ON HOMOTOPY SEVEN-SPHERES THAT ADMIT DIFFERENTIABLE PSEUDO-FREE CIRCLE ACTIONS

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1. INTRODUCTION

This paper treats differentiable pseudo-free circle actions on homotopy 7-spheres. In an earlier paper, we showed how to construct all of these actions, but left open the question which homotopy 7-spheres can occur [3]. This question is now answered by the following theorem.

THEOREM A. *Each of the* 28 *homotopy* 7-*spheres admits a differentiable pseudo-free circle action with exactly one exceptional orbit.*

We note that only 10 of 28 homotopy 7-spheres admit differentiable free circle actions [2].

2. CONSTRUCTION OF SOME ASSOCIATED MANIFOLDS

A differentiable action of the circle group G on a homotopy 7-sphere Σ^7 is said to be *pseudo-free* if it is an effective action for which every isotropy group is finite and the set of exceptional orbits (that is, the set of orbits where the isotropy group is not trivial) is finite but not void. Suppose that such an action is given, and let

$$Gb_1, \cdots, Gb_k$$

be the exceptional orbits in Σ^7. For $i = 1, \cdots, k$, the isotropy group G_{b_i} at b_i is a finite cyclic group $\mathbb{Z}_{q_i}$ of order q_i, where q_i is an integer greater than 1, and since Σ^7 has the integral homology of a 7-sphere, we see that the integers $q_1, \cdots, q_k$ are mutually relatively prime. We let

$$q = q_1 \cdots q_k.$$

In the following, $\mathbb{C}^n$ denotes the unitary n-space, D^{2n} denotes the closed unit (2n)-disk in $\mathbb{C}^n$, and S^{2n-1} denotes the boundary of D^{2n}, that is, the unit (2n - 1)-sphere in $\mathbb{C}^n$. Then

$$G = S^1,$$

and the orthogonal action of G on S^7 given by

$$g(z_1, z_2, z_3, z_4) = (g^q z_1, gz_2, gz_3, gz_4)$$

is pseudo-free and has exactly one exceptional orbit Gb, where $b = (1, 0, 0, 0)$ and $G_b = \mathbb{Z}_q$.

Received March 9, 1973.
C. T. Yang was supported in part by the National Science Foundation.

Michigan Math. J. 20 (1973).

Whenever A is a subset of a space on which G acts, we let A^* denote the image of A in the orbit space, that means,

$$A^* = GA/G.$$

The following has been proved in [3] and [4].

THEOREM B. *There exists an equivariant differentiable map*

$$f: \Sigma^7 \to S^7$$

of degree ± 1, *and this map induces a homotopy equivalence*

$$f: \Sigma^* \to S^*,$$

where Σ^* *and* S^* *denote the orbit spaces* Σ^7/G *and* S^7/G, *respectively. Moreover, we can require* f *to have the following additional property: There is a slice* D *at* b, *and for* $i = 1, \cdots, k$ *there is a slice* D_i *at* b_i, *such that*

(i) D *is a closed* 6-*disk of center* b, *and* $\mathbb{Z}_q$ *acts orthogonally on* D,

(ii) D_i *is a closed* 6-*disk of center* b_i, *and* $\mathbb{Z}_{q_i}$ *acts orthogonally on* D_i $(i = 1, \cdots, k)$,

(iii) $GD_1, \cdots, GD_k$ *are mutually disjoint, and* $f^{-1}(GD) = \bigcup_{i=1}^k GD_i$,

(iv) f *maps each radius of* D_i *homeomorphically onto a radius of* D $(i = 1, \cdots, k)$.

We now use this theorem to construct some related spaces and manifolds.

Let M^8 be the mapping cylinder of the projection of Σ^7 onto Σ^*. Then M^8 is a compact differentiable 8-manifold of boundary Σ^7 with singularities $b_1^*, \cdots, b_k^*$. Similarly, the mapping cylinder N^8 of the projection of S^7 onto S^* is a compact differentiable 8-manifold of boundary S^7 with a singularity b^*. Next, let G act on S^9 as follows:

$$g(z_1, z_2, z_3, z_4, z_5) = (g^q z_1, gz_2, gz_3, gz_4, gz_5).$$

Then we may identify the orbit space S^9/G with the space $N^8 \cup D^8$ obtained by pasting together N^8 and D^8 along their common boundary. Therefore

$$N^8 = S^9/G - \text{int } D^8.$$

Let $\mathbb{Z}_q$ act on complex projective 4-space $\mathbb{C}P^4$ as follows:

$$h(z_1 : z_2 : z_3 : z_4 : z_5) = (h^{-1} z_1 : z_2 : z_3 : z_4 : z_5).$$

Clearly, the action is differentiable and semifree, and its set of fixed points consists of a complex projective 3-space

$$\mathbb{C}P^3 : z_1 = 0$$

and an isolated point

$$b^* : z_2 = z_3 = z_4 = z_5 = 0.$$

Now the orbit space $\mathbb{C}P^4/\mathbb{Z}_q$ can be identified with S^9/G as follows. Whenever $(z_1: z_2: z_3: z_4: z_5) \in \mathbb{C}P^4$, we may assume that either $z_1 = 0$ or $z_1/|z_1| \in \mathbb{Z}_q$. Then we have an identification given by

$$\mathbb{Z}_q(z_1: z_2: z_3: z_4: z_5) = G(z_1, z_2, z_3, z_4, z_5).$$

With this identification,

$$N^8 = \mathbb{C}P^4/\mathbb{Z}_q - \text{int } D^8.$$

Since D^8 may be any closed 8-disk differentiably imbedded into $\mathbb{C}P^4/\mathbb{Z}_q$, we lose no generality by assuming that

$$D^8 \cap (\mathbb{C}P^3 \cup \{b^*\}) = \emptyset.$$

Let

$$\pi: \mathbb{C}P^4 \to \mathbb{C}P^4/\mathbb{Z}_q$$

be the projection. Then

$$\tilde{N}^8 = \pi^{-1}(N^8)$$

is a compact connected differentiable 8-manifold with

$$\partial\tilde{N}^8 = qS^7 \quad (= \text{disjoint union of } q \text{ copies of } S^7).$$

Moreover, there is a differentiable semifree action of $\mathbb{Z}_q$ on $\tilde{N}^8$ such that

$$\tilde{N}^8/\mathbb{Z}_q = N^8$$

and its set of fixed points is $\mathbb{C}P^3 \cup \{b^*\}$. Similarly,

$$\tilde{N}^6 = \pi^{-1}(S^*)$$

is a closed differentiable 6-manifold in $\tilde{N}^8 - \partial\tilde{N}^8$ diffeomorphic to $\mathbb{C}P^3$, and there is a differentiable semifree action of $\mathbb{Z}_q$ on $\tilde{N}^6$ such that

$$\tilde{N}^6/\mathbb{Z}_q = S^*$$

and its set of fixed points consists of b^* and

$$\mathbb{C}P^2 = S^* \cap \mathbb{C}P^3.$$

The equivariant differentiable mapping $f: \Sigma^7 \to S^7$ can be naturally extended to a mapping

$$f: M^8 \to N^8,$$

which is differentiable except at $\{b_1^*, \cdots, b_k^*\}$. Altering $f: (M^8, \Sigma^7) \to (N^8, S^7)$ by a homotopy, if necessary, we may assume that f is transverse regular at $\mathbb{C}P^2$ and $\mathbb{C}P^3$ and that

$$L^4 = f^{-1}(\mathbb{C}P^2) \quad \text{and} \quad L^6 = f^{-1}(\mathbb{C}P^3)$$

are both connected.

