BOUNDARY-REDUCIBLE 3-MANIFOLDS AND WALDHAUSEN'S THEOREM

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INTRODUCTION

Let M and N be compact, connected, irreducible, orientable and boundary-irreducible 3-manifolds, and suppose that N is sufficiently large (see Section 1) and $\pi_1(M) \neq 1$. In [6], F. Waldhausen proved that every mapping f: $(M, \partial M) \to (N, \partial N)$ that induces a monomorphism of fundamental groups is homotopic (boundary going to boundary during the homotopy) to a covering or a map of M into ∂N. In the latter case, M is a product $F \times I$, where F is a closed, orientable 2-manifold and I is the unit interval [0, 1]. An immediate consequence of this result is that for such manifolds M and N, a homotopy equivalence of pairs $(M, \partial M)$ and $(N, \partial N)$ is induced by a homeomorphism. In [3], W. Heil extended Waldhausen's Theorem to nonorientable, P^2-irreducible manifolds; there is then the possibility that M is a nontrivial I-bundle over an arbitrary closed surface with F(M) again contracting into ∂N. A noncompact version of Heil's Theorem was proved by E. M. Brown and this author in [1].

If M is boundary-reducible, it is easy to construct a counterexample to the theorem. Let D be the set of all complex numbers z with $|z| \leq 1$. Let $M = S^1 \times D$. Let f: $(M, \partial M) \to (M, \partial M)$ be defined by $f(\theta, z) = (\theta, z^2)$. Then f has a branch curve at $S^1 \times \{0\}$, and there is no homotopy that takes boundary to boundary and makes f a covering map (unbranched) or contracts f(M) into ∂M. We shall show, however, that this is in effect the only counterexample. Theorem 1 gives the boundary-reducible version of Waldhausen's Theorem. In Section 2, we state the nonorientable analogue and sketch a proof. In Section 3, we consider the problem of respecting peripheral structure in boundary-reducible manifolds.

1. THE BOUNDARY-REDUCIBLE WALDHAUSEN THEOREM

We recall some terminology used in [6]. A 3-manifold is *irreducible* if every embedded 2-sphere bounds a 3-cell. A *surface* F in a 3-manifold M is a 2-sided 2-manifold embedded so that $F \cap \partial M = \partial F$ or $F \subset \partial M$. A surface F in M is *incompressible* in M if $\pi_1(F) \to \pi_1(M)$ is monic and F is not a 2-sphere bounding a 3-cell (unlabeled mappings of fundamental groups are induced by inclusion). A compact, irreducible 3-manifold is *sufficiently large* if it contains an incompressible surface. We say M is *boundary-irreducible* if all boundary components of M are incompressible in M. All manifolds will be triangulated, and whenever possible all maps will be assumed to be piecewise linear.

We first state a glorified version of Lemma 1.4.3 of [6] (Nielsen's Theorem).

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LEMMA 1. *Let* $f: (M, \partial M) \to (N, \partial N)$ *be a mapping of compact, connected* 2-*manifolds, and suppose* M *is not a sphere or projective plane. Then either there is a simple, orientable, closed curve representing a non-trivial element of* $\ker(f_*: \pi_1(M) \to \pi_1(N))$, *or* f_* *is monic and there exists a homotopy* $f_t: (M, \partial M) \to (N, \partial N)$ *with* $f_0 = f$ *such that one of the following three cases occurs:*

(i) f_1 *is a covering map;*

(ii) M *is an annulus, Möbius strip, or disk, and* $f_1(M) \subset \partial N$;

(iii) M *and* N *are disks, and* $f_1 \mid \partial M$ *is a more-than-one-sheeted covering of* ∂N.

Proof. Simply mimic the proof of Waldhausen's Theorem in one less dimension. The details are cumbersome, but no problems arise.

Remark. We are interested in Lemma 1 mainly because it provides a simple closed curve representing a nontrivial element of $\ker f_*$ when f_* is not monic. This does not seem to be well-known. It is also interesting to compare the above statement of Nielsen's Theorem with our Theorem 1.

