ISOTROPY SUBGROUPS OF TORUS T^n-ACTIONS ON (n + 2)-MANIFOLDS

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1. INTRODUCTION

If a compact, connected Lie group G acts effectively on a simply connected, compact manifold M with codimension 2, and if at least one orbit is singular, then there exists no nontrivial finite isotropy subgroup of (G, M) [1, p. 211]. In fact, Theorem 4 in our paper states that if a torus group T^n acts effectively on a simply connected, closed (n + 2)-manifold M^{n+2} $(n \geq 2)$, then both T^1- and T^2-subgroups of T^n must appear as isotropy subgroups, and that these are the only possible nontrivial isotropy subgroups of T^n.

Numerous research papers discuss the importance of a thorough understanding of isotropy subgroups of the action (G, M). The following are typical examples of cases that are related to our work.

Let (G, M) be the action of a compact, connected Lie group G on a compact, connected, orientable, aspherical cohomology manifold M. Here, *aspherical* means that the universal covering space is contractible. P. E. Conner and D. Montgomery [2, Theorem 5.2] wrongly concluded that there are no nontrivial isotropy subgroups of (G, M). In this case, the action is principal, and this enabled them to prove their major theorem by using theorems about principal fiber bundles. Later, P. E. Conner and F. Raymond [3, Theorem 5.6] showed that the isotropy subgroups of (G, M) do not necessarily all reduce to the identity element e, but that each isotropy subgroup is finite. They succeeded in giving a correct proof of the theorem mentioned above [2, Theorem 5.2].

One more example we wish to mention is that given by D. Montgomery and G. D. Mostow [5], who showed that if the toroid T^r acts effectively on an n-Euclideanlike cohomology manifold M $(n \leq 2r + 1)$, then all the isotropy subgroups are connected and T^r has exactly 2^r isotropy subgroups: e, the circle subgroups $T^1_1, T^1_2, \cdots, T^1_r$, and their direct products. In other words, there are no nontrivial finite isotropy subgroups of (T^r, M^n), and the set of fixed points $F(T^r, M^n)$ is not empty. It is known that $F(T^r, M^n)$ is actually some Euclideanlike cohomology manifold.

The examples above suggest that the size of isotropy subgroups of G depend strongly upon the fundamental group $\Pi_1(M)$, on the higher homotopy groups of M, and on the codimension of the action (G, M).

Define a mapping f: (G, e) → (M, x) by the formula f(g) = gx. This mapping induces a homomorphism $f_*: \Pi_1(G, e) \to \Pi_1(M, x)$.

Received March 3, 1973.
Soon-Kyu Kim acknowledges partial support from National Science Foundation Grant GP 29117.

Michigan Math. J. 20 (1973).

The purpose of this paper is to show that if a torus group T^n acts effectively on a connected, orientable $(n+2)$-manifold M^{n+2} such that the image of f_* is $\{0\} \subset \Pi_1(M^{n+2})$, where $\Pi_1(M^{n+2})$ is a finite group containing a cyclic subgroup Z_p $(p \geq 2)$, then there exists a point $x \in M^{n+2}$ such that the isotropy subgroup T^n_x is finite. This extends a result of [8], and an immediate corollary says that if a circle group T^1 acts effectively on a 3-dimensional lens space $L(p, q)$ with $F(T^1, L(p, q)) \neq \emptyset$, then there exists a unique orbit whose isotropy group is Z_p [9, Theorem 3].

Let (G, X) and (G, Y) be connected group actions on connected spaces X and Y. Let $h: (X, x) \to (Y, h(x))$ be an equivariant mapping, and let $H \subset \Pi_1(X, x)$ and $K \subset \Pi_1(Y, h(x))$ be normal subgroups of their respective fundamental groups such that $\operatorname{Im} f_* \subset H$ and $h_*(H) \subset K$. Let $B(X)$ and $B(Y)$ be the covering spaces corresponding to $H \subset \Pi_1(X, x)$ and $K \subset \Pi_1(Y, h(x))$, respectively. Then we can lift G-actions on X and Y to $B(X)$ and $B(Y)$. We show here that the equivariant mapping h also can be lifted to $\bar{h}: (G, B(X)) \to (G, B(Y))$ equivariantly. Now let $B^*(X) = B(X)/G$ and $B^*(Y) = B(Y)/G$ be orbit spaces on which $\Pi_1(X)/H$ and $\Pi_1(Y)/K$ act properly discontinuously. Theorem 2 says that if $h^{\perp}: B^*(X) \to B^*(Y)$ is a mapping induced by $\bar{h}$, then there exists a monomorphism

$$\gamma: (\Pi_1(X)/H)_{b_*} \to (\Pi_1(Y)/K)_{h^{\perp}(b_*)},$$

where b_* is the orbit containing $b \in B(X)$. This result gives a criterion involving the fundamental groups of two actions, which in a number of cases can be used to exclude the possible existence of an equivariant mapping between them. We give an application of this (Corollary 1).