Imitating the construction of an induced fibre bundle, we shall construct a compact connected differentiable 8-manifold $\tilde{M}^8$ and a differentiable action of $\mathbb{Z}_q$ on $\tilde{M}^8$ such that

$$\tilde{M}^8/\mathbb{Z}_q = M^8$$

and the mapping $f\colon M^8 \to N^8$ is covered by a $\mathbb{Z}_q$-equivariant differentiable mapping $\tilde{f}\colon \tilde{M}^8 \to \tilde{N}^8$. Thus we have the commutative diagram

$$\begin{array}{ccc} \tilde{M}^8 & \xrightarrow{\tilde{f}} & \tilde{N}^8 \\ \downarrow \pi & & \downarrow \pi \\ M^8 & \xrightarrow{f} & N^8 . \end{array}$$

We may identify N^8 with the orbit space of the action of G on $S^7 \times D^2$ such that for each $g \in G$ and each $(x, y) \in S^7 \times D^2$,

$$g(x, y) = (gx, gy).$$

In fact, the mapping $\phi\colon S^7 \times [0, 1] \to (S^7 \times D^2)/G$ given by

$$\phi(x, t) = G(x, 1 - t)$$

induces an identification of N^8 with $(S^7 \times D^2)/G$. If we identify S^7 with $S^7 \times \{0\}$ in $S^7 \times D^2$ by setting $x = (x, 0)$ for all $x \in S^7$, it is clear that the action of G on $S^7 \times D^2$ is differentiable and pseudo-free and has Gb as the only exceptional orbit. Moreover, if D is a slice at b for the action of G on S^7 as described in Theorem B and if D is regarded as the closed 6-disk in $\mathbb{C}^3$ of center 0 and radius 1/2, then for the action of G on $S^7 \times D^2$,

$$E = \{(x, y) \in D \times D^2 \,\big|\, |x|^2 + |y|^2 \leq 1/4\}$$

is a slice at b that constitutes a closed 8-disk of center b contained in $\operatorname{int}(S^7 \times D^2)$ and on which $\mathbb{Z}_q$ acts orthogonally. Notice that $E \cap S^7 = D$. Similarly, we may regard M^8 as the orbit space of a differentiable pseudo-free action of G on $\Sigma^7 \times D^2$ with exceptional orbits $Gb_1, \cdots, Gb_k$. For $i = 1, \cdots, k$, let E_i be a slice at b_i for the action of G on $\Sigma^7 \times D^2$, constructed from D_i as E is constructed from D, where D_i is a slice at b_i for the action of G on Σ^7 as described in Theorem B. By Theorem B, we may assume that $E_1^*, \cdots, E_k^*$ are mutually disjoint subsets of $M^8 - \partial M^8$, that

$$f^{-1}(E^*) = \bigcup_{i=1}^{k} E_i^*,$$

and that for $i = 1, \cdots, k$, the mapping $f\colon E_i^* \to E^*$ is covered by a $\mathbb{Z}_{q_i}$-equivariant differentiable mapping

$$\tilde{f}_i\colon E_i \to E$$

that maps each radius of E_i homeomorphically onto a radius of E.

Let

$$X = \left\{ (x, y) \in \left(M^8 - \bigcup_{i=1}^{k} \operatorname{int} E_i^* \right) \times \tilde{N}^8 \;\middle|\; f(x) = \pi(y) \right\},$$

and let $\mathbb{Z}_q$ act on X so that

$$h(x, y) = (x, hy).$$

Then X is a compact differentiable 8-manifold, and the action of $\mathbb{Z}_q$ on X is differentiable and semifree. If we identify $X/\mathbb{Z}_q$ with $M^8 - \bigcup_{i=1}^{k} \operatorname{int} E_i^*$ by setting

$$\mathbb{Z}_q(x, y) = \{x\} \times \mathbb{Z}_q y = x,$$

then the set of fixed points of $\mathbb{Z}_q$ in X is L^6. Let $\pi: X \to M^8$ be the projection, and let $\tilde{f}: X \to \tilde{N}^8$ be the $\mathbb{Z}_q$-equivariant differentiable mapping given by $\tilde{f}(x, y) = y$ Then $\pi\tilde{f} = f\pi$.

Clearly, ∂X is the disjoint union of

$$\pi^{-1}(\partial M^8) = q\Sigma^7 \quad (= \text{disjoint union of } q \text{ copies of } \Sigma^7),$$

and

$$\pi^{-1}(\partial E_i^*) \qquad (i = 1, \cdots, k).$$

For $i = 1, \cdots, k$, the set $\pi^{-1}(\partial E_i^*)$ is the disjoint union of q/q_i copies of ∂E_i. Let $\tilde{M}^8$ be obtained from X by attaching one copy of E_i to each component of $\pi^{-1}(\partial E_i^*)$ $(i = 1, \cdots, k)$, that is, let

$$\tilde{M}^8 = X \cup \bigcup_{i=1}^{k} (q/q_i) E_i .$$

Then $\tilde{M}^8$ is a compact, connected, differentiable 8-manifold with

$$\partial\tilde{M}^8 = q\Sigma^7 .$$

Since $\mathbb{Z}_{q_i}$ leaves each component of $\pi^{-1}(\partial E_i^*)$ invariant and $\mathbb{Z}_{q/q_i}$ permutes the components cyclically, we have a natural differentiable action of $\mathbb{Z}_q$ on $\tilde{M}^8$ that constitutes an extension of the action of $\mathbb{Z}_q$ on X; moreover, for $i = 1, \cdots, k$, $\mathbb{Z}_{q_i}$ acts orthogonally on each copy of E_i. Notice that for each i $(i = 1, \cdots, k)$, the action of $\mathbb{Z}_{q_i}$ on $\tilde{M}^8$ is semifree, and that its set of fixed points consists of L^6 and q/q_i isolated points, namely the centers of the slices E_i in $(q/q_i)E_i$.

Let $\tilde{f}: \tilde{M}^8 \to \tilde{N}^8$ be the $\mathbb{Z}_q$-equivariant differentiable mapping such that $\tilde{f} \mid X$ is the one we had earlier and $\tilde{f} \mid E_i = \tilde{f}_i$ $(i = 1, \cdots, k)$, and let $\pi: \tilde{M}^8 \to M^8$ be the natural extension of $\pi: X \to M^8$. It is clear that $\pi\tilde{f} = f\pi$ remains valid.

3. USING THE ASSOCIATED MANIFOLDS

By means of the manifolds introduced in the preceding section, we can compute the modified Eells-Kuiper invariant [2] of $\partial\tilde{M}^8 = q\Sigma^7$ when q is odd. Later, we shall use this to verify Theorem A.

It can be seen that $H^r(S^9/G)$ is infinite cyclic for $r = 0, 2, 4, 6, 8$ and that it is trivial otherwise. Let β_1 denote the generator of $H^r(S^9/G)$ such that, if B is an oriented closed 2-cell in S^9 and ∂B is the exceptional orbit Gb, then the value of β_1 at the fundamental class $[B^*]$ of B^* is 1; that is, let

$$\beta_1[B^*] = 1.$$

Then, for $j = 2, 3, 4$, there is a generator β_j of $H^{2j}(S^9/G)$ such that

$$\beta_1\beta_{j-1} = q\beta_j.$$

Since for $j = 1, 2, 3$ the inclusion mapping of N^8 into S^9/G induces an isomorphism of $H^{2j}(S^9/G)$ onto $H^{2j}(N^8)$, we can regard β_j for $j = 1, 2, 3$ as a generator of $H^{2j}(N^8)$ also. We shall let $\mathbb{C}P^3$ and $\mathbb{C}P^2$ $(= S^* \cap \mathbb{C}P^3)$ be oriented so that

$$\beta_2[\mathbb{C}P^2] = q, \qquad \beta_3[\mathbb{C}P^3] = q,$$

where $[\mathbb{C}P^2]$ and $[\mathbb{C}P^3]$ denote the fundamental classes on $\mathbb{C}P^2$ and $\mathbb{C}P^3$, respectively. Since $f\colon M^8 \to N^8$ is a homotopy equivalence, $\alpha_j = f^*(\beta_j)$ is a generator of $H^{2j}(M^8)$ $(j = 1, 2, 3)$, and

$$\alpha_1\alpha_{j-1} = q\alpha_j \qquad (j = 2, 3).$$

Moreover, we let L^4 and L^6 be so oriented that

$$\alpha_2[L^4] = q, \qquad \alpha_3[L^6] = q$$

and hence the mappings

$$f\colon L^4 \to \mathbb{C}P^2, \qquad f\colon L^6 \to \mathbb{C}P^3$$

are of degree 1.

