TOPOLOGICAL PROPERTIES OF THE SPACE OF HOMEOMORPHISMS OF n-DIMENSIONAL EUCLIDEAN SPACE

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INTRODUCTION

Let R^n denote n-dimensional euclidean space with the usual topology. A continuous function $\delta\colon R^n \to (0, \infty)$ is called a *majorant* (on R^n). Let $H(R^n)$ denote the group of homeomorphisms of R^n. We define a topology for $H(R^n)$, called the *majorant topology*, as follows: A basis consists of all sets of the form

$$N_\delta(f) = \{g \in H(R^n) \mid d(g(x), f(x)) < \delta(x) \text{ for all } x \in R^n\},$$

where $f \in H(R^n)$ and δ is a majorant. $H(R^n)$ with the majorant topology is a topological group ([1], [2]). We denote by $H_c(R^n)$ the subspace of $H(R^n)$ consisting of all homeomorphisms of R^n that are the identity outside some compact set.

THEOREM 1. *$H_c(R^n)$ is the (path-) component of the identity homeomorphism in $H(R^n)$.*

THEOREM 2. *$H_c(R^n)$ is a nowhere dense, non-first-countable subspace of $H(R^n)$.*

A topological space X is called a *Fréchet space* if, whenever x is a limit point of a subset A of X, there exists a sequence in A converging to x. Clearly, all first-countable spaces are Fréchet spaces.

THEOREM 3. *$H(R^n)$ is not a Fréchet space.*

THEOREM 4. *If $n \neq 4$, then $H(R^n)$ is separable.*

COROLLARY 5. *$H(R^n)$ is not metrizable.*

COROLLARY 6. *$H(R^n)$ contains no connected open sets.*

1. PROOF OF THEOREM 1

LEMMA 7. *$H_c(R^n)$ is path-connected.*

Proof. It will suffice to join $F \in H_c(R^n)$ to the identity mapping id. by a path in $H_c(R^n)$. Since $F \in H_c(R^n)$, the mapping F is the identity outside some compact set, and therefore F is the identity outside a ball B of radius r, centered at the origin. For $0 \leq t \leq 1$, we define

$$\Phi(x, t) = \begin{cases} x & \text{if } \|x\| \geq r \text{ or } t = 0, \\ tF(x/t) & \text{if } \|x\| < r \text{ and } 0 < t \leq 1. \end{cases}$$

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The mapping $\Phi: R^n \times I \to R^n$ is continuous, and if $\Phi_t(x) = \Phi(x, t)$, then each Φ_t belongs to $H_c(R^n)$. Put $\hat{\Phi}(t) = \Phi_t$; then $\hat{\Phi}: I \to H_c(R^n) \subseteq H(R^n)$. Let $H_K(R^n)$ denote the space of homeomorphisms of R^n with the compact-open topology, and let $H(R^n, B)$ and $H_K(R^n, B)$ denote the space of homeomorphisms of R^n that are the identity outside B, with the majorant and compact-open topologies, respectively. Then $\hat{\Phi}: I \to H_K(R^n, B)$ is continuous, and since id. : $H_K(R^n, B) \to H(R^n, B)$ is a homeomorphism, it follows that $\hat{\Phi}$ yields a path in $H_c(R^n)$ from F to id.; this proves the lemma.

It follows that $H_c(R^n)$ is connected. Let δ be a majorant, and define

$$W_\delta = \{h \in H(R^n) \mid d(h(x), x) < M\,\delta(x) \text{ for some } M > 0 \text{ and all } x\}.$$

Clearly, W_δ is open in $H(R^n)$; but W_δ is also closed, for if $k \in H(R^n) - W_\delta$, then for each i there exists a point $x_i \in R^n$ with $d(k(x_i), x_i) \geq i\,\delta(x_i)$. But then, if $g \in N_{\delta/2}(k)$, it follows that $d(g(x_i), k(x_i)) < \delta(x_i)/2$, so that

$$\left(i - \frac{1}{2}\right)\delta(x_i) \leq d(g(x_i), x_i)$$

for each i. Thus $N_{\delta/2}(k) \subseteq H(R^n) - W_\delta$, and therefore W_δ is closed.