For completeness, we include a statement of Theorem 4 (whose proof appears in [4]). An immediate consequence of this theorem is that if T^2 acts effectively on a simply connected 4-manifold M^4, then $F(T^2, M^4) \neq \emptyset$, a result which appeared in [7].

In summary, a number of well-known results are more or less immediate corollaries of the rather elementary theorems of this paper.

2. DEFINITIONS

We consider an action (G, X) of a pathwise connected topological group G on a pathwise connected space X for which covering-space theory makes sense. The group $G_x = \{g \in G \mid gx = x\}$ is called an *isotropy subgroup* of (G, X) at $x \in X$. By $G(x) = \{g(x) \mid g \in G\}$ we shall denote the orbit corresponding to G_x, or the orbit of $x \in X$. The orbit space, the set of all orbits, will be denoted by $X^* = X/G$. The maximum orbit type for orbits in X is called the *principal orbit type* P, and orbits of this type are called *principal orbits*. If Q is another orbit type such that $\dim P > \dim Q$, then Q is called a *singular orbit type*. The *codimension* of (G, X) is defined to be $\dim X - \dim P$.

For technical reasons, we assume that (G, X) is at least a locally smooth action (see [1] for the definition).

From the well-known slice theorem, it follows that if (T^n, M^{n+2}) is an effective torus action on a closed, compact, orientable manifold M, then there exists a principal T^n-orbit, and the orbit space M^* is a compact 2-manifold. The set of principal orbits forms a dense open subset of M^*, and the boundary ∂M^*, possibly empty, consists of singular orbits.

3. ISOTROPY SUBGROUPS OF T^n-ACTIONS ON M^{n+2}

Let $P(e, G)$ be the space of paths in G issuing from the identity element e, and let $P(x, X)$ be the space of paths in X issuing from $x \in X$. Define a mapping $f: (G, e) \to (X, x)$ by $f(g) = gx$. This mapping induces a homomorphism $f_*: \Pi_1(G, e) \to \Pi_1(X, x)$. Let $H \subset \Pi_1(X, x)$, and let $B(X)$ be a covering space corresponding to H. The following two lemmas from [3] play a crucial role in the latter part of this paper. We assume that all spaces are path-connected.

LEMMA 1. *Let* (G, X) *be an action. If* $\operatorname{Im} f_* \subset H$, *then there exists an equivariant covering action* $P: (G, B(X)) \to (G, X)$. *Furthermore, if* H *is normal, then* $g(b\alpha) = (gb)\alpha$ *for all* $\alpha \in \Pi_1/H$.

Let $B(X)$ be a covering space corresponding to $H \subset \Pi_1(X, P(b))$, where H is invariant under $G_{P(b)}$ for some $b \in B$. Let $B^*(X) = B(X)/G$. Since Π_1/H acts freely on B, and $g(b\alpha) = (gb)\alpha$ for all $\alpha \in \Pi_1/H$, there is an action of Π_1/H on $B^*(X)$ defined by $G(b)\alpha = G(b\alpha)$, and the diagram

$$\begin{array}{ccc} B(X) & \xrightarrow{\quad P_1' \quad} & B^*(X) = B(X)/G \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ X & \xrightarrow{\quad P_1 \quad} & X^* = X/G, \end{array}$$

(where the P's are the obvious projection maps) commutes.

For each $b \in B(X)$, we have the relation $G_b \subset G_{P(b)}$; for if $gb = b$, then $P(gb) = P(b) = g\,P(b)$.

At each $b \in B(X)$ we can define a homomorphism $\eta_b: G_{P(b)} \to \Pi_1/H$. If $g \in G_{P(b)}$, define $\eta_b(g)$ to be the unique element in Π_1/H with $gb = b\,\eta_b(g)$. It is not difficult to see that η_b is a homomorphism for each $b \in B(X)$.