Let $\tilde{\beta}$ be the generator of $H^2(\tilde{N}^8)$ such that

$$\pi^*(\beta_1) = q\tilde{\beta}.$$

Then, for $j = 2, 3$, we see that $\tilde{\beta}^j$ is a generator of $H^{2j}(\tilde{N}^8)$ and

$$\pi^*(\beta_j) = q\tilde{\beta}^j.$$

Hence

$$\tilde{\beta}^2[\mathbb{C}P^2] = 1, \qquad \tilde{\beta}^3[\mathbb{C}P^3] = 1.$$

Since the inclusion mapping of $\tilde{N}^8$ into $(\tilde{N}^8, \partial\tilde{N}^8)$ induces an isomorphism of $H^2(\tilde{N}^8, \partial\tilde{N}^8)$ onto $H^2(\tilde{N}^8)$, we may regard $\tilde{\beta}$ as a generator of $H^2(\tilde{N}^8, \partial\tilde{N}^8)$, and thus $\tilde{\beta}^4$ is an element of $H^8(\tilde{N}^8, \partial\tilde{N}^8)$. We shall let $\tilde{N}^8$ be so oriented that

$$\tilde{\beta}^4[\tilde{N}^8, \partial\tilde{N}^8] = 1.$$

Let

$$\tilde{\alpha} = \tilde{f}^*(\beta).$$

Then, for $j = 1, 2, 3$, $\tilde{\alpha}^j$ is an element of $H^{2j}(\tilde{M}^8)$ with

$$\pi^*(\alpha_j) = q\tilde{\alpha}^j,$$

and

$$\tilde{\alpha}^2[L^4] = 1, \qquad \tilde{\alpha}^3[L^6] = 1.$$

Moreover, $\tilde{\alpha}$ is also regarded as an element of $H^2(\tilde{M}^8, \partial\tilde{M}^8)$, and $\tilde{M}^8$ is oriented so that

$$\tilde{\alpha}^4[\tilde{M}^8, \partial\tilde{M}^8] = 1.$$

LEMMA 1. *If* q *is odd, then the second Stiefel-Whitney class of* $\tilde{M}^8$ *is the reduction of an integral cohomology class modulo 2. In fact,*

$$w_2(\tilde{M}^8) = \tilde{\alpha} \pmod 2.$$

Proof. The action of $\mathbb{Z}_q$ on $\tilde{M}^8$ induces an action of $\mathbb{Z}_q$ on $H^2(\tilde{M}^8; \mathbb{Z}_2)$. Denote by $H^2(\tilde{M}^8; \mathbb{Z}_2)^{\mathbb{Z}_q}$ the set of fixed points of $\mathbb{Z}_q$ in $H^2(\tilde{M}^8; \mathbb{Z}_2)$. Then

$$w_2(\tilde{M}^8) \in H^2(\tilde{M}^8; \mathbb{Z}_2)^{\mathbb{Z}_q}.$$

In fact, each $h \in \mathbb{Z}_q$ is a diffeomorphism of $\tilde{M}^8$ onto $\tilde{M}^8$, so that it maps $w_2(\tilde{M}^8)$ into itself.

Since q is odd, the projection

$$\pi^*: H^2(M^8; \mathbb{Z}_2) \to H^2(\tilde{M}^8; \mathbb{Z}_2)$$

maps $H^2(M^8; \mathbb{Z}_2)$ isomorphically onto $H^2(\tilde{M}^8; \mathbb{Z}_2)^{\mathbb{Z}_q}$. Therefore, for some integer r,

$$w_2(\tilde{M}^8) = r\tilde{\alpha} \pmod 2.$$

We may let $f: M^8 \to N^8$ be transverse regular at the complex projective line $\mathbb{C}P^1 \subset \mathbb{C}P^2$ such that $L^2 = f^{-1}(\mathbb{C}P^1)$ is connected. Then $f: L^2 \to \mathbb{C}P^1$ is of degree ± 1. Let ν be the normal bundle of $\mathbb{C}P^1$ in $\tilde{N}^8$. Clearly, $\tilde{f}^*\nu$ is the normal bundle of L^2 in $\tilde{M}^8$, and

$$w_2(\tilde{f}^*\nu)[L^2] = w_2(\nu)[\mathbb{C}P^1] = 1.$$

Since L^2 is orientable, its tangent bundle is stably trivial. Therefore

$$w_2(\tilde{M}^8)[L^2] = w_2(\tilde{f}^*\nu)[L^2] = 1.$$

Hence $w_2(\tilde{M}^8) = \tilde{\alpha} \pmod 2$.

From now on, *we assume that* q *is odd.* Since the homomorphism $H^{2j}(\tilde{M}^8, \partial\tilde{M}^8) \to H^{2j}(\tilde{M}^8)$ induced by the inclusion mapping is an isomorphism for $j = 1, 2, 3$, it follows from Lemma 1 that the modified Eells-Kuiper invariant $\nu(\partial\tilde{M}^8)$ can be computed from $\tilde{M}^8$. This is carried out below.

LEMMA 2. *If* $I(L^4)$ *is the index of* L^4, *then the first rational Pontrjagin class of* $\tilde{M}^8$ *is given by*

$$p_1(\tilde{M}^8) = (3I(L^4) + 2)\tilde{\alpha}^2.$$

Proof. Using the same argument as in the first part of the proof of Lemma 1, we can show that

$$p_1(\tilde{M}^8) \in \pi^* H^4(M^8; \mathbb{Q}).$$

Therefore $p_1(\tilde{M}^8)$ is determined by the value $p_1(\tilde{M}^8)[L^4]$. In fact,

$$p_1(\tilde{M}^8) = p_1(\tilde{M}^8)[L^4] \cdot \tilde{\alpha}^2.$$

Let ν be the normal bundle of $\mathbb{C}P^2$ in $\tilde{N}^8$. Then $\tilde{f}^* \nu$ is the normal bundle of L^4 in $\tilde{M}^8$, so that

$$p_1(f^* \nu)[L^4] = p_1(\nu)[\mathbb{C}P^2] = 2.$$

It is well known that if τ is the tangent bundle of L^4, then

$$p_1(\tau)[L^4] = 3I(L^4).$$

Hence

$$p_1(\tilde{M}^8)[L^4] = p_1(\tau \oplus f^* \nu)[L^4] = (p_1(\tau) + p_1(f^* \nu))[L^4] = 3I(L^4) + 2.$$

Let

$$\tilde{M}^6 = \pi^{-1}(\Sigma^*).$$

Then $\tilde{M}^6$ is a closed, differentiable 6-manifold that can be so oriented that

$$\tilde{\alpha}^3[\tilde{M}^6] = 1.$$

Moreover, $\mathbb{Z}_q$ acts differentiably on $\tilde{M}^6$ so that

$$\tilde{M}^6/\mathbb{Z}_q = \Sigma^*,$$

and for each $i = 1, \cdots, k$, the action of $\mathbb{Z}_{q_i}$ on $\tilde{M}^6$ is semifree and has $L^4 \cup \pi^{-1}(b_i^*)$ as its set of fixed points.

Now we calculate the Atiyah-Singer invariant $\operatorname{Sign}(h, \tilde{M}^6)$ (see [1]), where $h \in \mathbb{Z}_q - \{1\}$. Whenever $h \in \mathbb{Z}_q$ and r is an integer such that $1 - h^r \neq 0$, we let

$$\theta_r(h) = (1 + h^r)/(1 - h^r).$$

Since the action of $\mathbb{Z}_q$ on $\mathbb{C}P^4$ is given by

$$h(z_1 : z_2 : z_3 : z_4 : z_5) = (h^{-1} z_1 : z_2 : z_3 : z_4 : z_5),$$

we see that for each $h \in \mathbb{Z}_q - \{1\}$, the term in $\operatorname{Sign}(h, \tilde{M}^6)$ associated with L^4 is

$$-\theta_1(h)(I(L^4) - 1 + \theta_1(h)^2).$$

Hence, for each $h \in \mathbb{Z}_q - \bigcup_{i=1}^k \mathbb{Z}_{q_i}$,

$$\operatorname{Sign}(h, \tilde{M}^6) = -\theta_1(h)(I(L^4) - 1 + \theta_1(h)^2).$$

