PERIODIC SOLUTIONS OF ANALYTIC FUNCTIONAL DIFFERENTIAL EQUATIONS ARE ANALYTIC

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In 1955, E. M. Wright [11] studied the nonlinear differential-difference equation

$$x'(t) = -\alpha x(t-1)(1+x(t)) \qquad (\alpha > 0),$$

and he proved among many other results that each real solution $x(t)$ of this equation that is defined, continuously differentiable, and bounded on the real axis has a complex analytic extension on the strip $|\Im(t)| < (\alpha e^{\alpha})^{-1}$. Wright's method is based on the observations that $x(t)$ is necessarily infinitely differentiable and that repeated differentiation of the defining equation leads to estimates on $|x^{(n)}(t)|$. The estimates on $|x^{(n)}(t)|$ imply the analyticity of $x(t)$. The same technique works for some other equations, for example, the equation

$$x'(t) = -\left(\sum_{j=1}^{n} \alpha_j x(t-\tau_j)\right)(1+x(t)),$$

where α_j and τ_j denote positive constants; but for more general equations it is by no means obvious how to obtain the appropriate estimates on $|x^{(n)}(t)|$.

In 1962, G. S. Jones [7] proved that if $\alpha > \pi/2$, then the equation

$$x'(t) = -\alpha x(t-1)(1+x(t))$$

has a nontrivial periodic solution; by Wright's work, this solution is necessarily analytic on a strip. Jones also proved in [8] that if $\alpha > \pi/2$, the equation $x'(t) = -\alpha x(t-1)(1-x^2(t))$ has a nontrivial periodic solution that is analytic on a strip. These results and a number of other special cases led him to ask the following question in [8]: *If* η *is a real-valued function of bounded variation, and if* α, a, b, *and* h *are constants, is each real-valued periodic solution* $x(t)$ *of the equation*

$$x'(t) = \left(-\alpha \int_{-h}^{0} x(t+\theta)\,d\eta(\theta)\right)(1+ax(t)+bx^2(t))$$

analytic on a neighborhood of $\mathbb{R}$ *in the complex plane?*

The answer is yes. We shall prove a much more general theorem. The proof is surprisingly straightforward. The idea is simply to apply some abstract methods that reduce the question to one of analytic solutions of an ordinary differential equation with values in a complex Banach space. We have enough information about the ordinary differential equation to prove the result.

Received December 6, 1972.

The author was partially supported by National Science Foundation Grant GP-20228 and a Rutgers Research Council Faculty Fellowship.

Michigan Math. J. 20 (1973).

We begin the details of our presentation by recalling some definitions and establishing some nomenclature. Let Y be a complex Banach space, D an open subset of the complex plane $\mathbb{C}$, and $u: D \to Y$ a continuous mapping. Then u is called analytic if for every $\lambda_0 \in D$,

$$\lim_{\lambda \to \lambda_0} \frac{u(\lambda) - u(\lambda_0)}{\lambda - \lambda_0} = u'(\lambda_0)$$

exists. If U is an open subset of Y, if Z is a complex Banach space, and if $f: U \to Z$ is a continuous mapping, then f is called analytic if for all $x_0 \in U$ and $h \in Y$, the mapping $g(\lambda) = f(x_0 + \lambda h)$ is analytic on some neighborhood of the origin in $\mathbb{C}$. Expositions of these and related ideas are given in [1, Chapter 9] and [6, Chapter 3].