LEMMA 2. *The sequence*

$$e \longrightarrow G_b \xrightarrow{\ i\ } G_{P(b)} \xrightarrow{\ \eta_b\ } (\Pi_1/H)_{P_1'(b)} \longrightarrow 0$$

is exact for every $b \in B(X)$.

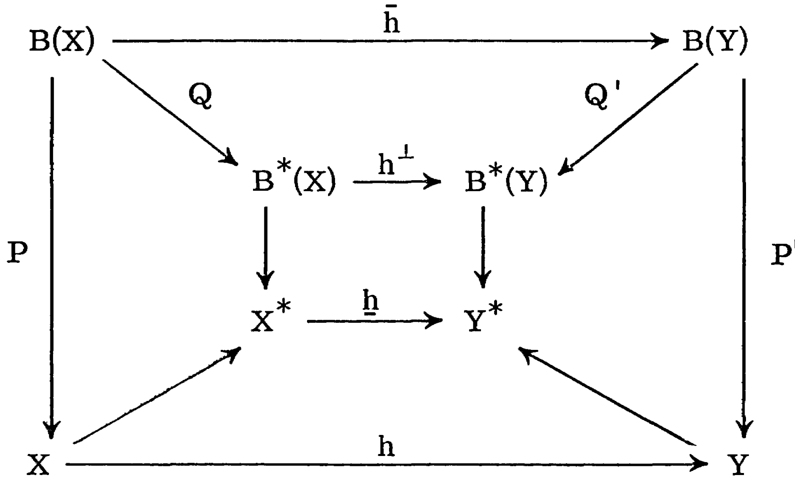
Let X and Y be G-spaces. A mapping $h: X \to Y$ is called a *G-equivariant* mapping if $h(g(x)) = g(h(x))$ for all $x \in X$ and $g \in G$. The immediate result is that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad h \quad} & Y \\ \downarrow & & \downarrow \\ X^* & \xrightarrow{\quad \underline{h} \quad} & Y^* \end{array}$$

(where $\underline{h}: X^* \to Y^*$ is a mapping induced by h) commutes. Also, since $gh(x) = h(g(x)) = h(x)$ if $g \in G_x$, it is easily seen that $G_x \subset G_{h(x)}$.

Let $H \subset \Pi_1(X, x)$ and $K \subset \Pi_1(Y, h(x))$ be normal subgroups such that $\mathrm{im}\, f_* \subset H$ and $h_*(H) \subset K$. If f': $(G, e) \to (Y, h(x))$ is defined by the equation $f'(g) = gh(x)$, then $f'_*(\Pi_1(G, e)) \subset K$, since $f' = hf$. Therefore, there exists a covering action $(G, B(Y))$ corresponding to K.

THEOREM 1. *Let* h: $(X, x) \to (Y, h(x))$ *be a* G-*equivariant mapping such that* $h_*(H) \subset K \subset \Pi_1(Y, h(x))$, *and let* $\mathrm{Im}\, f_* \subset H$. *Then there exists an equivariant mapping* $\bar{h}$: $B(X) \to B(Y)$ *covering* h *such that the diagram*



commutes. Here the P's *and* Q's *are natural projection mappings, and the* h's *are the obvious equivariant mappings.*

Proof. A G-action on $B(X)$ is defined as follows. Given $g \in G$ and $b \in B(X)$, we select first a path $g(t) \in P(e, G)$ with $g(1) = g$; then we choose a path $P(t)$ that represents $b \in B(X)$. We define gb to be the point represented by a path

$$\begin{cases} g(2t)x & (0 \leq t \leq 1/2), \\ g(1)\,P(2t - 1) & (1/2 \leq t \leq 1). \end{cases}$$

This is a well-defined action (for more details, we refer the reader to [3]). Similarly, there exists an action $(G, B(Y))$.

Now define $\bar{h}$: $B(X) \to B(Y)$ by defining $\bar{h}(b)$ to be the point represented by $h(P(t))$. Thus $\bar{h}(g(b))$ is represented by

$$\begin{cases} h(g(2t)x) & (0 \leq t \leq 1/2), \\ h(g(1)\,P(2t - 1)) & (1/2 \leq t \leq 1). \end{cases}$$

Since h is equivariant, this is the same as

$$\begin{cases} g(2t)\,h(x) & (0 \leq t \leq 1/2), \\ g(1)\,h(P(2t - 1)) & (1/2 \leq t \leq 1), \end{cases}$$

and this represents $g(\bar{h}(b))$. Since $h_*(\mathrm{Im}\, f_*) \subset h_*(H) \subset K$, the mapping $\bar{h}$ is well-defined and equivariant. Now the mapping $h^\perp$: $B^*(X) \to B^*(Y)$ induced by $\bar{h}$ is given by the equation $h^\perp(b_*) = Q'\bar{h}Q^{-1}(b_*)$, where $b_* = Q(G(b))$ for every $b \in B(X)$. The fundamental groups of X and Y act on $B^*(X)$ and $B^*(Y)$, respectively.