Let $W_0 = \bigcap_\delta W_\delta$. Let C be the component of id. in $H(R^n)$. Lemma 7 implies that $H_c(R^n) \subseteq C$. If $C \not\subseteq W_0$, then C contains a point in the complement of some W_δ. Since W_δ is open and closed, it must separate C. Thus we must have the inclusion $C \subseteq W_0$, and W_0 is closed.

LEMMA 8. $W_0 = H_c(R^n)$.

Proof. We need to show that $W_0 \subseteq H_c(R^n)$. Suppose $h \notin H_c(R^n)$. Then we can find a sequence $\{x_i\}$ in R^n, with $\|x_i\| < \|x_{i+1}\|$ and $\lim_{i\to\infty} \|x_i\| = \infty$, such that $h(x_i) \neq x_i$. Put $\eta_i = \min\{2^{-i}, d(h(x_i), x_i)\}$. Let δ be a majorant on R^n with $\delta(x_i) = \eta_i^2$. We claim that $h \notin W_\delta$. If $h \in W_\delta$, then $d(h(x), x) < M\,\delta(x)$ for some M and all x. But then

$$\eta_i \leq d(h(x_i), x_i) < M\,\delta(x_i) = M\eta_i^2,$$

so that $1/\eta_i < M$ for all i. Since $\eta_i \leq 2^{-i}$ for each i, it follows that $M > 2^i$ for all i, which is absurd. Hence $h \in W_\delta$, so that clearly $h \notin W_0$; this proves the lemma.

Since $H_c(R^n) \subseteq C \subseteq W_0$, Lemma 8 implies that the component of id. in $H(R^n)$ is $H_c(R^n)$. Since $H(R^n)$ is a topological group, the components of $H(R^n)$ are precisely the translates of $H_c(R^n)$. It also follows that the path-components of $H(R^n)$ are equal to the components. Finally, since $H_c(R^n) = W_0$, we see that $H_c(R^n)$ is closed in $H(R^n)$.

2. PROOF OF THEOREM 2

First we show that $H_c(R^n)$ is nowhere dense. Let $\delta: R^n \to (0, \infty)$ be a majorant. It suffices to show that $N_\delta(\text{id.}) \not\subseteq H_c(R^n)$. Put $i_1 = 1$, and let r_1 be a real number with $0 < r_1 < \min\{\delta(x) \mid x \in B_{i_1}\}$, where B_k denotes the closed ball about the origin of radius k. Inductively, find an integer i_n and a real number r_n such that

$$i_n > i_{n-1} + r_{n-1} \quad \text{and} \quad 0 < r_n < \min\{\delta(x) \mid x \in B_{i_n}\}.$$

Define the homeomorphism h by sending B_{i_1} radially to $B_{i_1+r_1}$ and sending the annular region $B_{i_n} - B_{i_{n-1}}$ radially to the annular region $B_{i_n+r_n} - B_{i_{n-1}+r_{n-1}}$. It is evident that $h \in N_\delta(\text{id.})$ and that $h \notin H_c(R^n)$.

To finish the proof of Theorem 2, we need the following result.

LEMMA 9. *The space* $H_c(R^n)$ *is homeomorphic to a closed subspace of* $H_c(R^{n+1})$.

Proof. Given $h \in H_c(R^n)$, define $h' \in H_c(R^{n+1})$ by

$$h'(x, t) = \begin{cases} (x, t) & \text{for } |t| \geq 1, \\ \left((1 - |t|)h\left(\frac{x}{1 - |t|}\right), t\right) & \text{for } |t| < 1. \end{cases}$$

(We regard R^{n+1} as $R^n \times R^1$; that is, $x \in R^n$, $t \in R^1$.) We shall show that the mapping $\Phi: H_c(R^n) \to H_c(R^{n+1})$ defined by $\Phi(h) = h'$ is an embedding onto a closed subspace.