Clearly the fixed-point set $\pi^{-1}(b_i^*)$ of $\mathbb{Z}_{q_i}$ contains q/q_i points that are permuted cyclically by $\mathbb{Z}_{q/q_i}$. From the construction of $\tilde{M}^8$, it is not hard to see that each point of $\pi^{-1}(b_i^*)$ has $D_i = E_i \cap \tilde{M}^6$ as a closed neighborhood in $\tilde{M}^6$ on which $\mathbb{Z}_{q_i}$ acts orthogonally. Let (z_1, z_2, z_3) be complex coordinates on D_i such that the orientation of $\tilde{M}^6$ is represented by the real coordinate system

$$(z_1 + \bar{z}_1, -\sqrt{-1}(z_1 - \bar{z}_1), z_2 + \bar{z}_2, -\sqrt{-1}(z_2 - \bar{z}_2), z_3 + \bar{z}_3, -\sqrt{-1}(z_3 - \bar{z}_3)),$$

and such that for some integers r_{i1}, r_{i2}, r_{i3} the action of $\mathbb{Z}_{q_i}$ on D_i is given by

$$h(z_1, z_2, z_3) = (h^{r_{i1}} z_1, h^{r_{i2}} z_2, h^{r_{i3}} z_3).$$

It has been shown in an earlier paper [3] that

$$r_{i1} r_{i2} r_{i3} \equiv q/q_i \pmod{2q_i}.$$

Therefore, for each $h \in \mathbb{Z}_{q_i} - \{1\}$, the term in $\operatorname{Sign}(h, \tilde{M}^6)$ associated with each point of $\pi^{-1}(b_i^*)$ is

$$\theta_{r_{i1}}(h)\, \theta_{r_{i2}}(h)\, \theta_{r_{i3}}(h).$$

Hence, for each $h \in \mathbb{Z}_{q_i} - \{1\}$,

$$\operatorname{Sign}(h, \tilde{M}^6) = -\theta_1(h)(I(L^4) - 1 + \theta_1(h)^2) + (q/q_i)\, \theta_{r_{i1}}(h)\, \theta_{r_{i2}}(h)\, \theta_{r_{i3}}(h).$$

Since q is odd, we can have integers r_{i1}, r_{i2}, r_{i3} such that $(q, r_{i1} r_{i2} r_{i3}) = 1$. Our results yield the following proposition.

LEMMA 3. *For each* $h \in \mathbb{Z}_q - \{1\}$,

$$\operatorname{Sign}(h, \tilde{M}^6) = -\theta_1(h)(I(L^4) - 1 + \theta_1(h)^2) + \sum_{i=1}^{k} \sum_{j=1}^{q/q_i} h^{(j-1)q_i}\, \theta_{r_{i1}}(h)\, \theta_{r_{i2}}(h)\, \theta_{r_{i3}}(h).$$

LEMMA 4. *Up to a congruence modulo* q, *the integer* $I(L^4)$ *is determined by the equality in Lemma* 3.

Proof. By the definition of $\operatorname{Sign}(h, \tilde{M}^6)$, there exist integers $\mu_1, \cdots, \mu_{q-1}$ such that

$$\mu_j = -\mu_{q-j} \qquad (j = 1, \cdots, q-1)$$

and such that for each $h \in \mathbb{Z}_q - \{1\}$,

$$\sum_{j=1}^{q-1} \mu_j h^j = \operatorname{Sign}(h, \tilde{M}^6)$$

$$= -\theta_1(h)(I(L^4) - 1 + \theta_1(h)^2) + \sum_{i=1}^{k} \sum_{j=1}^{q/q_i} h^{(j-1)q_i}\, \theta_{r_{i1}}(h)\, \theta_{r_{i2}}(h)\, \theta_{r_{i3}}(h).$$

If I is an integer with the property that there exist integers $\mu'_1, \cdots, \mu'_{q-1}$ such that

$$\mu'_j = -\mu'_{q-j} \quad (j = 1, \cdots, q-1),$$

and such that for each $h \in \mathbb{Z}_q - \{1\}$,

$$\sum_{j=1}^{q-1} \mu'_j h^j = -\theta_1(h)(I - 1 + \theta_1(h)^2) + \sum_{i=1}^{k} \sum_{j=1}^{q/q_i} h^{(j-1)q_i} \theta_{r_{i1}}(h)\theta_{r_{i2}}(h)\theta_{r_{i3}}(h),$$

then for each $h \in \mathbb{Z}_q - \{1\}$,

$$\sum_{j=1}^{q-1} (\mu_j - \mu'_j)h^j = \theta_1(h)(I - I(L^4)) = \sum_{j=1}^{q-1} \frac{1}{q}(q - 2j)(I - I(L^4))h^j,$$

and therefore

$$\frac{1}{q}(q - 2j)(I - I(L^4)) = \mu_j - \mu'_j \quad (j = 1, \cdots, k).$$

Since q is odd, it follows that

$$I - I(L^4) \equiv 0 \pmod q.$$

LEMMA 5. $I(L^4) \equiv 1 \pmod 8$.

Proof. Altering $f: (M^8, \partial M^8) \to (N^8, \partial N^8)$ by a homotopy, if necessary, we may assume that f is transverse regular at $\mathbb{C}P^1$ and $L^2 = f^{-1}(\mathbb{C}P^1)$ is a 2-sphere S^2 or a torus $S^1 \times S^1$, according as the associated Arf invariant in connection with the framed surgery used to alter f vanishes or not, and that $L^4 = f^{-1}(\mathbb{C}P^2)$ is simply connected. Let T be a closed tubular neighborhood of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. Then $\mathbb{C}P^2$ - int T is a closed 4-disk, and some closed differentiable 4-disk D^4 in S^7 is mapped diffeomorphically onto $\mathbb{C}P^2$ - int T by the projection of S^7 onto S^*. Since f is transverse regular at both $\mathbb{C}P^1$ and $\mathbb{C}P^2$, the inverse image $f^{-1}(T)$ is a closed tubular neighborhood of L^2 in L^4, and some compact differentiable 4-manifold $K^4 = f^{-1}(D^4)$ in Σ^7 is mapped diffeomorphically onto L^4 - int $f^{-1}(T)$. Since the normal bundle of D^4 in S^7 is trivial, so is that of K^4 in Σ^7. Hence K^4 is parallelizable.

If $L^2 = S^2$, then $\partial K^4 = S^3$, so that the index $I(K^4)$ of K^4 is a multiple of 16. Hence, in this case,

$$I(L^4) = I(K^4) + 1 \equiv 1 \pmod 8.$$

If $L^2 = S^1 \times S^1$, then ∂K^4 may be regarded as the closed differentiable 3-manifold obtained from $[0, 1] \times S^1 \times S^1$ by identification of $(0, z, z')$ with $(1, z, zz')$ for all $z, z' \in S^1$. The diffeomorphism of $[0, 1] \times S^1 \times S^1$ onto itself given by

$$(t, z, z') \to (1 - t, z^{-1}z'^{-1})$$

induces an orientation-reversing diffeomorphism ϕ of ∂K^4 onto ∂K^4, and hence $K^4 \cup_\phi K^4$ is a closed differentiable π-manifold of index $2I(K^4)$. Therefore $2I(K^4)$ is a multiple of 16, and hence

$$I(L^4) = I(K^4) + 1 \equiv 1 \pmod 8.$$

LEMMA 6. *Let* $\Sigma^7 = F \cup K$, *where* F *is a free* G-*manifold and* K *is a composite* G-*manifold. (For details of this decomposition, see* [3].) *Then* $I(L^4) \pmod{8q}$ *depends only on* K; *that means, it is independent of the pasting of* F *to* K. *Moreover, any integer in this residue class can be realized as* $I(L^4)$, *provided that the pasting of* F *to* K *is appropriately altered.*

Proof. By Lemma 4, $I(L^4) \pmod q$ depends only on K, and by Lemma 5, $I(L^4) \equiv 1 \pmod 8$. Since q is odd, it follows that $I(L^4) \pmod{8q}$ depends only on K, and hence it is independent of the pasting of F to K.

Let Φ be the group of all equivariant diffeomorphisms $\phi: \partial F \to \partial F$ such that $\phi: H_*(\partial F) \to H_*(\partial F)$ is the identity, and let $\mathscr{I}$ be the group of equivariant pseudo-isotopy classes $[\phi]$ $(\phi \in \Phi)$. It is known [3] that there exists an isomorphism

$$\lambda: \mathscr{I} \to \mathbb{Z}$$

such that each $\phi \in \Phi$ is equivariant homotopic to the identity if and only if $\lambda[\phi]$ is an even integer.

