is a UFD, we apply Theorem 3.2. Let N be the multiplicative system of D[X; S] generated by the set

$$\{X^{p_\alpha} \mid p_\alpha \text{ is a prime element of } S\}$$

of prime elements of D[X; S]. It follows that $D[X; S]_N \simeq D[X; G]$, where G is the quotient group of S. By Lemma 7.15, each element of G is of type (0, 0, 0, ⋯), and by Theorem 7.13, D[X; G] is a UFD. To complete the hypothesis of Theorem 3.2, we must show that no element $f \in D[X; S]$ is divisible by infinitely many primes X^{p_α} or by infinitely many powers of a fixed prime X^{p_α}. Let the canonical form of f be $r_1 X^{s_1} + r_2 X^{s_2} + \cdots + r_n X^{s_n}$, where $a = \gcd\{s_1, s_2, \cdots, s_n\}$. Then $f = X^a f_1$, where f_1 is e-primitive. Thus if X^{p_α} divides $f = X^a f_1$, then X^{p_α} divides X^a. This is so because $(X^q) \cap (f_1) = (X^q f_1)$ for each element q of S (see the proof of Theorem 6.4). Since a is not divisible by infinitely many primes of S or by infinitely many powers of a fixed prime of S, the hypothesis of Theorem 3.2 is satisfied. Thus $D[X; S]$ is a UFD.

Conversely, suppose that $D[X; S]$ is a UFD. By Theorem 7.5, D is a UFD. Since $D[X; G]$ is a UFD, where G is the quotient group of S, each element of G is of type $(0, 0, 0, \cdots)$. If $a \in S$, then X^a is a product of prime elements of $D[X; S]$, say $X^a = f_1 f_2 \cdots f_n$. By Lemma 4.1, each f_i is of the form $u_i X^{p_i}$, where u_i is a unit of D and p_i is an element of S; moreover, by Lemma 7.16, each p_i is prime in S. Thus each element of S is a finite sum of prime elements and S is a UFS.

We have investigated the analogue of Corollary 7.14 for semigroups, but our results in this direction are incomplete. If D is an integral domain with identity and S is a torsion-free semigroup, then sufficient conditions for a.c.c.p. to be satisfied in $D[X; S]$ are that a.c.c.p. is satisfied in D, the ascending chain condition for principal ideals of the semigroup S is satisfied, and each element of the quotient group G of S is of type $(0, 0, 0, \cdots)$. The conditions that a.c.c.p. is satisfied in D and in S are also necessary in order that a.c.c.p. is satisfied in $D[X; S]$, but the condition that G is of type $(0, 0, 0, \cdots)$ is not necessary; for example, if S is the additive semigroup consisting of 0 and the set of real numbers greater than 1, and if D is an integral domain with identity in which the a.c.c.p. is satisfied, then a.c.c.p. is satisfied in $D[X; S]$, but the quotient group of S is not of type $(0, 0, 0, \cdots)$. More generally, if $\{0\}$ is the maximal subgroup of S, then a.c.c.p. is satisfied in $D[X; S]$ if it is satisfied in D and in S.

8. THE CASE OF A PRINCIPAL IDEAL DOMAIN

In this section, we consider the following question. Under what conditions is a semigroup ring $R[X; S]$ a principal ideal domain? To answer this question, we need to consider the structure of a subsemigroup S of the infinite cyclic group Z of integers; we consider only the case where S properly contains the trivial semigroup $\{0\}$. Let d be the (positive) greatest common divisor of the set of elements of S; if $d = 1$, then S is called a *prime* subsemigroup of Z [33, p. 201]. It is clear that $S = dS_1$ for some prime subsemigroup S_1 of Z. It is known [33, Theorem 82, p. 201], that if S is contained in Z_0, the set of nonnegative integers, then there exists a positive integer K_0 such that $kd \in S$ for each $k \geq K_0$. Similarly, if S is a subset of the set $-Z_0$ of nonpositive integers, then there exists a negative integer K_0 such that $kd \in S$ for each $k \leq K_0$. If S contains both positive and negative integers, then we let $S_1 = S \cap Z_0$ and $S_2 = S \cap (-Z_0)$, and we let d_i be the (positive) greatest common divisor of the elements of S_i. The integers d_1 and d_2 are equal. To prove this, let K_1 be a positive integer such that $kd_1 \in S_1$ for each $k \geq K_1$, and let K_2 be a negative integer such that $kd_2 \in S_2$ for each $k \leq K_2$. We prove that d_1 divides d_2; the proof that d_2 divides d_1 is similar. Thus, choose $k \leq K_2$ such that $(k, d_1) = 1$. For sufficiently large r, $kd_2 + rd_1$ is in S_1, and hence d_1 divides $kd_2 + rd_1$; by choice of k, it follows that d_1 divides d_2.

In summary, the following statement gives a description of all subsemigroups of Z.

(8.1) *Let* S *be a subsemigroup of the additive group* Z *such that* $\{0\}$ *is properly contained in* S, *and let* d *be the positive greatest common divisor of the elements of* S.

