A GENERALIZATION OF EPSTEIN ZETA FUNCTIONS

Chungming An

In [1], we associated with certain polynomials a Dirichlet series that generalizes the Epstein zeta functions. In [2], we used various methods to study the analytic properties of the Dirichlet series. In this note, we obtain somewhat stronger results for certain special cases.

Let $F(X) = F(X_1, \cdots, X_n)$ be an integral form of degree δ such that the equation $F(x) = 0$ has no solutions in $\mathbb{R}^n$ except $x = 0$. We may assume that $F(x)$ is positive definite. It is obvious that for each k the equation $F(\gamma) = k$ has only finitely many solutions γ in $\mathbb{Z}^n$. Hence it makes sense to consider series of the type

$$\zeta(F, \alpha, s) = \sum_{\gamma \in \mathbb{Z}^n - \{0\}} F(\gamma)^{-s} e(\langle \alpha, \gamma \rangle),$$

where $s = \sigma + it$ is a complex number, $\alpha \in \mathbb{Z}^n$, the symbol $\langle\ ,\ \rangle$ indicates the standard inner product in $\mathbb{R}^n$, and $e(a) = \exp(2\pi i a)$ for $a \in \mathbb{R}$. If $F(x)$ is a quadratic form and $\alpha \in \mathbb{Z}^n$, then $\zeta(F, \alpha, s)$ is the well-known Epstein zeta function. The absolute convergence of the series for $\sigma > n/\delta$ in the general case and the analytic continuability for $\alpha \in \mathbb{Q}^n$ in certain special cases have been established in [1] and [2]. For $\alpha \in \mathbb{Q}^n$, we may apply C. L. Siegel's method [3] to continue the series analytically into the half-plane $\sigma > (n - 1)/\delta$ (see [2]).

In this paper, we shall prove the following result.

THEOREM. (a) *If* $\alpha \notin \mathbb{Z}^n$, *the function* $\zeta(F, \alpha, s)$ *can be continued analytically as an entire function of* s.

(b) *If* $\alpha \in \mathbb{Z}^n$, *the function* $\zeta(F, \alpha, s)$ *can be continued analytically as a meromorphic function of* s *with only a simple pole at* $s = n/\delta$; *the residue is*

$$\operatorname*{Res}_{s=n/\delta} \zeta(F, \alpha, s) = (2\pi)^{n/\delta}\, \Gamma(n/\delta)^{-1} \int_{\mathbb{R}^n} \exp(-2\pi F(x))\, dx.$$

Proof. Let us put $\xi(F, \alpha, s) = (2\pi)^{-s}\, \Gamma(s)\, \zeta(F, \alpha, s)$. By the Mellin transform, we get the integral representation

$$\xi(F, \alpha, s) = \int_0^\infty \sum_{\gamma \in \mathbb{Z}^n - \{0\}} \exp(-2\pi t F(\gamma))\, e(\langle \alpha, \gamma \rangle)\, t^{s-1}\, dt$$

$$= \int_0^\infty [\mathscr{O}(F, \alpha, it) - 1]\, t^{s-1}\, dt \qquad (s > n/\delta),$$

where, for $\tau \in H = \{z \in \mathbb{C}\colon \Im z > 0\}$,

Received September 28, 1973.

Michigan Math. J. 21 (1974).

$$\mathscr{O}(F, \alpha, \tau) = \sum_{\gamma \in \mathbb{Z}^n} e(\tau F(\gamma) + \langle \alpha, \gamma \rangle).$$

We call $\mathscr{O}(F, \alpha, \tau)$ the generalized theta function associated with F. We follow the standard method for the Riemann ζ-function. This gives the formula

$$\xi(F, \alpha, s) = I_1(s) + I(s),$$

where

$$I_1(S) = \int_0^1 [\mathscr{O}(F, \alpha, it) - 1] t^{s-1} dt, \qquad I(s) = \int_1^\infty [\mathscr{O}(F, \alpha, it) - 1] t^{s-1} dt.$$

It is easy to show that $I(s)$ is an entire function (see [2]). If we put

$$I_2(s) = \int_0^1 \mathscr{O}(F, \alpha, it) t^{s-1} dt,$$

then $I_1(s) = I_2(s) - 1/s$. But $g_t(x) = \exp(-2\pi t F(x) + 2\pi i \langle \alpha, x \rangle)$. Therefore the Fourier transform $\hat{g}_t(y)$ is given by the equation

$$\hat{g}_t(y) = \int_{\mathbb{R}^n} \exp(-2\pi t F(x) + 2\pi i \langle \alpha - y, x \rangle) dx.$$

The Poisson summation formula gives the relation

$$\sum_{\gamma \in \mathbb{Z}^n} g_t(\gamma) = \sum_{\gamma \in \mathbb{Z}^n} \hat{g}_t(\gamma).$$

Thus we may write

$$\begin{aligned} I_2(s) &= \int_0^1 \sum_{\gamma \in \mathbb{Z}^n} g_t(\gamma) t^{s-1} dt \\ &= \int_0^1 \left[\sum_{\gamma \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \exp(-2\pi t F(x) + 2\pi i \langle \alpha - \gamma, x \rangle) dx \right] t^{s-1} dt. \end{aligned}$$