Let S^2 be a differentiable 2-sphere in $\Sigma^* - \{b_1^*, \cdots, b_k^*\}$ representing a generator of $H_2(\Sigma^* - \{b_1^*, \cdots, b_k^*\})$. Then the intersection number of S^2 with L^4 is equal to q, so that we may assume that $S^2 \cap L^4$ contains exactly q points and that S^2 and L^4 intersect transversally at each of the q points. Since we may use a small closed tubular neighborhood of S^2 as F^* (see [3] for details), we lose no generality by assuming that $F^* \cap L^4$ contains q closed 4-disks. It is known that if $\phi \in \Phi$ is such that $\lambda[\phi]$ is even, say $\lambda[\phi] = 2m$, then each 3-sphere in $\phi^{-1}(\partial F^* \cap L^4)$ bounds in F^* a parallelizable compact differentiable 4-manifold of index 16m. Therefore, for the circle action on the homotopy 7-sphere $F \cup_\phi K$, the index $I(L^4)$ is increased by 16m. Conversely, if $\phi \in \Phi$ is such that for the circle action on $F \cup_\phi K$, the value of $I(L^4)$ is increased by 16m, then $\lambda[\phi] = 2m$. Hence, for each $\psi \in \Phi$ with odd $\lambda[\psi]$, the value of $I(L^4)$ for the circle action on $F \cup_\psi K$ is changed by an odd multiple of 8q, and each odd multiple of 8q can actually occur as the change, provided that an appropriate ψ is used. Combining these results, we have established our assertion.

LEMMA 7. *The Atiyah-Singer invariant for the action of* $\mathbb{Z}_q$ *on* $\tilde{M}^8$ *is given as follows: For each* $h \in \mathbb{Z}_q - \{1\}$,

$$\operatorname{Sign}(h, \tilde{M}^8) = I(L^4) + \theta_1(h) \operatorname{Sign}(h, \tilde{M}^6) = (I(L^4) + \theta_1(h)^2)(1 - \theta_1(h)^2)$$
$$+ \sum_{i=1}^{k} \sum_{j=1}^{q/q_i} h^{(j-1)q_i} \theta_1(h)\, \theta_{r_{i1}}(h)\, \theta_{r_{i2}}(h)\, \theta_{r_{i3}}(h).$$

Moreover, if $\nu_1, \cdots, \nu_{q-1}$ *are integers such that*

$$\nu_j = \nu_{q-j} \qquad (j = 1, \cdots, q-1),$$

and such that for each $h \in \mathbb{Z}_q - \{1\}$

$$\operatorname{Sign}(h, \tilde{M}^8) = 1 + \sum_{j=1}^{q-1} \nu_j h^j,$$

then the index of $\tilde{M}^8$ *is given by*

$$I(\tilde{M}^8) = 1 + \sum_{j=1}^{q-1} \nu_j .$$

Proof. As in the proof of Lemma 3, it can be shown that for each $h \in \mathbb{Z}_q - \{1\}$, the term in $\mathrm{Sign}(h, \tilde{M}^8)$ associated with L^6 is

$$(I(L^4) + \theta_1(h)^2)(1 - \theta_1(h)^2)$$

and that for each $h \in \mathbb{Z}_{q_i} - \{1\}$, the term in $\mathrm{Sign}(h, \tilde{M}^8)$ associated with each point of $\pi^{-1}(b_i^*)$ is

$$\theta_1(h)\,\theta_{r_{i1}}(h)\,\theta_{r_{i2}}(h)\,\theta_{r_{i3}}(h) .$$

Hence the first part of Lemma 7 follows.

In order to prove the second part of Lemma 7, let us first recall the definition of $\mathrm{Sign}(h, \tilde{M}^8)$. Let V be the real vector space

$$H^4(\tilde{M}^8; \mathbb{R}) \cong H^4(\tilde{M}^8, \partial\tilde{M}^8; \mathbb{R}) .$$

Clearly, V is finite-dimensional and $\mathbb{Z}_q$ acts linearly on V. Let $\langle\ ,\ \rangle$ be a $\mathbb{Z}_q$-invariant positive definite inner product on V; this means that with respect to $\langle\ ,\ \rangle$, $\mathbb{Z}_q$ acts orthogonally on V. Let B be the symmetric bilinear form on V defined by

$$B(u, v) = (uv)[\tilde{M}^8, \partial\tilde{M}^8],$$

and let A be the self-adjoint linear operator on V defined by

$$B(u, v) = \langle u, Av \rangle .$$

Then the eigenvalues of A are real numbers different from 0, so that there exists a decomposition

$$V = V_+ \oplus V_-,$$

where V_+ and V_- are the positive and negative eigenspaces of A. Since B is $\mathbb{Z}_q$-invariant, A commutes with the action of $\mathbb{Z}_q$, so that V_+ and V_- are $\mathbb{Z}_q$-invariant. For each $h \in \mathbb{Z}_q - \{1\}$, the Atiyah-Singer invariant $\mathrm{Sign}(h, \tilde{M}^8)$ is defined by

$$\mathrm{Sign}(h, \tilde{M}^8) = \mathrm{trace}(h \mid V_+) - \mathrm{trace}(h \mid V_-) .$$

Using the action of $\mathbb{Z}_q$, one can find a decomposition of V_+ into a number of invariant linear subspaces. In fact,

$$V_+ = V_0 \oplus \nu_1' V_1 \oplus \cdots \oplus \nu'_{(q-1)/2} V_{(q-1)/2},$$

where

$$V_0 = \{v \in V_+ \mid \mathbb{Z}_q v = v\},$$

where for each $j = 1, \cdots, (q-1)/2$, V_j denotes a 2-dimensional invariant linear space on which the action of $\exp 2\pi\sqrt{-1}/q$ may be represented by the matrix

$$\begin{pmatrix} \cos 2j\pi/q & -\sin 2j\pi/q \\ \sin 2j\pi/q & \cos 2j\pi/q \end{pmatrix},$$

where ν_j' is an integer ($\nu_j' \geq 0$), and where $\nu_j' V_j$ denotes the direct sum of ν_j' copies of V_j. Since π^*: $H^4(M^8; \mathbb{R}) \to H^4(\tilde{M}^8; \mathbb{R})$ is a monomorphism having V_0 as its image, we infer that V_0 is 1-dimensional. Hence, for each $h \in \mathbb{Z}_q - \{1\}$,

$$\operatorname{trace}(h \mid V_+) = 1 + \sum_{j=1}^{(q-1)/2} \nu_j'(h^j + h^{q-j}).$$

Similarly, there are nonnegative integers $\nu_1'', \cdots, \nu_{(q-1)/2}''$ such that

$$V_- = \nu_1'' V_1 \oplus \cdots \oplus \nu_{(q-1)/2}'' V_{(q-1)/2}$$

and

$$\operatorname{trace}(h \mid V_-) = \sum_{j=1}^{(q-1)/2} \nu_j''(h^j + h^{q-j}).$$

Hence, for each $h \in \mathbb{Z}_q - \{1\}$,

$$\operatorname{Sign}(h, \tilde{M}^8) = 1 + \sum_{j=1}^{q-1} \nu_j h^j,$$

where

$$\nu_j = \nu_{q-j} = \nu_j' - \nu_j'' \quad (j = 1, \cdots, (q-1)/2).$$

Notice that the integers $\nu_1, \cdots, \nu_{q-1}$ are completely determined by the equality above.

There exists an orthonormal basis $\{v_1, \cdots, v_n\}$ of V such that each v_j is an eigenvector of A, and such that for some integer m, V_+ is generated by $\{v_1, \cdots, v_m\}$ and V_- is generated by $\{v_{m+1}, \cdots, v_n\}$. Then, for each pair i, j ($i, j = 1, \cdots, n$), $v_i v_j \neq 0$ if and only if $i = j$, and for $i = 1, \cdots, n$, $(v_i v_i)[\tilde{M}^8, \partial M^8]$ is positive if and only if $i \leq m$. Hence

$$\begin{aligned} I(\tilde{M}^8) &= m - (n - m) = \dim V_+ - \dim V_- \\ &= 1 + \sum_{j=1}^{(q-1)/2} 2\nu_j' - \sum_{j=1}^{(q-1)/2} 2\nu_j'' = 1 + \sum_{j=1}^{q-1} \nu_j. \end{aligned}$$

This completes the proof of Lemma 7.