Now denote by X_0 the Banach space of bounded, continuous mappings $x: (-\infty, 0] \to \mathbb{R}^n$; as usual, $\|x\| = \sup_{-\infty < t \leq 0} |x(t)|$, where $|\cdot|$ denotes a fixed norm on $\mathbb{R}^n$. Define X to be the complex Banach space of continuous mappings $x: (-\infty, 0] \to \mathbb{C}^n$, also in the sup norm. Obviously, X_0 can be viewed as a closed subset of X. Let U denote the open subset

$$\{x \in X: \sup_{-\infty < t \leq 0} |\Im\, x(t)| < r\}$$

of X. If $y: \mathbb{R} \to \mathbb{R}^n$ is a bounded, continuous mapping, then for each $t \in \mathbb{R}$ define $y_t \in X_0$ by the formula $y_t(s) = y(t + s)$. This notation will be fixed from here on; in particular, the letters X, X_0, and U will always be used in the sense of this paragraph. Our goal is to prove the following theorem.

THEOREM 1. *Let* $f: U \to \mathbb{C}^n$ *be a continuous, analytic mapping such that* $f(X_0) \subset \mathbb{R}^n$, *and let* $y: \mathbb{R} \to \mathbb{R}^n$ *be a bounded, continuously differentiable mapping. Suppose in addition that*

(1) f *maps closed, bounded subsets of* U *into bounded subsets of* $\mathbb{C}^n$ *and*

(2) *if* $x \in U$ *and* $s \leq 0$, *and if* $s_n \to s$ *and* $s_n \leq 0$, *then* $f(x_{s_n}) \to f(x_s)$.

Then, if $y'(t) = f(y_t)$ *for all real* t, *the function* y *has a complex analytic extension that maps an open neighborhood of* $\mathbb{R}$ *in* $\mathbb{C}$ *into* $\mathbb{C}^n$.

Since periodic solutions are bounded, Theorem 1 immediately implies analyticity for periodic solutions. By a simple argument, the periodic solutions are actually analytic on a strip containing the real line.

The following five lemmas are elementary in the sense that in the proofs we only need the Cauchy integral formula for analytic mappings of $\mathbb{C}$ into a complex Banach space (see [1, Theorem 9.9.1] or [6, Theorem 3.11.3]).

LEMMA 1. *Let* M *be a closed, bounded subset of a complex Banach space* Y, *let* $N_{2\delta}(M) = \{x \in Y: d(x, M) < 2\delta\}$, *and let* $f: N_{2\delta}(M) \to \mathbb{C}^n$ *be a complex analytic mapping. Assume* $|f(x)| \leq A$ *for* $x \in N_{2\delta}(M)$. *Then* $|f(x) - f(y)| \leq (6A/\delta)\|x - y\|$ *whenever* $x, y \in M$ *and* $\|x - y\| < \delta$.

Proof. For $\lambda \in \mathbb{C}$ and $|\lambda| \leq 3\delta/2$, define $g(\lambda) = f(x + \lambda(y - x)/\|y - x\|)$. By definition, $g(\lambda)$ is analytic; moreover, if $\Gamma = \{\lambda \in \mathbb{C}: |\lambda| = 3\delta/2\}$, then the Cauchy integral formula implies that

$$g'(\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{g(\xi)}{(\xi - \lambda)^2}\,d\xi$$

for $|\lambda| \leq \|x - y\|$. Using this formula and the fact that $\|x - y\| < \delta$, we can easily see that $|g'(\lambda)| \leq 6A/\delta$. It follows that

$$|g(\|x - y\|) - g(0)| = |f(y) - f(x)| = \left|\int_0^{\|x-y\|} g'(\lambda)\,d\lambda\right| \leq (6A/\delta)\|x - y\|. \quad ■$$

LEMMA 2. *Let* $f\colon U \to \mathbb{C}^n$ *be a continuous, analytic mapping such that* $f(X_0) \subset \mathbb{R}^n$. *Assume in addition that conditions* 1 *and* 2 *of Theorem* 1 *on* f *hold. Define a mapping* $F\colon U \to X$ *by* $(Fx)(s) = f(x_s)$ *for* $x \in U$ *and* $-\infty < s \leq 0$. *Then* F *is a locally Lipschitzian mapping; that is, for each* $x \in U$ *there exists a neighborhood* N_x *such that* $F \mid N_x$ *is Lipschitzian.*

Proof. For each $x \in U$, $\text{cl}\{x_s\colon -\infty < s \leq 0\}$ is a bounded subset of U; therefore condition 1 on f implies that $\{f(x_s)\colon -\infty < s \leq 0\}$ is bounded. Condition 2 implies that the mapping $s \to f(x_s)$ is continuous, so that Fx is really an element of X.

