

# ON THE HESSIAN OF A FUNCTION AND THE CURVATURES OF ITS GRAPH

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## INTRODUCTION

In a well-known paper [3], S. S. Chern constructs certain complicated integrands, denoted by  $B_{m-h}$  (see [3, p. 84]), on a nonparametric hypersurface  $x_{m+1} = f(x_1, \dots, x_m)$  in  $\mathbb{R}^{m+1}$ . The purpose of this note is to interpret these forms in two ways. First, we show that they are closely related to the elementary invariants of the Hessian matrix  $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ . This is proved in two cases by Chern in [3] and by H. Flanders in [4]. For our second interpretation, valid in the cases when  $h$  is even, we require a concept of a Ricci tensor of order  $q$  in the theory of  $q$ -sectional curvature of J. A. Thorpe [10].

Several other results are scattered through the note. For example, we give a formula for the  $k$ th mean curvature function of a nonparametric hypersurface; it generalizes the well-known formula

$$m\sigma_1 = \sum_{j=1}^m \frac{\partial}{\partial x_j} \left( \frac{\partial f}{\partial x_j} / \left( 1 + \sum_{k=1}^m \left( \frac{\partial f}{\partial x_k} \right)^2 \right)^{1/2} \right).$$

## 1. THE INVARIANTS OF A SYMMETRIC TRANSFORMATION

In this section, we review some elementary facts. Let us consider a symmetric linear transformation  $A: V \rightarrow V$ , where  $V$  is an  $m$ -dimensional inner-product space. We denote the eigenvalues of  $A$  by  $\lambda_1, \lambda_2, \dots, \lambda_m$ .

**1.1 Definition.** If  $0 \leq q \leq m$ , then the  $q$ -th invariant  $S_q(A)$  of  $A$  is the  $q$ th elementary symmetric function of the numbers  $\lambda_1, \dots, \lambda_m$ . That is,

$$S_q(A) = \sum_{1 \leq i_1 < \dots < i_q \leq m} \lambda_{i_1} \cdots \lambda_{i_q}.$$

Furthermore, the  $q$ th Newton transformation  $T_q(A)$  associated with  $A$  is

$$T_q(A) = S_q(A)I - S_{q-1}(A) \cdot A + \cdots + (-1)^q A^q.$$

For the sake of convenience, we gather the principal facts concerning  $S_q(A)$  and  $T_q(A)$  into a single proposition.

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*Notation.* For  $1 \leq q \leq m$  and  $1 \leq i_1, \dots, i_q, j_1, \dots, j_q \leq m$ , the Kronecker symbol  $\delta \begin{pmatrix} i_1 & \dots & i_q \\ j_1 & \dots & j_q \end{pmatrix}$  has the value  $+1$  (respectively,  $-1$ ) if  $i_1, \dots, i_q$  are distinct and  $(j_1, \dots, j_q)$  is an even permutation (respectively, an odd permutation) of  $(i_1, \dots, i_q)$ . Otherwise, it has the value  $0$ .

**1.2 PROPOSITION.** *Suppose that, relative to some basis of  $V$ , the transformation  $A$  has matrix  $(A_i^j)$ . Then*

$$(a) S_q(A) = \frac{1}{q!} \sum \delta \begin{pmatrix} i_1 & \dots & i_q \\ j_1 & \dots & j_q \end{pmatrix} A_{i_1}^{j_1} \dots A_{i_q}^{j_q},$$

$$(b) T_q(A)_j^i = \frac{1}{q!} \sum \delta \begin{pmatrix} i_1 & \dots & i_q & i \\ j_1 & \dots & j_q & j \end{pmatrix} A_{i_1}^{j_1} \dots A_{i_q}^{j_q},$$

$$(c) \text{Trace}(T_q(A) \cdot A) = (q + 1)S_{q+1}(A),$$

$$(d) T_q(A) = S_q(A)I - T_{q-1}(A) \cdot A,$$

$$(e) \text{Trace } T_q(A) = (m - q)S_q(A).$$

*Proof.* First observe that both sides of the equation transform similarly under a change of basis in (a) and (b). Thus (a) and (b) follow easily if we choose an orthonormal basis of eigenvectors of  $A$ . Now (c) follows from (a) and (b), the relation (d) is trivial, and (e) follows from (c) and (d). ■

**1.3 Remark.** We shall encounter “mixed invariants” and “mixed Newton tensors” once or twice in this note. We get these objects by partially polarizing the homogeneous polynomials  $S_q(A)$  and  $T_q(A)$ . Precisely, if  $0 \leq r \leq q \leq m$  and  $A$  and  $B$  are symmetric transformations of  $V$ , we set

$$S_{qr}(A, B) = \frac{1}{q!} \sum \delta \begin{pmatrix} i_1 & \dots & i_q \\ j_1 & \dots & j_q \end{pmatrix} A_{i_1}^{j_1} \dots A_{i_r}^{j_r} B_{i_{r+1}}^{j_{r+1}} \dots B_{i_q}^{j_q}$$

and

$$T_{qr}(A, B)_j^i = \frac{1}{q!} \sum \delta \begin{pmatrix} i_1 & \dots & i_q & i \\ j_1 & \dots & j_q & j \end{pmatrix} A_{i_1}^{j_1} \dots A_{i_r}^{j_r} B_{i_{r+1}}^{j_{r+1}} \dots B_{i_q}^{j_q}.$$

Clearly, we can extend Proposition 1.2 (c) to the following result.