We shall need a slightly stronger product theorem than Waldhausen's Lemma 5.1. The following is all that is necessary to weaken his assumptions of boundary-irreducibility.

LEMMA 2. *Let* F *be a component of* ∂M. *Suppose every loop in* F *has a multiple that is freely homotopic to a loop in some other component* G *of* ∂M. *Then* F *is incompressible.*

Proof. Suppose F is compressible. By the Loop Theorem, there is a disk D embedded in M with $D \cap F = D \cap \partial M = \partial D = \alpha$, where α is a loop noncontractible in F. There are two cases.

(1) The loop α does not separate F. Let β be a loop intersecting α once. Some multiple of β is homotopic to a loop in G, and hence it has intersection number 0 with the disk D, a contradiction.

(2) The loop α separates F into components F_1 and F_2. Let β be an orientable, simple loop intersecting α only at the points p and q, and such that $\beta \cap F_i$ is nonseparating in F_i $(i = 1, 2)$. A multiple of β is homotopic to a loop in G. By the Generalized Loop Theorem [5], we may conclude that there exists an annulus A embedded in M with $A \cap \partial M = \beta \cup \beta'$, where $\beta' \subset G$. Let p and q separate α and β into components α_1, α_2 and β_1, β_2, respectively. Then, by looking at the intersection of A and D, we see clearly that at least one of the four loops $\alpha_i \cup \beta_j$ $(i, j = 1, 2)$ bounds a disk in M. Since all of these loops are nonseparating (by the choice of β), we are now back in case (1).

We are now ready to prove our main theorem.

THEOREM 1. *Let* M *and* N *be compact, connected, orientable, irreducible* 3-*manifolds, and suppose that* N *is sufficiently large and* $\pi_1(M) \neq 1$. *Let* $f: (M, \partial M) \to (N, \partial N)$ *be a map such that* $f_*: \pi_1(M) \to \pi_1(N)$ *is monic. Then there exists a homotopy* $f_t: (M, \partial M) \to (N, \partial N)$ *with* $f_0 = f$ *such that one of the following three cases occurs:*

(i) f_1 *is a covering map;*

(ii) $M = F \times I$, *where* F *is an orientable* 2-*manifold, and* $f_1(M) \subset \partial N$;

(iii) M *and* N *are solid tori,* $f_1 \mid \partial M$ *is a covering map, and*

$$\ker(\pi_1(\partial N) \to \pi_1(N)) \not\subset \operatorname{im}(f_1 \mid \partial M)_*.$$

Proof. The proof proceeds in three steps from the "best" situation to the "worst".

Step I. Assume that $(f \mid F)_*$ is monic for each component F of ∂M, and either $f \mid F$ is homotopic to a homeomorphism or $f(F)$ is incompressible in N.

By Lemma 1, $f \mid F$ is homotopic to a covering map. Constructing such homotopies in a regular neighborhood of ∂M, we may assume $f \mid \partial M$ is a covering map. If we now follow Waldhausen's proof of his original theorem, we find the only obstruction to homotoping f to a covering map is the existence of an arc γ in M that joins boundary components F and G (possibly $F = G$) and such that $f(\gamma)$ is a contractible closed curve in N. Assume we encounter such an arc. If $F = G$, then $f \mid F$ is not a homeomorphism, since f takes the distinct endpoints of γ to the same point in $f(F)$. Therefore, by our assumptions in Step I, $f(F)$ is incompressible in N. Because $f \mid F$ is a finite-sheeted covering and f_* is monic, F is also incompressible. The rest of Waldhausen's proof now applies, and we arrive at a contradiction as he does. If $F \neq G$, then the existence of γ, together with the fact that $f \mid F$ and $f \mid G$ are finite-sheeted covering maps, implies every loop in F (or G) has a multiple freely homotopic to a loop in G (or F). By Lemma 2, F and G are incompressible, and hence $f(F)$ and $f(G)$ are also incompressible. Again the rest of Waldhausen's proof applies, and we conclude that case (ii) must occur, where F is a closed, orientable surface.