Now we are in a position to describe the idea of the calculation to be carried out later. As was seen in [3], we can obtain each differentiable pseudo-free action of the circle group G on a homotopy 7-sphere Σ^7, up to an equivariant diffeomorphism, by pasting a free G-manifold F to a composite G-manifold K. Let

$$k;\ q_1, \cdots, q_k, q;\ r_{11}, r_{12}, r_{13}, \cdots, r_{k1}, r_{k2}, r_{k3}$$

be as before. Then

(i) $k \geq 1$,

(ii) $q_i > 1$ for $i = 1, \cdots, k$, and the integers $q_1, \cdots, q_k$ are mutually relatively prime, and

(iii) $q = q_1 \cdots q_k$, and for $i = 1, \cdots, k$,

$$r_{i1} r_{i2} r_{i3} \equiv q/q_i \pmod{2q_i}.$$

Moreover, up to an equivariant diffeomorphism the composite G-manifold K is determined by these integers, and conversely. Furthermore, if there are such integers satisfying (i), (ii), (iii) above, and if K is the determined composite G-manifold, then we can have a differentiable pseudo-free circle action on a homotopy 7-sphere obtained by pasting a free G-manifold F to K. By varying the pasting of F to K, we actually have a collection $\mathscr{S}$ of such actions that depends only on the integers k; $q_1, \cdots, q_k, q;\ r_{11}, r_{12}, r_{13}, \cdots, r_{k1}, r_{k2}, r_{k3}$. If q is relatively prime to 28, then the homotopy 7-spheres that appear in $\mathscr{S}$ can be determined as follows.

Suppose that in $\mathscr{S}$, we have a differentiable pseudo-free action of the circle group G on a homotopy 7-sphere. Let L^4 and $\tilde{M}^8$ be the associated manifolds constructed earlier. We first assert that the index $I(L^4)$ can be found. Since q is odd, we lose no generality by assuming that $(q, r_{i1} r_{i2} r_{i3}) = 1$ $(i = 1, \cdots, k)$. Therefore we have an integer I such that

(a) $I \equiv 1 \pmod 8$ and

(b) there exist integers $\mu_1', \cdots, \mu_{q-1}'$ such that

$$\mu_j' = -\mu_{q-j}' \qquad (j = 1, \cdots, q-1),$$

and for each $h \in \mathbb{Z}_q - \{1\}$,

$$(1)\qquad \sum_{j=1}^{q-1} \mu_j' h^j = -\theta_1(h)(I - 1 + \theta_1(h)^2) + \sum_{i=1}^{k} \sum_{j=1}^{q/q_i} h^{(j-1)q_i} \theta_{r_{i1}}(h)\, \theta_{r_{i2}}(h)\, \theta_{r_{i3}}(h).$$

In fact, the existence of I is guaranteed by Lemmas 3 and 5. By Lemma 6, there exists an integer m such that

$$(2)\qquad I(L^4) = I + 8mq.$$

By Lemma 3, there exist integers $\mu_1, \cdots, \mu_{q-1}$ such that

$$\mu_j = -\mu_{q-j} \qquad (j = 1, \cdots, q-1),$$

and such that for each $h \in \mathbb{Z}_q - \{1\}$,

$$(3)\qquad \sum_{j=1}^{q-1} \mu_j h^j = -\theta_1(h)(I(L^4) - 1 + \theta_1(h)^2) + \sum_{i=1}^{k-1} \sum_{j=1}^{q/q_i} h^{(j-1)q_i} \theta_{r_{i1}}(h)\, \theta_{r_{i2}}(h)\, \theta_{r_{i3}}(h).$$

Then, by Lemma 7, there exist integers $\nu_1, \cdots, \nu_{q-1}$ such that

$$\nu_j = \nu_{q-j} \qquad (j = 1, \cdots, q-1),$$

and such that

(4) $$1 + \sum_{j=1}^{q-1} \nu_j h^j = I(L^4) + \theta(h) \sum_{j=1}^{q-1} \mu_j h^j .$$

Since $\mu_1, \cdots, \mu_{q-1}$ are determined by (3) and $\nu_1, \cdots, \nu_{q-1}$ are then determined by (4), it follows from Lemma 7 that

(5) $$I(\tilde{M}^8) = 1 + \sum_{j=1}^{q-1} \nu_j$$

is determined.

As we remarked before Lemma 2, we can use $\tilde{M}^8$ to compute the modified Eells-Kuiper invariant $\nu(\partial\tilde{M}^8)$. In fact, we know from Lemma 2 that

$$p_1(\tilde{M}^8) = (3I(L^4) + 2)\tilde{\alpha}^2 .$$

Therefore

(6) $$\nu(\partial\tilde{M}^8) \equiv \frac{1}{896}(3I(L^4)+2)^2 - \frac{1}{192}(3I(L^4)+2) + \frac{1}{384} - \frac{1}{224}I(\tilde{M}^8) \pmod 1 .$$

We have seen that

$$\partial\tilde{M}^8 = \pm q\Sigma^7,$$

so that

(7) $$\nu(\partial\tilde{M}^8) = \pm q\nu(\Sigma^7) .$$

Since $\nu(\Sigma^7) \equiv \ell/28 \pmod 1$ for some integer ℓ, and since $(q, 28) = 1$, the modified Eells-Kuiper invariant $\nu(\Sigma^7)$ of Σ^7 is determined.

By Lemma 6, the number m in (2) can be any integer. By varying m in (2), we find the Eells-Kuiper invariant for homotopy 7-spheres appearing in $\mathcal{S}$.

4. A SPECIAL CASE

The computation described above is in general very complicated. Therefore we carry it out only for certain special cases.

Let q be an integer such that

$$q > 1, \qquad (q, 28) = 1 .$$

Then there is an orthogonal pseudo-free action of G on S^7 given by

$$g(z_1, z_2, z_3, z_4) = (g^q z_1, gz_2, gz_3, gz_4) .$$

Let

$$S^7 = F \cup K,$$

where F is a free G-manifold and K is a composite G-manifold, and let $\mathscr{S}$ be the collection of differentiable pseudo-free circle actions on homotopy 7-spheres obtained by pasting F to K. Then, for the actions in $\mathscr{S}$,

$$k = 1, \quad q_1 = q, \quad r_{11} = r_{12} = r_{13} = 1.$$

Suppose that in $\mathscr{S}$, we have a differentiable pseudo-free circle action on a homotopy 7-sphere Σ^7, and let L^4 and $\tilde{M}^8$ be as before. By (1), we may let

$$I = 1.$$

Then it follows from (2) that for some integer m,

$$I(L^4) = 1 + 8mq.$$

Using (3) and (4), we see that

$$1 + \sum_{j=1}^{q-1} \nu_j h^j = I(L^4) + \theta_1(h) \sum_{j=1}^{q-1} \mu_j h^j$$

$$= 1 + 8mq - 8mq\, \theta_1(h)^2$$

$$= 1 - 8m \sum_{j=1}^{q-1} (2jq - 2j^2)\, h^j.$$

Therefore, by (5),

$$I(\tilde{M}^8) = 1 - 8m \sum_{j=1}^{q-1} (2jq - 2j^2) = 1 - \frac{8}{3} mq(q^2 - 1).$$

Hence, by (6) and (7),

$$\pm q\nu(\Sigma^7) \equiv \frac{9}{14} m^2 q^2 + \frac{1}{7} mq + \frac{1}{84} mq(q^2 - 1) \pmod 1.$$

If q is not divisible by 3, then

$$\nu(\Sigma^7) \equiv \pm\left[\frac{9}{14} m^2 q + \frac{1}{7} m + \frac{1}{84} m(q^2 - 1) \right] \pmod 1; \tag{8}$$

otherwise,

$$\nu(\Sigma^7) \equiv \pm\left[\frac{9}{14} m^2 q + \frac{1}{7} m + \frac{1}{84} m(q^2 + 27) \right] \pmod 1. \tag{9}$$

LEMMA 8. *Let $\mathscr{S}$ be as above, and let θ^7 be the group of homotopy 7-spheres. Then each homotopy 7-sphere that appears in $\mathscr{S}$ is in the subgroup $2\theta^7$ of θ^7 of index 2. Moreover, every homotopy 7-sphere in $2\theta^7$ appears in $\mathscr{S}$ either for $q = 5$ or for $q = 11$.*

Proof. It is easily seen that the right side of (8) and (9) can be reduced to $\ell/14 \pmod 1$, for some integer ℓ. Hence the first part follows. To prove the second part, we have only to show that for $q = 5$,

m	0	4	3	6	1	5
$\nu(\Sigma^7)$ (mod 1)	0	$\pm\frac{2}{14}$	$\pm\frac{3}{14}$	$\pm\frac{4}{14}$	$\pm\frac{5}{14}$	$\pm\frac{7}{14}$

and that for $q = 11$,

m	9	2
$\nu(\Sigma^7)$ (mod 1)	$\pm\frac{1}{14}$	$\pm\frac{6}{14}$

This is so because m can be any integer, by Lemma 6.