**1.2 PROPOSITION (c').**  $\text{Trace}(T_{qr}(A, B) \cdot A) = (q + 1)S_{q+1, r+1}(A, B)$  and  $\text{Trace}(T_{qr}(A, B) \cdot B) = (q + 1)S_{q+1, r}(A, B)$ .

## 2. THE HESSIAN MATRIX

*Notation.* Throughout the rest of this paper,  $f$  will denote a real-valued function of class  $C^3$  defined on the closure  $\overline{\mathcal{D}}$  of a domain  $\mathcal{D} \subset \mathbb{R}^m$ . We assume that  $\overline{\mathcal{D}}$  is compact and that the boundary  $\partial\mathcal{D} = \overline{\mathcal{D}} - \mathcal{D}$  is a smooth closed hypersurface

embedded in  $\mathbb{R}^m$ . We denote the partial derivatives  $\frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial x_j \partial x_k}, \dots$ , by  $f_j, f_{jk}, \dots$ . We treat the Hessian matrix  $H(f) = (f_{jk})$  as a field of symmetric linear transformations in  $\mathbb{R}^m$ , but we write  $S_q(f)$  and  $T_q(f)$  instead of the more proper  $S_q(H(f))$  and  $T_q(H(f))$ . Finally, we set  $W = \sqrt{1 + \sum_k f_k^2}$ .

*Remarks.* (a) Observe that, by (a) and (b) of Proposition 1.2,  $S_q(f)$  and  $T_q(f)$  are  $C^1$  on  $\overline{\mathcal{D}}$ .

(b) From now on our index notation will not always agree with classical tensor notation. However, repeated indices will normally still be summed.

Let us consider a pair of smooth functions  $f$  and  $g$  on  $\overline{\mathcal{D}}$ . The mixed Newton tensors  $T_{qr}(f, g)$  ( $= T_{qr}(H(f), H(g))$ ) enjoy a pleasant differentiation property.

**2.1 PROPOSITION.** *For all  $r$  and  $q$  ( $0 \leq r \leq q \leq m$ ),  $\operatorname{div}(T_{qr}(f, g)) = 0$ . That is,  $\sum_i \frac{\partial}{\partial x_i} (T_{qr}(f, g)_j^i) = 0$  ( $j = 1, \dots, m$ ).*

*Proof.* We have the relations

$$\begin{aligned} \sum_i \frac{\partial}{\partial x_i} (T_{qr}(f, g)_j^i) &= \sum_i \frac{\partial}{\partial x_i} \frac{1}{q!} \sum \delta \binom{i_1 \dots i_q i}{j_1 \dots j_q j} f_{i_1 j_1} \dots f_{i_r j_r} g_{i_{r+1} j_{r+1}} \dots g_{i_q j_q} \\ &= \sum_i \frac{1}{q!} \sum \delta \binom{i_1 \dots i_q i}{j_1 \dots j_q j} \\ &\quad \cdot (r f_{i_1 j_1} \dots f_{i_r j_r} g_{i_{r+1} j_{r+1}} \dots g_{i_q j_q} + (q - r) f_{i_1 j_1} \dots f_{i_r j_r} g_{i_{r+1} j_{r+1}} \dots g_{i_q j_q} i). \end{aligned}$$

Now  $f_{i_r j_r i}$  and  $g_{i_q j_q i}$  are symmetric in  $i_r i$  and  $i_q i$ , respectively, while the Kronecker symbols are skew-symmetric in those indices. Thus the sums over  $i_r i$  and  $i_q i$  vanish. ■

**2.2 COROLLARY.** *Suppose that  $f$  and  $g$  are smooth functions on  $\overline{\mathcal{D}}$  such that  $\operatorname{grad} f = \operatorname{grad} g$  on the boundary  $\partial \mathcal{D}$ . Then*

$$\int_{\mathcal{D}} S_{q+1}(f) dx_1 \dots dx_m = \int_{\mathcal{D}} S_{q+1}(g) dx_1 \dots dx_m.$$