Clearly, Φ is one-to-one and $\Phi(H_c(R^n)) \subseteq H_c(R^{n+1})$. To show that Φ is continuous, let ε be an arbitrary majorant on R^{n+1}, and define the majorant δ on R^n by

$$\delta(x) = \inf\{\varepsilon(z, t) \mid \|z\| \leq \|x\|, \ |t| \leq 1\}.$$

We show that $\Phi N_\delta(g) \subseteq N_\varepsilon(\Phi(g))$. Let $h \in N_\delta(g)$; then $d(h(x), g(x)) < \delta(g(x))$ for all $x \in R^n$, so that (for $|t| < 1$)

$$\begin{aligned} d(h'(x, t), g'(x, t)) &= d\left(\left[(1 - |t|)h\left(\frac{x}{1 - |t|}\right), t\right], \left[(1 - |t|)g\left(\frac{x}{1 - |t|}\right), t\right]\right) \\ &= (1 - |t|)d\left(h\left(\frac{x}{1 - |t|}\right), g\left(\frac{x}{1 - |t|}\right)\right) \leq (1 - |t|)\delta\left(\frac{x}{1 - |t|}\right) \\ &\leq (1 - |t|)\delta(x) \leq \delta(x) \leq \varepsilon(x, t). \end{aligned}$$

Thus $\Phi(h) \in N_\varepsilon(\Phi(g))$. Similarly, Φ is open onto its image. (It is easy to show that $N_\eta(h') \cap \Phi(H(R^n)) \subset \Phi N_\delta(h)$, where $\eta(x, t) = \delta(x)$.)

Let $Z = \Phi H_c(R^n)$. We show that Z is closed in $H_c(R^{n+1})$. Let h^* be a limit point of Z; clearly, $h^* \mid R^n: R^n \to R^n$. Let $h = h^* \mid R^n$ and $\Phi(h) = h'$. We shall show that $h^* = h' \in \Phi(H_c(R^n))$.

Let ε be an arbitrary majorant on R^{n+1}, and let δ be the majorant on R^n given by

$$\delta(x) = \frac{1}{2}\inf\{\varepsilon(z, t) \mid \|z\| \leq \|x\|, \ |t| \leq 1\}.$$

Let η be the majorant on R^{n+1} defined by $\eta(x, t) = \delta(x)$ for all $(x, t) \in R^{n+1}$. By hypothesis, there is some $g' \in N_\eta(h^*) \cap Z$. Note that this implies $g' \in N_{\varepsilon/2}(h^*)$. Let $g = \Phi^{-1}(g')$; then $g \in N_\delta(h)$, and this implies $g' \in N_{\varepsilon/2}(h')$. Thus

$g' \in N_{\varepsilon/2}(h^*) \cap N_{\varepsilon/2}(h')$, which implies $h' \in N_\varepsilon(h^*)$. Since ε is arbitrary, we conclude that $h^* = h'$.

Proof that $H_c(R^n)$ *is not first-countable.* We begin by showing that $H_c(R^1)$ is not first-countable; it suffices to show that $H_c(R^1)$ is not first-countable at id. Suppose that $\{\delta_i\}$ is a sequence of majorants in R^1. Choose r_i $(1/8 > r_i > 0)$ so that $\delta_i(x) > 4r_i$ if $|x - i| < 4r_i$. Let δ be a majorant for R^1, with $\delta(i) = r_i$ for each i.

Define h_i by

$$h_i(x) = \begin{cases} x & \text{if } x \leq i - 2r_i \text{ or } x \geq i + 3r_i, \\ 2x - i + 2r_i & \text{if } i - 2r_i \leq x \leq i, \\ \dfrac{x}{3} + \dfrac{2i}{3} + 2r_i & \text{if } i \leq x \leq i + 3r_i. \end{cases}$$

Then $h_i \in H_c(R^1)$ for each i. Since $|h_i(i) - i| = 2r_i > \delta(i)$, we conclude that $h_i \notin N_\delta(\text{id.})$ for each i; but it is easy to see that $h_i \in N_{\delta_i}(\text{id.})$. Thus none of the $\{N_{\delta_i}(\text{id.})\}$ is contained in $N_\delta(\text{id.})$, so that $\{N_{\delta_i}(\text{id.})\}$ is not a countable basis for $H_c(R^1)$ at id.; this proves the lemma.