5. A RESULT ON $I(L^4)$

We shall need the following result.

THEOREM C. *Let* q, r_1, r_2 *be integers such that*

$$q > 1, \qquad (q, r_1 r_2) = 1.$$

Let $\mathbb{Z}_q$ *be the group of complex numbers* h *with* $h^q = 1$, *and for each* $h \in \mathbb{Z}_q - \{1\}$ *and each integer* r *relatively prime to* q, *let*

$$\theta_r(h) = (1 + h^r)/(1 - h^r).$$

Then there exist integers $\lambda_1, \cdots, \lambda_{q-1}$ *such that*

$$\lambda_j = \lambda_{q-j} \qquad (j = 1, \cdots, q-1),$$

and such that for each $h \in \mathbb{Z}_q - \{1\}$,

$$r_1 r_2 (\theta_{r_1}(h)\, \theta_{r_2}(h) - \theta_{r_1 r_2}(h)\, \theta_1(h)) = \sum_{j=1}^{q-1} \lambda_j h^j .$$

Moreover,

$$\sum_{j=1}^{q-1} \lambda_j \equiv \frac{1}{3}(q^2 - 1)(r_1^2 - 1)(r_2^2 - 1) \pmod q .$$

Proof. Let r be an integer with $(q, r) = 1$. For $j = 1, \cdots, q-1$, let $a(j)$ and $b(j)$ be integers such that

$$jr = a(j)q + b(j), \qquad 0 < b(j) < q.$$

Then

$$a(j) + a(q-j) = r - 1 \quad \text{and} \quad b(j) + b(q-j) = q,$$

and

$$\begin{pmatrix} 1 & \cdots & q-1 \\ b(1) & \cdots & b(q-1) \end{pmatrix}$$

is a permutation. Therefore

$$r - 1 - 2a(j) = a(q-j) - a(j) \qquad (j = 1, \cdots, q-1),$$

so that

$$\sum_{j=1}^{q-1} (r - 1 - 2a(j)) = 0 .$$

For $m = 0, \cdots, q-1$, let

$$\mu_m = \sum_{\substack{i+b(j)\equiv m \pmod q \\ 0<i<q}} \frac{1}{q}(q - 2i)(r - 1 - 2a(j))$$

$$= \sum_{b(j)<m} \frac{1}{q}(q - 2m + 2b(j))(r - 1 - 2a(j)) + \sum_{b(j)>m} \frac{1}{q}(-q - 2m + 2b(j))(r - 1 - 2a(j))$$

$$= \sum_{b(j)<m} (r - 1 - 2a(j)) - \sum_{b(j)>m} (r - 1 - 2a(j)) - \frac{2}{q}\sum_{j=1}^{q-1} (m - b(j))(r - 1 - 2a(j)) .$$

Then for $\ell = 1, \cdots, q-1$, the number

$$\mu_\ell - \mu_0 = \sum_{b(j)<\ell} (r - 1 - 2a(j)) + \sum_{b(j)\le\ell} (r - 1 - 2a(j)) - \frac{2\ell}{q}\sum_{j=1}^{q-1} (r - 1 - 2a(j))$$

$$= \sum_{b(j)<\ell} (r - 1 - 2a(j)) + \sum_{b(j)\le\ell} (r - 1 - 2a(j))$$

is an integer. Moreover,

$$\sum_{\ell=1}^{q-1} (\mu_\ell - \mu_0) = \sum_{j=1}^{q-1} (2q - 1 - 2b(j))(r - 1 - 2a(j))$$

$$= \sum_{j=1}^{q-1} (-2b(j))(r - 1 - 2a(j))$$

$$= 2\sum_{j=1}^{q-1} b(j)(a(j) - a(q-j))$$

$$= 2\sum_{j=1}^{q-1} a(j)(b(j) - b(q-j))$$

$$= 2 \sum_{j=1}^{q-1} a(j)(2b(j) - q)$$

$$= 4 \sum_{j=1}^{q-1} a(j)b(j) - q(q-1)(r-1).$$

Since

$$r^2 \sum_{j=1}^{q-1} j^2 = \sum_{j=1}^{q-1} (a(j)q + b(j))^2$$

$$= q^2 \sum_{j=1}^{q-1} a(j)^2 + 2q \sum_{j=1}^{q-1} a(j)b(j) + \sum_{j=1}^{q-1} b(j)^2,$$

we infer that

$$4 \sum_{j=1}^{q-1} a(j)b(j) = \frac{2}{q}(r^2 - 1) \sum_{j=1}^{q-1} j^2 - 2q \sum_{j=1}^{q-1} a(j)^2$$

$$= \frac{1}{3}(q-1)(2q-1)(r^2-1) - 2q \sum_{j=1}^{q-1} a(j)^2$$

$$= -\frac{1}{3}(q^2-1)(r^2-1) + q((q-1)(r^2-1) - 2 \sum_{j=1}^{q-1} a(j)^2).$$

Hence

$$\sum_{\ell=1}^{q-1} (\mu_\ell - \mu_0) \equiv -\frac{1}{3}(q^2-1)(r^2-1) \pmod q.$$

Similar to $a(j)$, $b(j)$, μ_m for r, let

$$a_1(j),\ b_1(j),\ \mu_{m1};\ a_2(j),\ b_2(j),\ \mu_{m2};\ a_{12}(j),\ b_{12}(j),\ \mu_{m12}$$

be the corresponding numbers for r_1, r_2, $r_1 r_2$, respectively. Then, modulo q,

$$\sum_{\ell=1}^{q-1} [(\mu_{\ell 1} - \mu_{01}) + (\mu_{\ell 2} - \mu_{02}) - (\mu_{\ell 12} - \mu_{012})]$$

$$\equiv -\frac{1}{3}(q^2-1)[(r_1^2-1) + (r_2^2-1) - (r_1^2 r_2^2 - 1)]$$

$$\equiv \frac{1}{3}(q^2-1)(r_1^2-1)(r_2^2-1).$$

For $m = 0, \cdots, q-1$, let

$$\mu_{m0} = \sum_{b_1(j)+b_2(k)\equiv m \pmod q} (r_1 - 1 - 2a_1(j))(r_2 - 1 - 2a_2(k)).$$

Then $\mu_{00}, \cdots, \mu_{q-1,0}$ are integers, and

$$\sum_{m=0}^{q-1} \mu_{m0} = \sum_{j=1}^{q-1} (r_1 - 1 - 2a_1(j)) \cdot \sum_{k=1}^{q-1} (r_2 - 1 - 2a_2(k)) = 0,$$

so that

$$\sum_{\ell=1}^{q-1} (\mu_{\ell 0} - \mu_{00}) = -q\mu_{00} \equiv 0 \pmod q.$$

For each $h \in \mathbb{Z}_q - \{1\}$,

$$\theta_1(h) = (1+h)/(1-h) = \sum_{i=1}^{q-1} \frac{1}{q}(q-2i)h^i,$$

so that

$$\begin{aligned} r\,\theta_r(h) &= r \sum_{j=1}^{q-1} \frac{1}{q}(q-2j)h^{rj} \\ &= \sum_{j=1}^{q-1} \frac{1}{q}(rq - 2a(j)q - 2b(j))h^{b(j)} \\ &= \sum_{j=1}^{q-1} (r - 1 - 2a(j))h^{b(j)} + \sum_{j=1}^{q-1} \frac{1}{q}(q - 2b(j))h^{b(j)} \\ &= \sum_{j=1}^{q-1} (r - 1 - 2a(j))h^{b(j)} + \theta_1(h). \end{aligned}$$

Using this equality, we see that

$$r_1 r_2(\theta_{r_1}(h) - \theta_{r_2}(h) - \theta_{r_1 r_2}(h)\theta_1(h)) = \sum_{m=0}^{q-1} (\mu_{m0} + \mu_{m1} + \mu_{m2} - \mu_{m12})h^m = \sum_{\ell=1}^{q-1} \lambda_\ell h^\ell,$$

where

$$\lambda_\ell = (\mu_{\ell 0} - \mu_{00}) + (\mu_{\ell 1} - \mu_{01}) + (\mu_{\ell 2} - \mu_{02}) - (\mu_{\ell 12} - \mu_{012})$$

is an integer. Since h is any root of $\sum_{i=0}^{q-1} h^i = 0$, the numbers $\lambda_1, \cdots, \lambda_{q-1}$ are uniquely determined. Moreover,

$$\sum_{\ell=1}^{q-1} \lambda_\ell \equiv \frac{1}{3}(q^2 - 1)(r_1^2 - 1)(r_2^2 - 1) \pmod{q}.$$

If h is replaced by h^{-1}, the equality we have proved remains unchanged. Hence

$$\lambda_\ell = \lambda_{q-\ell} \qquad (\ell = 1, \cdots, q - 1).$$

This completes the proof.