Step II. Assume $(f \mid F)_*$ is monic for each component F of ∂M.

As in Step I, since $(f \mid F)_*$ is monic, we may in fact assume $f \mid \partial M$ is already a covering map. Let $p \colon N' \to N$ be a covering of N such that $p_*(\pi_1(N')) = f_*(\pi_1(M))$. Then f lifts to a map $f' \colon (M, \partial M) \to (N', \partial N')$. We shall show that either case (ii) occurs, or N' is compact and f' satisfies the hypothesis of Step I, or N' is compact and case (iii) occurs.

Consider the commutative diagram

$$\begin{array}{ccc} H_3(M, \partial M) & \xrightarrow{\partial} & H_2(\partial M) \\ \Big\downarrow f'_* & & \Big\downarrow (f' \mid \partial M)_* \\ H_3(N', \partial N') & \xrightarrow{\partial} & H_2(\partial N') . \end{array}$$

It is important that here all homology has coefficients in Z, the integers. Suppose $f'_* \colon H_3(M, \partial M) \to H_3(N', \partial N')$ is trivial. Then $(f' \mid \partial M)_* \left(\sum [F]\right) = 0$ in $H_2(\partial N')$, where $[F]$ is the homology class of the component F of ∂M, and the summation is taken over all boundary components of M. Since $f' \mid \partial M$ is a covering map, $(f' \mid \partial M)_*([F]) \neq 0$. Hence f' must map two distinct components F and G of ∂M to the same component of N'. Let α be any arc joining a point p of F to a point q of G such that $f'(p) = f'(q)$. Then $f'(\alpha)$ is a loop in N'. Since $f'_* \colon \pi_1(M) \to \pi_1(N')$ is an isomorphism, there exists a loop β in M such that $f'(\beta)$ is homotopic to $f'(\alpha)^{-1}$. Let $\gamma = \beta \cdot \alpha$. Then $f'(\gamma)$ is a contractible loop in N', and hence $f(\gamma)$ is a contractible loop in N. As in Step I, the existence of γ tells us every loop in F (or G) has a multiple freely homotopic to a loop in G (or F). Hence Lemma 2 and Waldhausen's results may be applied, and we conclude as before that case (ii) must occur, where F is a closed surface.

Thus we may assume that $f'_* \colon H_3(M, \partial M) \to H_3(N', \partial N')$ is nontrivial. In particular, f' must be onto, and N' must be compact. By the Sphere Theorem [4] and the irreducibility of M and N, we see that $\pi_i(M) = \pi_i(N) = \pi_i(N') = 0$ for all $i > 1$.

Therefore f' is a homotopy equivalence, and $\chi(M) = \chi(N')$, where $\chi(M)$ is the Euler characteristic of M. Poincaré duality in the doubles of M and N' implies that $\chi(\partial M) = \chi(\partial N')$. Since $f' \mid \partial M$ is a covering map and f' is onto, $f' \mid F$ must be a homeomorphism for every component F of ∂M with $\chi(F) < 0$. (Since M is orientable and irreducible, all components of ∂M have negative Euler characteristic, except the torus components.) If f'(F) is incompressible for each torus component of ∂N, then by Step I, case (i) of our theorem must occur. Thus f' and hence f is homotopic to a covering map. If f'(F) is compressible, the irreducibility of M and N forces M, N', and N to be solid tori. If $\ker(\pi_1(\partial N) \to \pi_1(N)) \subset \mathrm{im}(f \mid \partial M)_*$, it is a simple matter to cut N apart along a disk D, homotope f to be transverse to D and $f \mid f^{-1}(D)$ to be a covering map, and to conclude that in fact case (i) of our theorem holds. Otherwise, we have case (iii).

Step III. Suppose $(f \mid F)_*$ is not monic for some component F of ∂M.