*Proof.* Consider the vector field  $Y = (Y_1, \dots, Y_m)$  on  $\overline{\mathcal{D}}$ , where

$$Y_i = \sum_{j=1}^m \sum_{r=0}^q T_{qr}(f, g)_j^i (f_j - g_j).$$

Then Proposition 2.1 and Proposition 1.1(c') imply that

$$\begin{aligned} \operatorname{div} Y &= \sum_i \frac{\partial}{\partial x_i} Y_i = \sum_{ij} \sum_r T_{qr}(f, g)_j^i (f_{ij} - g_{ij}) \\ &= \sum_r ((q + 1) S_{q+1, r+1}(f, g) - (q + 1) S_{q+1, r}(f, g)) \end{aligned}$$

$$= (q + 1)S_{q+1,q+1}(f, g) - (q + 1)S_{q+1,0}(f, g) = (q + 1)S_{q+1}(f) - (q + 1)S_{q+1}(g) .$$

Now apply Stokes' theorem to  $\int_{\mathcal{D}} ((q + 1)S_{q+1}(f) - (q + 1)S_{q+1}(g)) dx_1 \cdots dx_m$  and use the hypothesis that  $Y = 0$  on  $\partial\mathcal{D}$  . ■

2.3 *Remarks.* (a) By setting  $g = 0$  in the proof of Corollary 2.2, we get the general formula

$$\int_{\mathcal{D}} (q + 1)S_{q+1}(f) dx_1 \cdots dx_m = \int_{\partial\mathcal{D}} \sum_{ij} T_q(f)_j^i (f_j t_i) dA ,$$

where  $t = (t_1 \cdots t_m)$  is the outward unit normal to  $\partial\mathcal{D}$  and  $dA$  is the volume element for the hypersurface  $\partial\mathcal{D}$  .

(b) By breaking up  $\text{grad}(f - g)|_{\partial\mathcal{D}}$  into its components normal to  $\partial\mathcal{D}$  and tangent to  $\partial\mathcal{D}$  , we see that the hypothesis  $\text{grad}(f - g)|_{\partial\mathcal{D}} = 0$  in Corollary 2.2 means  $(f - g)|_{\partial\mathcal{D}} = \text{constant}$  and  $\frac{\partial}{\partial n}(f - g) = 0$  on  $\partial\mathcal{D}$  (where  $\frac{\partial}{\partial n}$  denotes the outward normal derivative). Thus  $\int_{\mathcal{D}} S_q(f) dx_1 \cdots dx_m$  depends only on the values of

$f|_{\partial\mathcal{D}}$  and  $\frac{\partial f}{\partial n}$  along  $\partial\mathcal{D}$  . We can easily make this dependence explicit. Indeed, let us set  $z = f|_{\partial\mathcal{D}}$  and  $z_m = \frac{\partial f}{\partial n}$  along  $\partial\mathcal{D}$  . It is a simple task to express the quantities  $f_{ij}$  and  $f_k$  appearing in the boundary integral in Remark (a) in terms of the components  $z, \alpha, z_{m,\alpha}$ , and  $z, \alpha\beta$  of the covariant derivatives of  $z$  and  $z_m$  (relative to  $\partial\mathcal{D}$ ) and the components  $A_{\alpha\beta}$  of the second fundamental form of  $\partial\mathcal{D}$  in  $\mathbb{R}^m$  . All these components are computed relative to an orthonormal frame field tangent to  $\partial\mathcal{D}$  . The indices  $\alpha, \beta, \dots$  run from 1 to  $m - 1$  . For the sake of completeness, we state the formula.

2.4 PROPOSITION. *Let  $f, z, z_m, z, \alpha\beta, z_{m,\alpha}$  have the same meaning as in Remark 2.3 (b) above. Set*

$$\tilde{S}_q(z) = \frac{1}{q!} \sum \delta \begin{pmatrix} \alpha_1 & \cdots & \alpha_q \\ \beta_1 & \cdots & \beta_q \end{pmatrix} (z, \alpha_1\beta_1 - z_m A_{\alpha_1\beta_1}) \cdots (z, \alpha_q\beta_q - z_m A_{\alpha_q\beta_q})$$

and

$$\tilde{T}_q(z)_{\alpha\beta} = \frac{1}{q!} \sum \delta \begin{pmatrix} \alpha_1 & \cdots & \alpha_q & \alpha \\ \beta_1 & \cdots & \beta_q & \beta \end{pmatrix} (z, \alpha_1\beta_1 - z_m A_{\alpha_1\beta_1}) \cdots (z, \alpha_q\beta_q - z_m A_{\alpha_q\beta_q}) .$$

Then

$$\int_{\mathcal{D}} (q + 1)S_{q+1}(f) dx_1 \cdots dx_m = \int_{\partial\mathcal{D}} \left( \tilde{S}_q(z) z_m - \sum_{\alpha\beta} \tilde{T}_{q-1}(z)_{\alpha\beta} z, \alpha z_{m,\beta} \right) dA .$$

2.5 COROLLARY. *Suppose that  $z = f|_{\partial\mathcal{D}}$  is constant on  $\partial\mathcal{D}$  . Then*

$$\int_{\mathcal{D}} (q + 1) S_{q+1}(f) dx_1 \cdots dx_m = (-1)^q \binom{m-1}{q} \int_{\partial\mathcal{D}} z_m^{q+1} \sigma_q dA,$$

where  $\sigma_q$  is the  $q$ th mean curvature function on  $\partial\mathcal{D}$ . In particular, if  $q + 1$  is even and  $\partial\mathcal{D}$  is strictly convex, then  $\int_{\mathcal{D}} S_{q+1}(f) dx_1 \cdots dx_m \geq 0$ , with equality if and only if  $z_m \equiv 0$ .