If u is a fixed element of U and $\sup_{-\infty < t \leq 0} |\Im u(t)| = r_0 < r$, define $4\delta = r - r_0$ and define

$$M = \text{cl}\{x_s\colon \|x - u\| < \delta \text{ and } -\infty < s \leq 0\}.$$

Clearly, M is a closed, bounded subset of U, and $\{x \in X\colon d(x, M) < 2\delta\} = N_{2\delta}(M)$ is a bounded subset of U whose closure lies in U. It follows by condition 1 on f that there exists a constant A such that $|f(v)| < A$ for all $v \in N_{2\delta}(M)$.

Now suppose that x and y are elements of U and that $\|x - u\| < \delta/2$ and $\|y - u\| < \delta/2$. For $-\infty < s \leq 0$, it follows that $x_s \in M$, $y_s \in M$, and $\|x_s - y_s\| < \delta$. By Lemma 1, we have the estimate

$$|f(x_s) - f(y_s)| \leq (6A/\delta)\|x_s - y_s\| \leq 6A/\delta\|x - y\|.$$

This implies that $\|Fx - Fy\| \leq (6A/\delta)\|x - y\|$. ■

LEMMA 3. *Assume that* $f\colon U \to \mathbb{C}^n$ *is a continuous analytic mapping such that* $f(X_0) \subset \mathbb{R}^n$. *Suppose in addition that* f *satisfies conditions* 1 *and* 2 *of Theorem* 1. *Then, if* $F\colon U \to X$ *is defined by* $(Fx)(s) = f(x_s)$ *for* $x \in U$ *and* $-\infty < s \leq 0$, *it follows that* F *is analytic.*

Proof. Because we have already shown that F is locally Lipschitzian, it suffices to show that for each $z \in U$ and each $h \in X$, $\lim_{\lambda \to 0} [F(\lambda h + z) - F(z)]/\lambda$ exists (λ denotes a complex number).

Define $w(s) = \lim_{\lambda \to 0} [f(\lambda h_s + z_s) - f(z_s)]/\lambda$; the limit exists for all nonpositive s, by the assumption that f is analytic. By condition 1 on f, there exist a constant $\delta > 0$ and a constant A such that $|f(\lambda h_s + z_s)| < A$ for $|\lambda| \leq \delta$ and $-\infty < s \leq 0$. If $\Gamma = \{\xi \in \mathbb{C}\colon |\xi| = \delta\}$, the Cauchy integral formula implies that

$$w(s) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi h_s + z_s)}{\xi^2}\,d\xi.$$

Using this formula, we immediately see that w(s) is bounded, and using Lebesgue's dominated-convergence theorem and condition 2 on f, we find that w is continuous.

Our next claim is that $\lim_{\lambda \to 0} [F(\lambda h + z) - F(z)]/\lambda = w$, and to prove this it suffices to show that

$$\lim_{\lambda \to 0} \left(\sup_{-\infty < s \leq 0} \left(\frac{f(\lambda h_s + z_s) - f(z_s)}{\lambda} - w(s) \right) \right) = 0.$$

If $|\lambda| < \delta$, the Cauchy integral formula implies

$$\frac{f(\lambda h_s + z_s) - f(z_s)}{\lambda} = \frac{1}{2\pi i} \left[\int_\Gamma \frac{f(\xi h_s + z_s)}{\lambda(\xi - \lambda)} d\xi - \int_\Gamma \frac{f(\xi h_s + z_s)}{\lambda \xi} d\xi \right].$$