By Lemma 1, there exists an orientable, simple closed curve α in $(\ker f \mid F)_*$ that is not contractible in F. Since f_* is monic, α is contractible in M. By Dehn's Lemma [4], there exists a disk D embedded in M with $D \cap \partial M = \alpha$. Because $f(\alpha)$ is contractible in ∂N and $\pi_2(N)$ is trivial, f(D) contracts into ∂N. In fact, if R(D) is a regular neighborhood of D that meets ∂M in a regular neighborhood of α, we can homotope f to a map that takes R(D) into ∂N. Let M' be the closure of M - R(D). Then $(f \mid M')(\partial M') \subset \partial N$. Suppose $(f \mid F)_*$ is now monic for every component F of $\partial M'$. If M' is a 3-cell, then M is a handlebody, and thus it is homeomorphic to $F \times I$ for some 2-manifold F. Since we can clearly homotope $f \mid M'$ to map M' into ∂N, we must be in case (ii) of the theorem. (Note that all homotopies mentioned can even be chosen to remain fixed on ∂M.) Assume then that M' is not a 3-cell. We may apply Step II of our theorem to $f \mid M'$. If case (i) or (iii) occurs for a component M_1' of M' ($M' = M_1' \cup M_2'$ if D separates M and $M' = M_1'$ otherwise), then $(f \mid M_1')_*(\pi_1(M_1'))$ has finite index in $\pi_1(N)$. Since f_* is monic, $\pi_1(M_1')$ has finite index in $\pi_1(M)$. But $\pi_1(M)$ is a free product of $\pi_1(M_1')$ with either $\pi_1(M_2')$ or Z. In either case, $\pi_1(M_1')$ does not have finite index in $\pi_1(M)$. If case (ii) occurs for all components of M', then f is homotopic to a map f_1 such that $f_1(M)$ is contained in a component G of ∂M. (The homotopy takes $\partial M'$ into ∂N and is fixed on R(D).) But $(f \mid F)_*$ is monic for every component F of $\partial M'$, by assumption. Thus $(f_1 \mid F)$ is homotopic to a covering map, and $(f_1 \mid F)_*(\pi_1(F))$ has finite index in $\pi_1(G)$. Considering f_1 as a map of M into G, we see that $(f_1)_*(\pi_1(M_1'))$ also has finite index in $\pi_1(G)$. Because f_* is monic, $(f_1)_*$ is monic, and therefore $\pi_1(M_1')$ must have finite index in $\pi_1(M)$. As above, this is a contradiction.

We conclude that either M' is a 3-cell, and we are in case (ii) with M a handlebody, or $(f \mid F)_*$ is not monic for some component F of $\partial M'$. If the latter happens, we may repeat the procedure above. Since $\chi(\partial M)$ is finite, the process must terminate so that our last M' is a collection of 3-cells. Hence case (ii) must apply, where F is a surface with boundary.

We obtain as a corollary the following result, which B. Evans established in [2] using quite different methods.

COROLLARY 1. *Let* M *and* N *be as in Theorem* 1. *If* $f: (M, \partial M) \to (N, \partial N)$ *is a mapping such that* f_* *is an isomorphism, then* f *is homotopic to a homeomorphism* (*it may be that* ∂M *is not taken into* ∂N *during the homotopy*).

Proof. If case (i) of Theorem 1 holds, there is nothing more to prove. If case (iii) holds, it is easy to construct the desired homotopy. If case (ii) holds with F closed, Waldhausen's Lemma 5.1 of [6] implies that N is also a product. Then Lemma 1 provides a homotopy of $f \mid F$ that can be extended to the required homotopy of f. If case (ii) occurs with M a handlebody, then N must also be a handlebody, and a theorem of Zieschang [7] tells us that f is homotopic to a homeomorphism.

2. THE NONORIENTABLE CASE

We state the nonorientable version of Theorem 1 and indicate the proof. Following Heil, we say a 3-manifold is P^2-irreducible if it is irreducible and contains no two-sided projective planes.