By the change of variables $x \to t^{-1/\delta} x$, we obtain the formula

$$\begin{aligned} I_2(s) &= \int_0^1 \left[\sum_{\gamma \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \exp(-2\pi F(x) + 2\pi i \langle \alpha - \gamma, t^{-1/\delta} x \rangle) dx \right] t^{s-1-n/\delta} dt \\ &= \int_0^1 \sum_{\gamma \in \mathbb{Z}^n} \hat{\phi}(t^{-1/\delta}(\alpha - \gamma)) t^{s-1-n/\delta} dt, \end{aligned}$$

where $\hat{\phi}(y)$ is the Fourier transform of $\phi(x) = \exp(-2\pi F(x))$. We see that $\phi(x)$ and $\hat{\phi}(y)$ are in the Schwartz space $S(\mathbb{R}^n)$. It is known that for each Schwartz function and each positive integer N there exists a constant C such that $|y|^{2N}|\phi(y)| \leq C$ for all $y \in R^n$; that is, $|\hat{\phi}(y)| \leq C|y|^{-2N}$ for all $y \neq 0$. We shall choose N large enough.

Case a. If $\alpha \notin \mathbb{Z}^n$, that is, if $\alpha - \gamma \neq 0$ for all $\gamma \in \mathbb{Z}^n$, then

$$|\hat{\phi}(t^{-1/\delta}(\alpha - \gamma))| \leq C t^{2N/\delta}|\alpha - \gamma|^{-2N}$$

for all $\gamma \in \mathbb{Z}^n$. Thus the integral form of $I_2(s)$ is majorized by the series

$$\frac{1}{s + (2N - n)/\delta} \sum_{\gamma \in \mathbb{Z}^n} |\alpha - \gamma|^{-2N}.$$

Here $|\alpha - X|^2$ is a polynomial of degree 2 whose highest homogeneous part is a positive definite quadratic form. By [1] we see that the series converges for $N > n/2$.

Case b. If $\alpha \in \mathbb{Z}^n$, we put $\eta = \alpha - \gamma \in \mathbb{Z}^n$. Then

$$I_2(s) = \int_0^1 \hat{\phi}(0)\, t^{s-1-n/\delta}\, dt + \int_0^1 \sum_{\substack{\eta \neq 0 \\ \eta \in \mathbb{Z}^n}} \hat{\phi}(t^{-1/\delta}\eta)\, t^{s-1-n/\delta}\, dt$$

$$= \frac{\hat{\phi}(0)}{s - n/\delta} + \int_0^1 \sum_{\substack{\eta \neq 0 \\ \eta \in \mathbb{Z}^n}} \hat{\phi}(t^{-1/\delta}\eta)\, t^{s-1-n/\delta}\, dt.$$

The second term (the integral) is majorized by the series

$$\frac{1}{s + (2N - n)/\delta} \sum_{\substack{\eta \neq 0 \\ \eta \in \mathbb{Z}^n}} |\eta|^{-2N},$$

which converges whenever $N > n/2$.

In each case, the majorized series converges. Hence in each case the integral represents a holomorphic function, for $\sigma \geq k$. Since k can be an arbitrary negative integer, the integrals represent entire functions.

In case $\alpha \in \mathbb{Z}^n$, the residue of $\xi(F, \alpha, s)$ at $s = n/\delta$ is

$$\hat{\phi}(0) = \int_{\mathbb{R}^n} \exp(-2\pi F(x))\, dx.$$

The conclusion about $\zeta(F, \alpha, s)$ follows trivially. This completes the proof.

Remarks. 1. One may verify that the point $s = 0$ is a removable singularity of the function ζ and that moreover

$$\lim_{s \to 0} \zeta(F, \alpha, s) = -1$$

regardless of F and α.

2. In [2], we define a generalized zeta function parametrized by $\rho \in \mathbb{R}$ and $\alpha \in \mathbb{R}^n$; that is, we write

$$\zeta(F, \rho, \alpha, s) = \sum_{\gamma \in \mathbb{Z}^n - \{0\}} F(\gamma)^{-s} e(\rho F(\gamma) + \langle \alpha, \gamma \rangle).$$

Here we can only handle the cases where $\rho \in \mathbb{Q}$ and $\alpha \in \mathbb{Q}^n$ or where $\rho = 0$ and $\alpha \in \mathbb{R}^n$, for $\sigma > (n - 1)/\delta$. The same method can be applied to the case $\rho \in Q$. We have also obtained some information about the generalized Gaussian sum defined in [2]. The results will appear elsewhere.

3. We conjecture that if ρ is irrational, then $\zeta(F, \rho, \alpha, s)$ is an entire function. Although we have some evidence to support this, much work remains to be done.

4. For binary quadratic forms, there are first and second Kronecker limit formulas corresponding to $\alpha \in \mathbb{Z}^n$ and $\alpha \notin \mathbb{Z}^n$ (see [3]). We may ask a similar question about $\zeta(F, \alpha, s)$. For quadratic forms in more than two variables, A. A. Terras [4] has obtained some generalizations of the first Kronecker limit formula.

REFERENCES

1. C. An, *On a generalization of the gamma function and its application to certain Dirichlet series.* Bull. Amer. Math. Soc. 75 (1969), 562-568.

2. ———, *On the analytic continuation of certain Dirichlet series.* J. Number Theory (to appear).

3. C. L. Siegel, *Lecture on advanced analytic number theory.* Tata Institute of Fundamental Research, Bombay, India, 1961.

4. A. A. Terras, *Bessel series expansion of the Epstein zeta function and the functional equation.* Trans. Amer. Math. Soc. (to appear).

Seton Hall University
South Orange, New Jersey 07079