*Proof of Corollary.* If  $z = \text{constant}$ , then all terms involving the derivatives  $z, \alpha$  and  $z, \alpha\beta$  in  $\tilde{S}_q$  and  $\tilde{T}_q$  will vanish. What remains, namely the expression

$$\frac{1}{q!} \sum \delta \begin{pmatrix} \alpha_1 & \cdots & \alpha_q \\ \beta_1 & \cdots & \beta_q \end{pmatrix} (-z_m A_{\alpha_1 \beta_1}) (-z_m A_{\alpha_2 \beta_2}) \cdots (-z_m A_{\alpha_q \beta_q})$$

in  $\tilde{S}_q(z)$ , multiplied by  $z_m$ , yields the formula. As for the inequality, if  $q + 1$  is even, then  $z_m^{q+1} \geq 0$ ,  $(-1)^q = -1$ , and (since  $\partial\mathcal{D}$  is strictly convex)  $\sigma_q < 0$ . The inequality now follows. ■

*Remark.* S. Bernstein [2] proved the corollary in the special case where  $m = 2$ ,  $q + 1 = 2$ , and  $\mathcal{D}$  is the ball of radius  $R$ .

### 3. THE CURVATURES OF A GRAPH

Suppose that  $M$  is the graph  $x_{m+1} = f(x_1 \cdots x_m)$ . We denote the natural coordinatization of  $M$  by

$$X(x_1 \cdots x_m) = (x_1, \cdots, x_m, f(x_1 \cdots x_m)) = \sum_{j=1}^m x_j A_j + f(x_1 \cdots x_m) A,$$

where  $A_j = (0, \cdots, 0, 1, \cdots, 0)$  (1 in the  $j$ th place) and  $A = (0, \cdots, 0, 1)$ . We recall the standard calculations, without proof, in a single proposition.

**3.1 PROPOSITION.** (a) *The natural frame field on  $M$  is  $X_1, \cdots, X_m$ , where  $X_i = \frac{\partial X}{\partial x_i} = A_i + f_i A$ , and the unit normal is*

$$N = \sum_{j=1}^m -\frac{f_j}{W} A_j + \frac{1}{W} A.$$

(b) *The matrix of the first fundamental form  $I$  of the induced metric on  $M$  (relative to the natural frame) is  $g_{ij} = \langle X_i, X_j \rangle = \delta_{ij} + f_i f_j$ .*

(c) *The volume element on  $M$  is*

$$dV = \sqrt{\det g_{ij}} dx_1 \wedge \cdots \wedge dx_m = W dx_1 \wedge \cdots \wedge dx_m.$$

(d) *The inverse matrix  $(g^{ij})$  of  $(g_{ij})$  is  $g^{ij} = \delta_{ij} - \frac{f_i f_j}{W^2}$ .*

(e) The matrix of the second fundamental form  $\Pi$  on  $M$  is  $b_{ij} = \frac{\partial^2 X}{\partial x_i \partial x_j} \cdot N = \frac{f_{ij}}{W}$ .

(f) The matrix of the shape operator  $B$  is  $b_i^j = \sum_k b_{ik} g^{kj} = \frac{f_{ij}}{W} - \sum_k \frac{f_{ik} f_k f_j}{W^3}$ .

3.2 Remark. The concepts from Section 1 are quite familiar in the context of the field  $B$  of symmetric linear transformations of the tangent spaces of  $M$ . Thus  $S_q(B) = \binom{m}{q} \sigma_q$ , where  $\sigma_q$  is the  $q$ th mean curvature function for  $M$ . Similarly, the Newton tensors  $T_q(B)$  have also been studied, for example in [6], [8], and [9]. We shall generally write  $S_q(M)$  and  $T_q(M)$ , or, if no confusion can arise, simply  $S_q$  and  $T_q$ , in place of  $S_q(B)$  and  $T_q(B)$ .

In [6] we proved that  $\text{div}_M T_q(B) = 0$ . Because our proof in [6] was quite awkward, we reprove the result here.

3.3 PROPOSITION. If  $\text{div}_M$  denotes the divergence operator in the induced metric on  $M$ , then  $\text{div}_M T_q(M) = 0$ ; that is,  $\sum_i T_{qj,i}^i = 0$  (where the comma denotes covariant differentiation).