Since $H_c(R^n)$ can be embedded as a closed subset of $H_c(R^{n+1})$, it follows by induction that for each n the space $H_c(R^n)$ is not first-countable.

3. PROOF OF THEOREM 3

LEMMA 10. *If a sequence* $\{h_i\}$ *in* $H(R^n)$ *converges to* id., *then there exist a compact subset* K *of* R^n *and an integer* N *such that* $h_i(x) = x$ *for each* $i > N$ *and all* $x \notin K$.

Proof. Suppose the lemma is false. Let K_1 be the closed unit ball in R^n. Then there exist an h_{i_1} and a point $x_1 \notin K_1$ with $h_{i_1}(x_1) \neq x_1$. Put $r_1 = \|x_1\|$ and $K_2 = B_{3r_1}$. Then we can find h_{i_2} and $x_2 \notin K_2$ with $i_1 < i_2$ and $h_{i_2}(x_2) \neq x_2$. We proceed inductively to get a subsequence $\{h_{i_j}\}$ $(i_1 < i_2 < \cdots)$ and a sequence of points $\{x_j\}$ with $\|x_{j+1}\| > 3\|x_j\|$ and $h_{i_j}(x_j) \neq x_j$. Put $\varepsilon_j = d(h_{i_j}(x_j), x_j)$.

Let N_k denote the open ball about the origin in R^n of radius k; let $U_1 = N_{2\|x_1\|}$. For $i \geq 2$, let $U_i = N_{2\|x_i\|} - B_{\|x_{i-1}\|}$. The $\{U_i\}$ are open sets in R^n, and clearly $x_i \in U_i$. Also, if $i \neq j$, then $x_j \notin U_i$. Each U_i meets at most two other sets U_j, so that $\{U_i\}$ is a locally finite open cover of R^n. Let $\{\pi_i\}$ be a partition of unity subordinate to the $\{U_i\}$, and let $\delta(x) = \frac{1}{2}\sum_i \varepsilon_i \pi_i(x)$. The function $\delta\colon R^n \to (0, \infty)$ is continuous, so that $N_\delta(\text{id.})$ is an open set containing id. But $d(x_j, h_{i_j}(x_j)) = \varepsilon_j > \varepsilon_j/2 = \delta(x_j)$, and therefore $h_{i_j} \notin N_\delta(\text{id.})$, for each j. This contradiction proves the lemma.

COROLLARY 11 (Siebenmann [3]). *If* $W\colon [0, 1] \to H(R^n)$ *is a path in* $H(R^n)$, *with* $W(t) = h_t$, *then there exists a compact subset* K *of* R^n *such that* $h_t(x) = h_0(x)$ *for all* $t \in [0, 1]$ *and* $x \notin K$.

Proof that $H(R^n)$ *is not a Fréchet space.* Let $X = H(R^n) - H_c(R^n)$. We shall show that id. ϵ cl X; but by Lemma 10, there is no sequence in X converging to the id. In other words, given a majorant δ, we must produce $h \in X \cap N_\delta(\text{id.})$.

Let $B_i(N_i)$ denote the closed (open) ball about the origin (= 0) in R^n of radius $i + 1$. Let $a_0 = 1$ and

$$a_i = \frac{1}{2}\min\{a_{i-1},\ \min\{\delta(x) \mid x \in B_i\}\} \qquad \text{for } i = 1, 2, 3, \cdots.$$

Then $0 < a_i < a_{i-1}$, and $a_i < 2^{-i}$. Let h_0 be the radial homeomorphism taking B_0 onto B_{a_1} with $h_0(0) = 0$. In general, let h_i be the radial homeomorphism of the annulus $A_i = B_i - B_{i-1}$ onto the annulus $B_{i+a_{i+1}} - B_{i+a_i-1}$. It is easy to verify that if $y \in A_i$, then $d(h_i(y), y) < \delta(y)$.

Now define the homeomorphism h of R^n by $h(y) = h_i(y)$ if $y \in A_i$. Clearly, $h \in X$ (since the only fixed point of h is the origin), and $h \in N_\delta(\text{id.})$ (since for each $y \in R^n$, there is some i such that $y \in A_i$, and $h(y) = h_i(y)$ implies $d(h(y), y) < \delta(y)$).