Using this formula and the Cauchy integral formula for w(s) and simplifying, we obtain the relation

$$\frac{f(\lambda h_s + z_s) - f(z_s)}{\lambda} - w(s) = \frac{\lambda}{2\pi i} \int_\Gamma \frac{f(\xi h_s + z_s)}{(\xi - \lambda)\xi^2} d\xi .$$

It follows that for $|\lambda| < \delta/2$, $\left| \frac{f(\lambda h_s + z_s) - f(z_s)}{\lambda} - w(s) \right| \leq K |\lambda|$, where K is a constant independent of s. ■

LEMMA 4. *Assume that* $f: U \to \mathbb{C}^n$ *is a continuous, analytic mapping such that* $f(X_0) \subset \mathbb{R}^n$ *and such that* f *satisfies conditions* 1 *and* 2 *of Theorem* 1. *Suppose that* $y: \mathbb{R} \to \mathbb{R}^n$ *is a bounded, continuously differentiable mapping such that* $y'(t) = f(y_t)$ *for all* t. *Then if* $Z: \mathbb{R} \to X_0$ *is defined by* $Z(t) = y_t$, *the function* Z *is continuously differentiable and* $Z'(t) = (y')_t$; *that is,* $(Z'(t))(s) = y'(t+s)$ *for* $-\infty < s \leq 0$.

Proof. Since $y'(t+s) = f(y_{t+s})$ for $-\infty < s \leq 0$ and $\mathrm{cl}\{y_{t+s}: -\infty < s \leq 0\}$ is a closed, bounded subset of U, the derivative $y'(t+s)$ is bounded and the mapping $s \to y'(t+s)$ gives an element of X_0. To prove the lemma, it therefore suffices to show that

$$\lim_{\delta \to 0} \left[\sup_{-\infty < s \leq 0} \frac{y(t+\delta+s) - y(t+s)}{\delta} - y'(t+s) \right] = 0.$$

Making the obvious substitution, we see that

$$\frac{y(t+\delta+s) - y(t+s)}{\delta} - y'(t+s) = \frac{1}{\delta} \int_0^\delta (f(y_{t+s+\rho}) - f(y_{t+s}))\, d\rho .$$

Since $\sup_{-\infty < \sigma < \infty} |y'(\sigma)| = B < \infty$, $\|y_{t+s+\delta} - y_{t+s+\rho}\| \leq B|\delta|$ for $0 \leq \rho \leq \delta$, and it follows by Lemma 2 that there exists a constant K such that

$$|f(y_{t+s+\rho}) - f(y_{t+s})| \leq K \|y_{t+s+\rho} - y_{t+s}\| .$$

whenever δ is small enough. Applying once again the fact that

$$\sup_{-\infty<\sigma<\infty} |y'(\sigma)| = B < \infty,$$

we see that $\|y_{t+s+\rho} - y_{t+s}\| \leq B\rho$. Using these estimates, we find that

$$\left|\frac{y(t+\delta+s) - y(t+s)}{\delta} - y'(t+s)\right| \leq \frac{KB}{\delta}\int_0^\delta \rho\, d\rho = \frac{KB\delta}{2}.$$

Since K and B are independent of s, for $-\infty < s \leq 0$, the last inequality implies the result. ■

If assumptions and notation are as in Lemma 4, then for $-\infty < t < \infty$ and $-\infty < s \leq 0$ we have the relations

$$y'(t+s) = (Z'(t))(s) = f(y_{t+s}) = (F(y_t))(s) = (F(Z(t)))(s),$$

or equivalently, $Z'(t) = F(Z(t))$.

Our next lemma comprises the heart of our proof. Essentially, the lemma is Theorem 10.4.5 of [1]. Some work is to be done, since one must show that our definition of analyticity, which is apparently weaker than that given by Dieudonné, is actually equivalent to his. However, this (and the continuity of the Fréchet derivative) follows by arguments with the Cauchy integral formula such as we have already used, and we omit it.