THEOREM 2. *If* M *and* N *are* P^2*-irreducible, compact, sufficiently large* 3-*manifolds and* $f: (M, \partial M) \to (N, \partial N)$ *is such that* f_* *is monic, then there exists a homotopy* $f_t: (M, \partial M) \to (N, \partial N)$ *such that* $f_0 = f$ *and one of the three following cases occurs:*

(i) f_1 *is a covering map;*

(ii) M *is an* I-*bundle over a* 2-*manifold* F *and* $f_1(M) \subset \partial N$;

(iii) M *and* N *are disk-bundles over* S^1 *(solid tori or Klein bottles),* $f_1 \mid \partial M$ *is a covering map, and* $\ker(\pi_1(\partial N) \to \pi_1(N)) \not\subset \operatorname{im}(f_1(\partial M)_*)$.

Proof. Since Lemma 2 has no restriction on orientability, Step I of the proof of Theorem 1 presents no difficulty. We simply appeal to Heil's nonorientable version [3] of Waldhausen's result. Assuming Step II can be carried out, we have no problems at Step III. Indeed, since Lemma 1 guarantees orientable simple closed curves in $\ker(f \mid \partial M)_*$, the desired surgery can be applied. Of course, we may now obtain a "twisted" handle body, that is, an I-bundle over F that is not a product. The only trouble arises in Step II. We used coefficients in Z to show N' is compact, and the argument as stated will not work with Z_2-coefficients. Instead, we proceed as follows. Let f' and N' be as in Step II. Recall that $f'_*: \pi_1(M) \to \pi_1(N)$ is an isomorphism. Let 0(M) be the subgroup of $\pi_1(M)$ whose elements have orientable loops as representatives. Let P be the subgroup $f'_*(0(M)) \cap 0(N')$, which has index at most 4 in $\pi_1(N')$. Let $\tilde{N}$ be the covering of N' corresponding to P, and $\tilde{M}$ the covering of M corresponding to $(f'_*)^{-1}(P)$. The mapping $f': M \to N'$ lifts to a mapping $\tilde{f}: \tilde{M} \to \tilde{N}$ that satisfies the same conditions as f'. Since $\tilde{M}$ and $\tilde{N}$ are orientable, we may use the argument from Step II to show that $\tilde{f}$ is onto or $\tilde{M} = F \times I$, where F is closed. In the first case, f' must also be onto. As before, we may then conclude that N' is compact and that $f' \mid \partial M$ is a homeomorphism except possibly on components of Euler characteristic zero (Klein bottles are now a possibility). The rest of the proof should be clear. In the second case when $\tilde{M} = F \times I$ where F is closed, $\tilde{M}$ is boundary-irreducible. It follows that M is boundary-irreducible, and hence we may use Heil's proof.

3. THE PERIPHERAL STRUCTURE OF BOUNDARY-REDUCIBLE MANIFOLDS

If we want to obtain information about the homeomorphism-type of a 3-manifold M simply from algebraic information about $\pi_1(M)$, in a manner similar, say, to Corollary 6.5 of [6], we need to know exactly how the peripheral groups $\pi_1(F_k)$ are mapped into $\pi_1(M)$, where F_k is a component of ∂M. In the boundary-irreducible case, $\pi_1(F_k)$ can be considered in some sense to be a subgroup of $\pi_1(M)$; but in general, this is not possible. We must actually know what the map $i_k: \pi_1(F_k) \to \pi_1(M)$ is, not simply what its image is. We therefore define a *peripheral group system* $\mathcal{G}$ to be a collection of groups G and G_k, together with homomorphisms $i_k: G_k \to G$.