We would like to show that $h^{\perp}$ is an equivariant mapping. Let $\alpha \in \Pi_1(X)/H$. Then $h^{\perp}(b_* \alpha) = Q'\bar{h}Q^{-1}(b_* \alpha)$, where $b_* \alpha = Q(G(b\alpha))$. Thus

$$\begin{aligned} h^{\perp}(b_* \alpha) &= Q'\bar{h}(G(b\alpha)) = Q'G(\bar{h}(b\alpha)) = Q'G(\bar{h}(b)h_*(\alpha)) = Q'(G(\bar{h}(b))h_*(\alpha)) \\ &= Q'(\bar{h}G(b))h_*(\alpha) = (Q'\bar{h}Q^{-1}(b_*))h_*(\alpha) = (h^{\perp}(b_*))h_*(\alpha). \end{aligned}$$

Some standard diagram-chasing completes the proof.

Let $h^{\perp}: B^*(X) \to B^*(Y)$ be the mapping induced by $\bar{h}: B(X) \to B(Y)$. For each point $b_* \in B^*(X)$, define $\gamma: (\Pi_1(X, x)/H)_{b_*} \to (\Pi_1(Y, h(x))/K)_{h^{\perp}(b_*)}$ by the rule $\gamma([\alpha]) = [h_*(\alpha)]$. This is a well-defined homomorphism, since

$$[h_*(\alpha)](h^{\perp}(b_*)) = h^{\perp}([\alpha](b_*)) = h^{\perp}(b_*),$$

and

$$\begin{aligned} \gamma([\alpha] + [\beta])(h^{\perp}(b_*)) &= [h_*([\alpha] + [\beta])](h^{\perp}(b_*)) = [h_*(\alpha) + h_*(\beta)](h^{\perp}(b_*)) \\ &= [h_*(\alpha)](h^{\perp}(b_*)) + [h_*(\beta)](h^{\perp}(b_*)) = \gamma[\alpha](h^{\perp}(b_*)) + \gamma[\beta](h^{\perp}(b_*)) \\ &= (\gamma[\alpha] + \gamma[\beta])(h^{\perp}(b_*)). \end{aligned}$$

THEOREM 2. *For a point* $b \in B$ *such that* $G_{\bar{h}(b)} = 0$, *the mapping* $\gamma: (\Pi_1(X, x)/H)_{b_*} \to (\Pi_1(Y, h(x))/K)_{h^{\perp}(b_*)}$ *is a monomorphism.*

Proof. Let $g \in G_{P(b)}$. Then there exists a unique element $\eta_b(g) \in \Pi_1(X, x)/H$ such that $gb = b\eta_b(g)$.

The relations

$$\bar{h}(b)\eta_{\bar{h}(b)}(jg) = j(g)\bar{h}(b) = g\bar{h}(b) = \bar{h}(gb) = \bar{h}(b\eta_b(g)) = \bar{h}(b)h_*\eta_b(g) = \bar{h}(b)\gamma\eta_b(g)$$

show that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_b & \longrightarrow & G_{P(b)} & \longrightarrow & (\Pi_1(X, x)/H)_{b_*} & \longrightarrow & 0 \\ & & \downarrow i & & \downarrow j & & \gamma \downarrow & & \\ 0 & \longrightarrow & G_{\bar{h}(b)} & \longrightarrow & G_{P'(\bar{h}(b))} & \xrightarrow{\eta_{\bar{h}(b)}} & (\Pi_1(Y, h(x))/K)_{h^{\perp}(b_*)} & \longrightarrow & 0 \end{array}$$

is commutative. The mappings i and j are inclusion homomorphisms. Lemma 2 shows that the rows are exact. By taking a point $b \in B$ such that $G_{\bar{h}(b)} = 0$, we see that γ is a monomorphism.

COROLLARY 1. *Let* $(T^1, L_3(p, q))$ *and* $(T^1, L_3(p', q'))$ *be two circle actions (with fixed points) on 3-dimensional lens spaces. If* p *does not divide* p', *then there is no equivariant mapping* h *from* $(T^1, L_3(p, q))$ *to* $(T^1, L_3(p', q'))$.