LEMMA 5 (Theorem 10.4.5 of [1]). *Let* Y *be a complex Banach space,* V *an open subset of* Y, *and* $F: V \to Y$ *a continuous, analytic mapping. Then, if* $W_0 \in V$ *and* $t_0 \in \mathbb{C}$, *there exist a* $\delta > 0$ *and a unique analytic mapping*

$$W: \{t \in \mathbb{C}: |t - t_0| < \delta\} \to V$$

such that $W(t_0) = W_0$ *and* $W'(t) = F(W(t))$ *for* $|t - t_0| < \delta$.

Proof of Theorem 1. Define the function $Z: \mathbb{R} \to X_0$ as in Lemma 4 and the mapping $F: U \to X$ as in Lemma 3. We have already shown that Z is continuously differentiable, that F is analytic, and that $Z'(t) = F(Z(t))$ for all real t. Given $t_0 \in \mathbb{R}$, define $Z_0 = Z(t_0)$. Since F is analytic, Lemma 5 implies that there exist $\delta > 0$ and a unique analytic mapping $W: \{t \in \mathbb{C}: |t - t_0| < \delta\} \to U$ such that $W(t_0) = Z_0$ and $W'(t) = F(W(t))$ for $|t - t_0| < \delta$. By the uniqueness of local solutions of $Y'(t) = F(Y(t))$ and $Y(t_0) = Z_0$ when t is real, we see that $W(t) = Z(t)$ for $t \in \mathbb{R}$ and $|t - t_0| < \delta$. Using standard arguments and the uniqueness part of Lemma 5, we see that if W_0 and W_1 are local analytic solutions as above, defined on $\{t: |t - t_0| < \delta_0\}$ and $\{t: |t - t_1| < \delta_1\}$, respectively, they must agree on the intersection of their domains. It follows that Z(t) has an analytic extension W(t) defined on a neighborhood of the real line in $\mathbb{C}$. Consequently, $y(t) = (Z(t))(0)$ has the analytic extension $(W(t))(0)$. ■

Remark 1. With somewhat more care concerning the size of δ in Lemma 5, one can actually prove that the solution y in Theorem 1 has an analytic extension defined on a strip $\{t \in \mathbb{C}: |\Im(t)| < c\}$, where c is an appropriate positive constant.

Remark 2. For equations less general than ours, we can reduce the problem to that of an ordinary differential equation in the complex Banach space $\ell^\infty(\mathbb{C}^n)$ (the space of bounded sequences $(x_0, x_1, \cdots, x_j, \cdots)$ with $x_j \in \mathbb{C}^n$ and with the sup norm). For instance, suppose $f: \mathbb{C}^{mn} \to \mathbb{C}^n$ is complex analytic and $f(\mathbb{R}^{mn}) \subset \mathbb{R}^n$. Assume that $y: \mathbb{R} \to \mathbb{R}^n$ is a bounded C^1-function such that

$$y'(t) = f(y(t), y(t - a_1), \cdots, y(t - a_{m-1}))$$

for all t. Assume that the a_j are commensurable constants, in other words, that $a_j = k_j d$, where each k_j is a positive integer and d is a positive number. Define a mapping $F: \ell^\infty(\mathbb{C}^n) \to \ell^\infty(\mathbb{C}^n)$ by the formula

$$F(y_0, y_1, \cdots, y_r, \cdots) = (u_0, u_1, \cdots, u_r, \cdots),$$

where $u_r = f(y_r, y_{k_1+r}, \cdots, y_{k_{m-1}+r})$. Then, if we take $y_j(t) = y(t - jd)$ and define $Y: \mathbb{R} \to \ell^\infty(\mathbb{C}^n)$ by the equation $Y(t) = (y_0(t), y_1(t), \cdots, y_r(t), \cdots)$, it is not hard to show that $Y'(t) = F(Y(t))$.