(1) *If* $S \subseteq Z_0$, *then there is a positive integer* K_1 *such that* $kd \in S$ *for each* $k \geq K_1$.

(2) *If* $S \subseteq (-Z_0)$, *then there is a negative integer* K_2 *such that* $kd \in S$ *for each* $k \leq K_2$.

(3) *If* S *contains both positive and negative integers, then there is a positive integer* K *such that* $kd \in S$ *for each* k *such that* $|k| \geq K$.

8.2. PROPOSITION. *Assume that* S *is a subsemigroup of the additive group* Z *such that* $\{0\}$ *is properly contained in* S, *and let* d *be the positive greatest common divisor of the elements of* S. *If* D *is an integral domain with identity, then the following conditions are equivalent.*

(1) *The semigroup ring* D[X; S] *is integrally closed.*

(2) D *is integrally closed, and either* $S = dZ_0$, $S = d(-Z_0)$, *or* $S = dZ$.

Proof. If (2) is satisfied, then D is integrally closed and D[X; S] is isomorphic to D[Y] or to $D[Y, Y^{-1}] = D[Y]_{\{Y^n\}_{n=1}^{\infty}}$. Therefore, (2) implies (1) [26], [17, Section 10].

We prove that (1) implies (2). By (8.1), there is an integer t such that td and $(t+1)d$ are in S. Hence $X^d = X^{(t+1)d}/X^{td}$ belongs to the quotient field of D[X; S]. If $S \subseteq Z_0$, then a positive power of X^d belongs to D[X; S], and hence $X^d \in D[X; S]$; consequently, $S = dZ_0$. Similarly, if $S \subseteq -Z_0$, then a positive power of X^{-d} is in D[X; S] and $S = d(-Z_0)$. If S contains both positive and negative integers, then X^d and X^{-d} are in D[X; S] and $S = dZ$. It follows that to within isomorphism, D[X; S] is D[X] or $D[X, X^{-1}]$. In either case, D[X; S] meets the quotient field of D in D, and hence D is integrally closed.

If D is integrally closed, then our proof of Proposition 8.2 shows that condition (1) of that result is equivalent to the condition that D[X; S] is *root-closed*—that is, D[X; S] contains each element t of its quotient field such that some positive power of t belongs to D[X; S]. An integrally closed domain is, of course, root-closed, but the converse fails in the general case [3, Exercise 15, p. 72], [17, Exercise 6, p. 184].

A principal ideal domain has (Krull) dimension at most 1. We shall see presently that the condition that a semigroup ring has dimension at most 1 imposes rather stringent conditions on the coefficient ring and the semigroup.

8.3. PROPOSITION. *Let* R *be a commutative ring with identity, and let* S *be a nonzero, torsion-free, cancellative semigroup with zero. Then*

$$\dim R[X; S] \geq \dim R + 1.$$

Proof. If $P_1 \subset \cdots \subset P_n$ is a chain of proper prime ideals of R, then $P_1[X; S] \subset \cdots \subset P_n[X; S]$ is a chain of proper primes of R[X; S]; this follows since $R[X; S]/P_i[X; S] \simeq (R/P_i)[X; S]$ is an integral domain for each i between 1 and n. Moreover, $P_n[X; S]$ is not maximal in R[X; S], for Corollary 4.2 implies that $1 - X^s$, for each $s \neq 0$, is not a unit modulo $P_n[X; S]$. It follows that $\dim R[X; S] \geq \dim R + 1$.

We are now in position to prove the main result of this section.

8.4. THEOREM. *Assume that* D *is an integral domain with identity and that* S *is a nonzero, torsion-free, cancellative semigroup with zero. The following conditions are equivalent.*

(1) D *is a field and* S *is isomorphic to* Z_0 *or to* Z.

(2) *The semigroup ring* D[X; S] *is a Euclidean domain.*

(3) D[X; S] *is a* PID.

(4) D[X; S] *is a Dedekind domain.*

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4) are clear [16, p. 10], [36, p. 287]. We prove that (4) implies (1). By Proposition 8.3, $\dim D[X; S] \geq \dim D + 1$. Hence (4) implies that $\dim D[X; S] = 1$, and D is a field. If G is the quotient group of S, then D[X; G] is a quotient ring of D[X; S], and hence D[X; G] is a Dedekind domain. In particular, D[X; G] is Noetherian, so that G is finitely generated [25, p. 154], [10, p. 658]. Since G is torsion-free, G is a direct sum of k copies of Z, and

$$D[X; G] \simeq D[X_1, \cdots, X_k, X_1^{-1}, \cdots, X_k^{-1}].$$

But $D[X_1, \cdots, X_k, X_1^{-1}, \cdots, X_k^{-1}]$ has dimension k (for example, $(X_1 + 1, \cdots, X_k + 1)$ is a maximal ideal of $D[X_1, \cdots, X_k]$ that contains no X_i), and therefore $k = 1$. It follows that S is a subsemigroup of Z, and D[X; S] is integrally closed. By Proposition 8.2, S is isomorphic to Z_0 or to Z, and our proof of Theorem 8.4 is complete.

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