Proof. Use a proof of the same style as that in Proposition 2.1, with the Codazzi equations  $b_{i,k}^j = b_{k,i}^j$  replacing the symmetry  $f_{ijk} = f_{ikj}$ .

3.4 Remark. If  $q$  is even, then  $S_q(M)$  and  $T_q(M)$  actually depend only on the induced metric on  $M$ . In fact, if we use the Gauss curvature equations  $R_{hi}^{jk} = b_i^j b_h^k - b_h^j b_i^k$  together with the skew-symmetries of the Kronecker symbols, we can easily verify that

$$S_q(M) = \left(-\frac{1}{2}\right)^{q/2} \frac{1}{q!} \sum \delta \binom{i_1 \dots i_q}{j_1 \dots j_q} R_{i_1 i_2}^{j_1 j_2} \dots R_{i_{q-1} i_q}^{j_{q-1} j_q}$$

and

$$T_q(M)_j^i = \left(-\frac{1}{2}\right)^{q/2} \frac{1}{q!} \sum \delta \binom{i_1 \dots i_q i}{j_1 \dots j_q j} R_{i_1 i_2}^{j_1 j_2} \dots R_{i_{q-1} i_q}^{j_{q-1} j_q}$$

In particular,  $-T_2$  is the classical Einstein tensor, and if  $q$  is even, then up to a constant factor  $T_q(M)$  is the  $q$ th generalized Einstein tensor of D. Lovelock [5].

Now we relate  $S_q(f)$  to  $S_q(M)$  and  $T_{q-1}(M)$ . Let  $A^T$  denote the vector field on  $M$  obtained by projecting the unit vector  $\bar{A} = (0, 0, \dots, 0, 1)$  orthogonally onto  $M$ . For later use, we state the following result.

3.5 PROPOSITION. (a)  $A^T = \sum_j \lambda^j X_j$ , where  $\lambda^j = f_j/W^2$ ,

(b)  $\text{div}_M(T_q(M)(A^T)) = (q+1)S_{q+1}(M) \cdot \langle A, N \rangle = (q+1)S_{q+1}(M) \cdot (1/W)$ .

Proof. (a) Clearly,  $f_k = \langle A, X_k \rangle = \langle A^T, X_k \rangle = \sum_j \lambda^j g_{jk}$ . Thus

$$\begin{aligned} \lambda^i &= \sum_k f_k g^{kj} = \sum_k f_k (\delta_{kj} - f_k f_j / W^2) \\ &= f_j - \left( \sum_k f_k^2 \right) f_j / W^2 = f_j (1 - (W^2 - 1) / W^2) = f_j / W^2. \end{aligned}$$

(b) Use Proposition 3.3 (see [6]).

3.6 THEOREM.  $S_q(f) = W^q(S_q(M) + W^2 \langle (T_{q-1}(M) \cdot B)(A^T), A^T \rangle)$ .

*Proof.* We know that

$$\begin{aligned} S_q(f) &= \frac{1}{q!} \sum \delta \binom{i_1 \dots i_q}{j_1 \dots j_q} f_{i_1 j_1} \dots f_{i_q j_q} = W^q \cdot \frac{1}{q!} \sum \delta \binom{i_1 \dots i_q}{j_1 \dots j_q} b_{i_1 j_1} \dots b_{i_q j_q} \\ &= W^q \cdot \frac{1}{q!} \sum \delta \binom{i_1 \dots i_q}{j_1 \dots j_q} b_{i_1}^{k_1} \dots b_{i_q}^{k_q} g_{k_1 j_1} \dots g_{k_q j_q} \\ &= W^q \frac{1}{q!} \sum \delta \binom{i_1 \dots i_q}{j_1 \dots j_q} b_{i_1}^{k_1} \dots b_{i_q}^{k_q} (\delta_{k_1 j_1} + f_{k_1 j_1}) \dots (\delta_{k_q j_q} + f_{k_q j_q}) \\ &= W^q S_q(M) + q \cdot W^q \cdot \frac{1}{q!} \sum \delta \binom{i_1 \dots i_q}{j_1 \dots j_q} b_{i_1}^k f_k f_{j_1} b_{i_2}^{j_2} \dots b_{i_q}^{j_q} \\ &= W^q \left( S_q(M) + \sum T_{q-1}(M)_j^i b_i^k f_k f_j \right). \end{aligned}$$

By the proof of Proposition 3.5 (a), we can write  $f_k = \sum_{\ell} \lambda^{\ell} g_{\ell k}$ , while  $f_j = W^2 \lambda^j (A^T = \sum_j \lambda^j X_j)$ . Thus,  $S_q(f) = W^q(S_q(M) + \langle (T_{q-1}(M) \cdot B)(A^T), A^T \rangle)$ . ■