Remark 3. If under the assumptions of Remark 2, the function y is periodic of period $p > \max\{a_j: 1 \leq j \leq m - 1\}$ and p is commensurable with the a_j, so that $p = Nd$ for some integer N, then the infinite system of differential equations above reduces to a finite system involving $y_0, y_1, \cdots, y_{N-1}$. This remark has actually proved useful in the study of periodic solutions of the differential equation $y'(t) = f(y(t - 1))$, where f is an odd function (see [9, Section 6]). Unfortunately, even for simple equations such as $x'(t) = -\alpha x(t - 1)(1 + x(t))$ $(\alpha > \pi/2)$, it seems likely that the period of a periodic solution may be irrational.

COROLLARY 1. *Let* α, a, b, *and* h *be constants* $(h > 0)$, *and let* $\eta: [-h, 0] \to \mathbb{R}$ *be a real-valued function of bounded variation. Then, if* $x: \mathbb{R} \to \mathbb{R}$ *is a bounded* C^1*-function such that*

$$x'(t) = \left[-\alpha \int_{-h}^{0} x(t + \theta)\, d\eta(\theta) \right] [1 + ax(t) + b(x(t))^2],$$

x *has an extension to a complex analytic map defined on an open neighborhood of* $\mathbb{R}$.

Proof. Let X denote the Banach space of bounded, continuous, complex-valued mappings $x: (-\infty, 0] \to \mathbb{C}$, and define $f: X \to \mathbb{C}$ by

$$f(y) = \left[-\alpha \int_{-h}^{0} y(\theta)\, d\eta(\theta) \right] [1 + ay(0) + b(y(0))^2] .$$

In this notation, $x'(t) = f(x_t)$ for all real t. Thus it suffices to show that f satisfies the hypotheses of Theorem 1. It is easy to see that f is continuous and satisfies conditions 1 and 2 of Theorem 1. To prove analyticity, let y and h be any fixed functions in X, and consider the mapping $\lambda \to f(y + \lambda h)$ $(\lambda \in \mathbb{C})$. Clearly, this mapping is determined by a cubic polynomial in λ; hence it is certainly analytic. ■

The proof of the next corollary is even more straightforward, and we omit it.

COROLLARY 2. *Let* a_j $(1 \leq j \leq m)$ *be positive constants, and let* $g: \mathbb{C}^{(m+1)n} \to \mathbb{C}^n$ *be an analytic mapping such that* $g(\mathbb{R}^{(m+1)n}) \subset \mathbb{R}^n$. *If* $y: \mathbb{R} \to \mathbb{R}^n$ *is a bounded, continuously differentiable mapping such that*

$$y'(t) = g(y(t), y(t - a_1), \cdots, y(t - a_m))$$

for all t, *then* y *has a complex analytic extension defined on a neighborhood of* $\mathbb{R}$.

COROLLARY 3. *Suppose that* f *and* g *are complex analytic mappings of the strip* $\{t \in \mathbb{C}: |\Im(t)| < c\}$ *into* $\mathbb{C}$ (c *denotes a positive constant*) *such that* $g(\mathbb{R}) \subset \mathbb{R}$ *and* $f(\mathbb{R}) \subset \mathbb{R}$. *Then each bounded, continuously differentiable solution of the generalized Liénard equations*

$$x'(t) = y(t) - f(x(t)), \qquad y'(t) = -g(x(t - r)) \quad (r > 0)$$

that is defined for all t *has a complex analytic extension defined on a neighborhood of* $\mathbb{R}$.

R. B. Grafton [2], [3] has shown that under certain hypotheses on f and g, these equations have nonconstant periodic solutions, and this author [10] has also proved the existence of periodic solutions under different hypotheses on f and g. If f and g are analytic, Corollary 3 implies that these periodic solutions are analytic on a strip containing $\mathbb{R}$. Of course, in the case where no time lag is involved $(r = 0)$, these equations have been extensively studied, for example, in [5, Chapter 7, Section 10].

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