Before proving the separability of $H(R^n)$, we need to state a few preliminaries. Let $f: X \to Y$ be a continuous function between metric spaces, and let $\varepsilon: Y \to (0, \infty)$ be continuous. The continuous function $\delta: X \to (0, \infty)$ is called a *continuous modulus of continuity* (CMC) for f and ε if $d(f(x), f(y)) < \varepsilon(f(x))$ whenever $x, y \in X$ and $d(x, y) < \delta(x)$.

THEOREM (see [2]). *For each continuous function* $f: X \to Y$ *and each continuous* $\varepsilon: Y \to (0, \infty)$, *there exists a* CMC $\delta: X \to (0, \infty)$.

Definition. If K is a simplicial complex and ε is a majorant on $|K|$, we say that the *mesh of* K *is less than* ε if for each simplex $\sigma \in K$, the diameter of σ is less than $\inf\{\varepsilon(x) \mid x \in \text{star } \sigma\}$.

4. PROOF OF THEOREM 4

Given $f \in H(R^n)$ and a majorant δ, we apply a theorem of [1, page 1] for $n > 4$ and [3, page 273] for $n < 4$ to obtain a PL (piecewise linear) homeomorphism g of R^n such that $d(f(x), g(x)) < \delta(x)$ for all $x \in R^n$. Let $\varepsilon(x) = \delta(g^{-1}(x))$. We may assume that g is simplicial with respect to triangulations K and L, each of mesh less than $\varepsilon/4$ (and less than 1/2). Let ε' be a CMC for g and $\varepsilon/4$; we may assume $\varepsilon'(x) \leq \varepsilon(x)$ for all $x \in R^n$. By taking subdivisions if necessary, we may assume that the mesh of K is less than $\varepsilon'/2$, and that the mesh of L is less than $\delta/8$. We now define new triangulations K' and L' of R^n such that all coordinates of the vertices of K' and L' are rational, and K' is obtained from K by "shifting" the vertices a small amount (similarly, L is shifted to L'). That is, we move the vertex v to a rational point v' in the star of v, with

$$d(v, v') < \frac{1}{4}\min\{\varepsilon'(x) \mid x \in \text{star } v\},$$

and extend conewise on $\text{st}(v, K) = v * \text{lk}(v, K) \approx v' * \text{lk}(v, K)$. This process works easily for compact polyhedra, and we can apply it to R^n by using alternate annular regions. On L, we shift points as in K, but replace $\varepsilon'(x)$ by $\min\{\varepsilon(x), \delta(x)/4\}$.

Now define the simplicial homeomorphism h of R^n (simplicial with respect to K' and L') by the composition $v' \to v \to g(v) \to (g(v))'$ and linear extension. It will suffice to show that $d(h(x), g(x)) < \delta(x)$ for all $x \in R^n$, since there are only countably many such simplicial maps h between complexes with rational vertices. We do this as follows: Given $x \in R^n$, pick a vertex $v' \in K'$ with $x \in st(v', K')$; then

$$d(h(x), g(x)) \leq d(h(x), h(v')) + d(h(v'), g(v')) + d(g(v'), g(x)).$$

Since the mesh of L' is less than $\delta/8 + 2\delta/16$, we have the inequality $d(h(x), h(v')) \leq \delta(x)/4$. Since the mesh of K' is less than $\varepsilon'/2 + 2\varepsilon'/4$, we also see that $d(g(v'), g(x)) < \varepsilon(g(x))/4 = \delta(x)/4$. To show that $d(h(v'), g(v')) < \delta(v')/2$ for all vertices $v' \in K'$, we observe that

$$d(h(v'), g(v')) = d((gv)', g(v')) \leq d((gv)', g(v)) + d(g(v), g(v'))$$

$$< \frac{\varepsilon(g(v'))}{4} + \frac{\varepsilon(g(v'))}{4} = \frac{\varepsilon(g(v'))}{2} = \frac{\delta(v')}{2}.$$

The proof of Theorem 2 actually establishes the following result.

COROLLARY 12. *The space of* PL *homeomorphisms of* R^n (*with the majorant topology*) *is separable.*

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