3.7 Remarks. (a) It is easy to verify that the form  $B_q$ , introduced by Chern on p. 84 of [3], can be identified with

$$\begin{aligned} &(-1)^q q! (m - q)! W^{q+1} (S_q(M) \cdot W^{-2} + \langle (T_{q-1}(M) \cdot B)(A^T), A^T \rangle) dV \\ &= (-1)^q q! (m - q)! W^q (S_q(M) + W^2 \langle (T_{q-1}(M) \cdot B)(A^T), A^T \rangle) dx_1 \wedge \dots \wedge dx_m \end{aligned}$$

(since  $dV = W dx_1 \wedge \dots \wedge dx_m$ ). Thus, by Theorem 3.6,

$$B_q = (-1)^q q! (m - q)! S_q(f) dx_1 \wedge \dots \wedge dx_m.$$

(b) We can now see that bounds on the curvatures of the nonparametric hypersurface  $M$  imply bounds on the invariants of the Hessian. Suppose, for example, that there exists a positive constant  $C$  such that no eigenvalue of  $T_{q-1}(M) \cdot B$  is less than  $C$ . Then

$$S_q(M) = \frac{1}{q} \text{Trace} (T_{q-1}(M) \cdot B) \geq \frac{m}{q} C$$

and

$$\langle (T_{q-1}(M) B)(A^T), A^T \rangle \geq C |A^T|^2.$$

Thus, using Theorem 3.6, we see that

$$\frac{S_q(f)}{W^{q+2}} \geq \frac{m}{q} C \langle A, N \rangle^2 + C |A^T|^2 \geq C (\langle A, N \rangle^2 + |A^T|^2) = C$$

(since  $\langle A, N \rangle = 1/W$  and  $|A| = 1$ ). Similarly, if no eigenvalue of  $T_{q-1}(M) \cdot B$  is greater than  $-C$ , we see that  $S_q(f)/W^{q+2} \leq -C$ .

(c) Suppose now that  $\mathcal{D}$  is a disc of radius  $R$ . In Theorem 4 of [3], Chern proves that a bound on the eigenvalues of  $T_1(M) \cdot B$  (as in Remark (b)) determines an upper bound for  $R$ . Similarly, in [4] Flanders proves that a condition of the form  $\left| \frac{S_2(f)}{W^{2+\varepsilon}} \right| \geq C > 0$  places an upper bound on  $R$ . Thus, by our remark (b), we see that the result of Flanders implies Chern's result. More importantly, an extension of Flanders' analytical result to inequalities such as  $\left| \frac{S_q(f)}{W^{q+2}} \right| \geq C > 0$  would automatically extend Chern's geometric theorem.

(d) By Proposition 1.2 (d) and Remark 3.4, we see that if  $q$  is even, then  $T_{q-1}(M) \cdot B$  depends only on the induced metric on  $M$ . It is easy to see that  $T_1 B$  is simply the Ricci tensor on  $M$ . (Later, we shall see that in general  $T_{q-1} \cdot B$  can be interpreted as a kind of Ricci tensor, if  $q$  is even.) Thus we can rephrase Chern's Theorem 4 [3] in a more striking form: if no eigenvalue of the Ricci tensor of  $M$  is greater than  $-C$ , and in addition the domain is a disc of radius  $R$ , then  $R \leq K/\sqrt{C}$ , where  $K$  is independent of  $C$ . In this formulation, Chern's Theorem is highly relevant to the following question: is it true that there is no complete hypersurface in  $\mathbb{R}^{m+1}$  such that all the eigenvalues of the Ricci tensor are negative and remain uniformly bounded away from 0? The celebrated Hilbert-Efimov theorem provides an affirmative answer to this question in the case  $m = 2$ .

We now interpret the tensor  $T_{q-1}(M) \cdot B$  ( $q$  even) in terms of the higher-order sectional curvatures (see [10]).

**3.8 Definition.** Let  $M$  be a Riemannian manifold of dimension  $m \geq 2$ . Let  $q$  be an even integer ( $2 \leq q \leq m$ ). If  $L$  is a  $q$ -dimensional subspace of the tangent space  $M_x$  at  $x \in M$ , and if  $(e_1, \dots, e_m)$  is an orthonormal frame at  $x$  with  $e_1, \dots, e_q \in L$ , then we define the  $q$ -sectional curvature  $\gamma(L)$  of  $M$  at  $L$  by

$$\gamma(L) = C_q \cdot \sum_1^q \delta \begin{pmatrix} i_1 & \dots & i_q \\ j_1 & \dots & j_q \end{pmatrix} R_{i_1 j_1}^{j_2} \dots R_{i_{q-1} j_{q-1}}^{j_q},$$

where  $C_q = \left(-\frac{1}{2}\right)^{q/2} \cdot \frac{1}{q!}$ . Notice that this expression is actually independent of the choice of frame, as long as  $e_1, \dots, e_q$  lie in  $L$ . We further define the *Ricci tensor of degree  $q$* , denoted by  $R_q$ , to be the symmetric tensor of type  $(1, 1)$  such that if  $v$  is a unit tangent vector at  $x \in M$  and  $(e_1, \dots, e_m)$  is an orthonormal basis at  $x$  such that  $v = e_m$ , then

$$\langle R_q(v), v \rangle = \sum_{1 \leq i_1 < \dots < i_{q-1} \leq (m-1)} \gamma(e_{i_1} \wedge \dots \wedge e_{i_{q-1}} \wedge v).$$

The independence of  $R_q$  from the choice of frame will become evident. Our construction generalizes the usual one for the classical Ricci tensor  $R_2$ .