The distinguished group G is called the *main* group, and $\{G_k\}$ is the set of *peripheral* groups. A *morphism* $\Phi: \mathscr{G} \to \mathscr{H}$ of two such systems consists of a main homomorphism $\phi: G \to H$ and homomorphisms $\phi_k: G_k \to H_{\ell(k)}$ (one for each k), where $H_{\ell(k)}$ is a peripheral group of $\mathscr{H}$, and where the diagram

$$(*) \qquad \begin{array}{ccc} G_k & \xrightarrow{\phi_k} & H_{\ell(k)} \\ {\scriptstyle i_k}\downarrow & & \downarrow{\scriptstyle j_{\ell(k)}} \\ G & \xrightarrow{\phi} & H \end{array}$$

conjugate-commutes—that is, there exists an inner automorphism ψ_k of H such that $\phi i_k = \psi_k j_{\ell(k)} \phi_k$.

We obtain a peripheral group system for a manifold M by choosing base-points $p \in M$ and $p_k \in F_k$ (one for each component F_k of ∂M), together with paths α_k joining p to p_k. Then $G = \pi_1(M, p)$, $G_k = \pi_1(F, p_k)$, and i_k is the inclusion-induced mapping $\pi_1(F, p_k) \to \pi_1(M, p_k)$ followed by the base-point-changing automorphism of $\pi_1(M)$ induced by α_k. A mapping f: $(M, \partial M) \to (N, \partial N)$ induces a morphism of peripheral group systems of M and N. Between any two peripheral systems for M, there is a morphism Φ induced by the identity map of M such that ϕ and the ϕ_k are all inner automorphisms.

LEMMA 3. *Let* $\mathscr{G}$ *and* $\mathscr{H}$ *be peripheral group systems for connected, compact* 3-*manifolds* M *and* N, *respectively, with* $\pi_2(N) = 0$. *Then each morphism* $\Phi: \mathscr{G} \to \mathscr{H}$ *is induced by a map* f: $(M, \partial M) \to (N, \partial N)$.

Proof. Choose base-point systems p, p_k, α_k, and q, q_ℓ, β_ℓ to obtain the group systems of M and N, respectively. Let T_k be a maximal tree in the 1-skeleton of F_k, for each k, and extend $\bigcup T_k$ to a maximal tree T in the 1-skeleton of M. Let T' be the collection of all edges in T having no vertices in ∂M. For each k, let e_k be the edge joining T' to T_k, and let ε_k be any path in T from p to p_k. We construct f as follows. Let $f(T_k) = q_{\ell(k)}$ and $f(T') = q$. Let δ_k be any loop in N that represents $\phi([\alpha_k^{-1} \cdot \varepsilon_k])$. Let γ_k be any loop in N based at q such that the inner automorphism in the diagram (*) is conjugation by the class of γ_k. Let $f(e_k) = \beta_{\ell(k)} \cdot \gamma_k \cdot \delta_k$. Now define f over the other edges in ∂M by using the peripheral maps ϕ_k. Define f over the other edges of M by using ϕ. We can extend f to the 2-skeleton of M, because ϕ and ϕ_k are homomorphisms. We can extend f to all of M, because $\pi_2(N) = 0$. We leave it to the reader to verify that f is the desired map.

Using Lemma 3, we may summarize our results as follows.

THEOREM 3. *Let* M *and* N *be compact, connected,* P^2-*irreducible, sufficiently large* 3-*manifolds. Let* Φ *be a morphism of group systems for* M *and* N *such that the main map* ϕ *is a monomorphism. Then* Φ *is induced by a mapping* f: $(M, \partial M) \to (N, \partial N)$ *of type* (i), (ii), *or* (iii) *as listed in Theorem* 2. *In particular, if* M *is not an* I-*bundle over an annulus or closed* 2-*manifold, and if* ϕ_k *is also a monomorphism for each* k, *then* Φ *is induced by a covering map. Or, if* ϕ_k *is arbitrary, but* M *and* N *are orientable and* ϕ *is an isomorphism, then* ϕ *is induced by a homeomorphism.* (Note that in this case, Φ may not be induced by the homeomorphism when M is an I-bundle over a 2-manifold.)

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