Proof. We deduce this from Theorem 2 by examining the fundamental groups of the lens spaces.

THEOREM 3. *Let* T^n *act effectively on a compact, connected, orientable* $(n+2)$-*manifold* M^{n+2} *whose fundamental group* $\Pi_1(M^{n+2})$ *is a nontrivial finite group containing a cyclic group* Z_p $(p \geq 2)$. *If* $\operatorname{Im} f_*^x = 0$ *for some* $x \in M^{n+2}$, *then* *there exists a point* $x \in M^{n+2}$ *such that* T^n_x *is a nontrivial finite group* (*for* $n = 1$, *we assume the existence of a singular orbit*).

Proof. Let B be the universal covering manifold corresponding to $H = 0$, as in Lemma 1. Thus $\Pi_1(M)$ is a free deck-transformation group. Since $\Pi_1(M)$ is finite, B is a compact, simply connected $(n+2)$-manifold, and T^n can be lifted to B. Let $P: B \to M$ denote the projection mapping. Since B is a simply connected $(n+2)$-manifold, $B^* = B/T^n$ is a simply connected 2-manifold [6]; in fact, we can assume that B^* is the two-dimensional disk D^2. Let $P': B \to D^2$ be the projection mapping. The free action of $\Pi_1(M)$ on B induces an action of $\Pi_1(M)$ on D^2, as in Lemma 2. By Lemma 2, we have for each $b \in B$ the exact sequence

$$e \longrightarrow T^n_b \xrightarrow{\ i\ } T^n_{P(b)} \xrightarrow{\ \eta_b\ } \Pi_1(M)_{p'(b)} \longrightarrow 0 .$$

By assumption, there exists $Z_p \subset \Pi_1(M)$ $(p \geq 2)$. Now, by the well-known fixed-point theorem, we can assume that there exists a point d in the interior of D^2 such that $d \in F(Z_p, D^2)$. Since every interior point of D^2 corresponds to a principal orbit of the T^n-action on B, we see that $T^n_b = e$ for $P'(b) = d$. The exact sequence shows that $\eta_b: T^n_{P(b)} \to \Pi_1(M)_{P'(b)=d}$ is an isomorphism. Since $\Pi_1(M)$ contains a nontrivial cyclic group Z_p, the proof is complete.

Remarks. (a) M^{n+2} is not an aspherical manifold [3].

(b) By Theorem 4, there exist isotropy subgroups T^1 and T^2 in the action of T^n on B. Therefore, in the T^n-action on M, there exists an isotropy subgroup containing T^1 and T^2. However, we know that for $n > 2$ the action (T^n, M^{n+2}) cannot have a fixed point. Therefore there exist singular orbits.

(c) There exist at least three different orbit types--principal, exceptional, and singular.

(d) Isotropy subgroups depend not only on $\Pi_1(M)$, but also on higher homotopy groups of M.

COROLLARY 2. *Let* $(T^1, L_3(p, q))$ *be an effective circle group action on a* 3-*dimensional lens space* $L_3(p, q)$ *with* $F(T^1; L_3(p, q)) \neq \emptyset$. *Then there exists exactly one orbit whose isotropy subgroup is* Z_p.

This corollary and Corollary 3 (following Theorem 4) were given as theorems in [9] and [7], respectively.

For completeness, we give the following theorem, which appears in [4]:

THEOREM 4. *Let* (T^n, M^{n+2}) *be an effective* T^n-*action on a simply connected, compact, closed* $(n+2)$-*manifold* M^{n+2}. *Then every isotropy subgroup is a* T^1- *or* T^2-*subgroup of* T^n, *and each isotropy subgroup* T^1 *is a subgroup of some isotropy subgroup* T^2 *in* T^n (*for* $n = 1$, *we assume the existence of a singular orbit*). *Furthermore, two or more* T^2-*subgroups of* T^n (*but a finite number of them*) *must appear as isotropy subgroups of* T^n.

We omit the proof of this theorem. The theorem generalizes [7, Lemma 5.2].

COROLLARY 3. *Let* (T^2, M^4) *be an effective action of the torus group* T^2 *on a simply connected closed* 4-*manifold* M^4. *Then the fixed-point set* $F(T^2, M^4)$ *is not empty.*

Remark. This result can fail to be true if the codimension is larger than 2.

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