It is easy to relate these generalized Ricci tensors to the generalized Einstein tensors of Lovelock. Indeed, let  $(e_1, \dots, e_m)$  be as above, with  $e_m = v$ . Then



$$\begin{aligned} \langle T_q(v), v \rangle &= T_{qm}^m = C_q \sum_1^m \delta \binom{i_1 \cdots i_q \ m}{j_1 \cdots j_q \ m} R_{i_1 i_2}^{j_1 j_2} \cdots R_{i_{q-1} i_q}^{j_{q-1} j_q} \\ &= C_q \sum_1^m \delta \binom{i_1 \cdots i_q}{j_1 \cdots j_q} R_{i_1 i_2}^{j_1 j_2} \cdots R_{i_{q-1} i_q}^{j_{q-1} j_q} \\ &\quad - \sum_1^m C_q \delta \binom{i_1 \cdots i_q}{j_1 \cdots j_q} R_{i_1 i_2}^{j_1 j_2} \cdots R_{i_{q-1} i_q}^{j_{q-1} j_q}. \end{aligned}$$

Here  $\sum'$  means "sum over values of the indices such that  $m \in \{i_1 \cdots i_q\}$ ". Now the first sum of the right-hand side clearly gives us  $S_q(M)$ . The  $\sum'$ -sum can be decomposed conveniently into summands as follows. For each  $q$ -tuple  $k_1 < k_2 < \cdots < k_{q-1} < k_q = m$ , gather all the terms in the  $\sum'$ -sum such that  $\{i_1 \cdots i_q\} = \{k_1 \cdots k_q\}$ . These terms add up to  $\gamma(e_{k_1} \wedge \cdots \wedge e_{k_{q-1}} \wedge v)$ . Now sum over all such  $q$ -tuples  $k_1 < k_2 < \cdots < k_q = m$ . The total sum is  $\langle R_q(v), v \rangle$ . Thus we have proved the following result.

**3.9 PROPOSITION.** *If  $M$  is an  $m$ -dimensional Riemannian manifold and  $q$  is even ( $2 \leq q \leq m$ ), then  $T_q(M) = S_q(M)I - R_q(M)$ .*

**3.10 COROLLARY.** *Suppose that  $M$  is an Einstein manifold of degree  $q$  in the sense that  $R_q = \lambda I$ . Then  $S_q$  is constant if  $q < m$ .*

*Proof.* If  $R_q = \lambda I$ , then

$$\text{Trace } R_q = \text{Trace}(S_q I - T_q) = m S_q - (m - q) S_q = q S_q = m \lambda,$$

so that  $\lambda = \frac{q}{m} S_q$ . Thus  $T_q = (S_q - \lambda)I = S_q(1 - q/m)I$ . Now the fact that  $\text{div}_M T_q = 0$  implies that  $\text{grad}_M(S_q(1 - q/m)) = 0$ , in other words, that  $S_q = \text{constant}$  if  $q/m \neq 1$ . ■

**3.11 COROLLARY.** *If  $M$  is a nonparametric hypersurface and  $q$  is even, then  $T_{q-1}(M) \cdot B = R_q$ .*

#### 4. MISCELLANEOUS RESULTS

Suppose that  $M$  is a nonparametric hypersurface, as before. We shall derive a formula in the coordinates  $x_1, \dots, x_m$  for the  $q$ th mean curvature. It will be in divergence form.

**4.1 PROPOSITION.** *If  $M$  is a nonparametric hypersurface, then*

$$(q + 1) S_{q+1}(M) = \sum_{ij} \frac{\partial}{\partial x_i} \left( \frac{1}{W} T_q(M)_{ij}^i f_j \right).$$

*Remark.* The point of this proposition is that we can express the right-hand side in terms of  $f$  and its derivatives, using Proposition 3.1 (f). Notice that when

$q = 1$ , we recover the well-known formula  $m\sigma_1 = \sum \frac{\partial}{\partial x_j} \left( \frac{f_j}{W} \right)$ . The proof of the proposition requires the following lemma (see [1, p. 77]).

**4.2 LEMMA.** *Let  $N_1$  and  $N_2$  be smooth manifolds of the same dimension, and let  $\alpha: N_1 \rightarrow N_2$  be a diffeomorphism. Suppose that  $N_1$  and  $N_2$  are equipped with volume elements  $\Omega_1$  and  $\Omega_2$ , respectively, and that*

$$\alpha^*(\Omega_2) = \rho \cdot \Omega_1, \quad (\alpha^{-1})^*(\Omega_1) = \lambda \Omega_2$$

(so that  $\lambda = (\rho \circ \alpha)^{-1}$ ). Let  $Z_1$  and  $Z_2$  be vector fields on  $N_1$  and  $N_2$ , respectively, such that  $Z_2 = \alpha_*(Z_1)$ . Also let  $\text{div}_1$  and  $\text{div}_2$  be the divergence operators associated with the volume elements  $\Omega_1$  and  $\Omega_2$ . Then  $\text{div}_1(\rho Z_1) = \rho((\text{div}_2 Z_2) \circ \alpha)$ .

In our situation, we have  $N_1 = \mathcal{D}$  and  $N_2 = M$ , while  $\Omega_1 = dx_1 \wedge \dots \wedge dx_m$  and  $\Omega_2 = dV$ . Also,  $\alpha = X$  and  $\rho = \sqrt{g} = W$ . We set  $Z_2 = T_q(A^T)$  and  $Z_1 = (\alpha^{-1})_*(Z_2)$ . The components of the vector  $Z_2$  relative to the natural frame  $X_1, \dots, X_m$  on  $M$  are  $\sum_j T_q(M)_j^i f_j / W^2$  ( $i = 1, \dots, m$ ) (see Proposition 3.5 (a)). Thus, since  $\alpha_*(A_j) = X_j$ , the vector field  $Z_1$  has the same components. Now, applying the lemma and Proposition 3.5 (b), we get the equation

$$\sum_i \frac{\partial}{\partial x_i} \left( W \cdot \left( \sum_j T_{qj}^i f_j / W^2 \right) \right) = W \cdot ((q + 1) S_{q+1}(M) \cdot \langle A, N \rangle) = (q + 1) S_{q+1}(M)$$

(since  $\langle A, N \rangle = 1/W$ ). ■

**4.2 COROLLARY.** *If  $\text{grad } f = 0$  on  $\partial \mathcal{D}$ , then  $\int_{\mathcal{D}} S_q(M) dx_1 \dots dx_m = 0$ .*

*Proof.* Use Proposition 4.1, Stokes' theorem and the fact that  $\text{grad } f = 0$  implies  $A^T = 0$ . ■

We end this note by proving a uniqueness theorem.

**4.3 THEOREM.** *Suppose that  $f_0$  and  $f_1$  are functions on  $\overline{\mathcal{D}}$  with graphs  $M_0$  and  $M_1$ . Suppose further that*

- (a)  $f_0 = f_1$  and  $\text{grad } f_0 = \text{grad } f_1$  on  $\partial \mathcal{D}$ ,
- (b)  $f_1 \geq f_0$  on  $\mathcal{D}$ ,
- (c)  $f_0$  and  $f_1$  are convex functions,
- (d)  $\int_{\mathcal{D}} S_q(M_0) dV_0 = \int_{\mathcal{D}} S_q(M_1) dV_1$  for some  $q < m$ .

Then  $f_0 \equiv f_1$  on  $\mathcal{D}$ .

*Proof.* Consider the 1-parameter family of functions  $f_t = tf_1 + (1 - t)f_0$  ( $0 \leq t \leq 1$ ). Corresponding to these functions we have the graphs  $M_t$ , curvatures  $S_q(t)$ , volume elements  $dV_t$ , normals  $N_t$ , and so forth. Let

$$X_t = (x_1 \dots x_m, f_t(x_1 \dots x_m))$$

be the corresponding family of immersions. Then  $\xi = \frac{\partial X_t}{\partial t}$  is the deformation vector.

Hypothesis (a) implies that  $\xi$  and  $\text{grad } \xi$  vanish on  $\partial \mathcal{D}$ . Thus we can use the variational formulas in [7] and assert that

$$\frac{d}{dt} \int_{\mathcal{D}} S_q(t) dV_t = -(q+1) \int_{\mathcal{D}} S_{q+1}(t) \langle \xi, N_t \rangle dV_t.$$

Now  $\langle \xi, N_t \rangle = \frac{f_1 - f_0}{W_t} \geq 0$ , by (b), and  $S_{q+1}(t)$  never vanishes, by (c). However by (d),

$$0 = \int_{\mathcal{D}} S_q(M_1) dV_1 - \int_{\mathcal{D}} S_q(M_0) dV_0 = \int_0^1 \frac{d}{dt} \int_{\mathcal{D}} S_q(t) dV_t dt.$$

Since we have just seen that the integrand for the  $t$ -integral never changes sign, and that it vanishes only if  $f_1 \equiv f_0$ , the result follows. ■

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