LEMMA 9. *Suppose that we have a differentiable pseudo-free action of the circle group* G *on a homotopy* 7*-sphere* Σ^7 *with exactly one exceptional orbit* Gb *such that* $G_b = \mathbb{Z}_q$, *where* q *is an odd integer* $(q > 1)$ *and there is a slice* D *at* b *that may be regarded as the closed unit* 6*-disk in* $\mathbb{C}^3$ *such that for some integers* r_1 *and* r_2 *with*

$$r_1 r_2 \equiv 1 \pmod{2q},$$

the action of $\mathbb{Z}_q$ *on* D *is given by*

$$h(z_1, z_2, z_3) = (h^{r_1} z_1, h^{r_2} z_2, hz_3).$$

If L^4 *is one of the associated manifolds constructed in Section* 2, *then*

$$I(L^4) \equiv 1 + \frac{1}{3}(q^2 - 1)(r_1^2 - 1)(r_2^2 - 1) \pmod{8q}.$$

Proof. Let $\xi_1, \cdots, \xi_{q-1}$ be the real numbers such that for each $h \in \mathbb{Z}_q - \{1\}$,

$$\theta_{r_1}(h)\,\theta_{r_2}(h) - \theta_1(h)^2 = \sum_{j=1}^{q-1} \xi_j h^j.$$

Since $r_1 r_2 \equiv 1 \pmod{q}$, it follows from Theorem C that $r_1 r_2 \xi_1, \cdots, r_1 r_2 \xi_{q-1}$ are integers and

$$\sum_{j=1}^{q-1} (r_1 r_2 \xi_j) \equiv \frac{1}{3}(q^2 - 1)(r_1^2 - 1)(r_2^2 - 1) \pmod{q}.$$

Therefore $\xi_1, \cdots, \xi_{q-1}$ are integers and

$$\sum_{j=1}^{q-1} \xi_j \equiv \frac{1}{3}(q^2 - 1)(r_1^2 - 1)(r_2^2 - 1) \pmod{q}.$$

By Lemma 3,

$$\theta_1(h)^{-1}\,\mathrm{Sign}(h, \tilde{M}^6) = -I(L^4) + 1 + \sum_{j=1}^{q-1} \xi_j h^j .$$

Since $\theta_1(h)^{-1} = \sum_{j=1}^{q-1} (-1)^j h^j$, and since $\mathrm{Sign}(h, \tilde{M}^6) = \sum_{j=1}^{q-1} \mu_j h^j$ for some integers $\mu_1, \cdots, \mu_{q-1}$, we infer that

$$-I(L^4) + 1 + \sum_{j=1}^{q-1} \xi_j \equiv 0 \pmod q .$$

By Lemma 5, we know also that $I(L^4) \equiv 1 \pmod 8$. Hence our assertion follows from the fact that q is odd.

6. ANOTHER SPECIAL CASE

Suppose that we have a differentiable pseudo-free action of the circle group G on a homotopy 7-sphere Σ^7 with exactly one exceptional orbit Gb such that $G_b = \mathbb{Z}_q$, where q is an integer greater than 1 and relatively prime to 28, and that some slice D at b may be regarded as the closed unit 6-disk in $\mathbb{C}^3$ such that for some integers r_1 and r_2 with

$$r_1 r_2 \equiv 1 \pmod{2q},$$

the action of $\mathbb{Z}_q$ on D is given by

$$h(z_1, z_2, z_3) = (h^{r_1} z_1, h^{r_2} z_2, hz_3) .$$

Let $\Sigma^7 = F \cup K$, where F is a free G-manifold and K is a composite G-manifold, and let $\mathscr{S}$ be the collection of differentiable pseudo-free circle actions on homotopy 7-spheres obtained by pasting F to K.

Let $h \in \mathbb{Z}_q - \{1\}$. As we saw in the proof of Lemma 9, there are integers $\xi_1, \cdots, \xi_{q-1}$ such that

$$\xi_j = \xi_{q-j} \qquad (j = 1, \cdots, q-1)$$

and

$$\theta_{r_1}(h)\,\theta_{r_2}(h) - \theta_1(h)^2 = \sum_{j=1}^{q-1} \xi_j h^j .$$

Therefore, by Lemma 7,

$$\begin{aligned}\mathrm{Sign}(h, \tilde{M}^8) &= I(L^4) + \theta_1(h)^2\,(-I(L^4) + 1 + \theta_{r_1}(h)\,\theta_{r_2}(h) - \theta_1(h)^2)\\ &= I(L^4) + \sum_{j=1}^{q-1} \frac{1}{q}((2j-1)q - 2j^2)\,h^j \cdot (-I(L^4) + 1 + \sum_{j=1}^{q-1} \xi_j h^j)\end{aligned}$$

$$= I(L^4) + \sum_{j=1}^{q-1} \frac{1}{q}((2j-1)q - 2j^2)\xi_j + \sum_{j=1}^{q-1} \eta_j h^j,$$

where $\eta_1, \cdots, \eta_{q-1}$ are numbers such that

$$\sum_{j=1}^{q-1} \eta_j = \sum_{j=1}^{q-1} \frac{1}{q}((2j-1)q - 2j^2)\left(-I(L^4) + 1 + \sum_{j=1}^{q-1} \xi_j\right) - \sum_{j=1}^{q-1} \frac{1}{q}((2j-1)q - 2j^2)\xi_j .$$

Hence

$$\begin{aligned} I(\tilde{M}^8) &= 1 - (q-1)(I(L^4) - 1 + \sum_{j=1}^{q-1} \frac{1}{q}((2j-1)q - 2j^2)\xi_j) + \sum_{j=1}^{q-1} \eta_j \\ &= 1 - \frac{1}{3}(q^2-1)(I(L^4)-1) + \frac{1}{3}(q-1)(q-2)\sum_{j=1}^{q-1} \xi_j - \sum_{j=1}^{q-1} ((2j-1)q - 2j^2)\xi_j . \end{aligned} \tag{10}$$

Assume now that

$$q = 5, \quad r_1 = 3, \quad r_2 = -3 .$$

By Lemma 9,

$$I(L^4) \equiv -7 \pmod{40},$$

so that for some integer m,

$$I(L^4) = -7 + 40m .$$

It is easily seen that for each $h \in \mathbb{Z}_5 - \{1\}$,

$$\theta_3(h)\theta_{-3}(h) - \theta_1(h)^2 = -2h - 2h^2 - 2h^3 - 2h^4,$$

so that $\xi_1 = \xi_2 = \xi_3 = \xi_4 = -2$. By (10),

$$I(\tilde{M}^8) = 73 - 320m.$$

Hence, by (6) and (7),

$$\pm 5\nu(\Sigma^7) \equiv \frac{1}{896}(-19 + 120m)^2 - \frac{1}{192}(-19 + 120m) + \frac{1}{384} - \frac{1}{224}(73 - 320m) \pmod 1$$

and consequently

$$\nu(\Sigma^7) \equiv \pm\left[\frac{1}{28} + \frac{1}{14}m(3m+2)\right] \pmod 1 .$$

This shows

m	0	5	3	9	1	7
$\nu(\Sigma^7)$ (mod 1)	$\pm\frac{1}{28}$	$\pm\frac{3}{28}$	$\pm\frac{5}{28}$	$\pm\frac{9}{28}$	$\pm\frac{11}{28}$	$\pm\frac{13}{28}$

Assume next that

$$q = 11, \quad r_1 = 3, \quad r_2 = -7.$$

Then, for some integer m,

$$I(L^4) = -39 + 88m, \qquad I(\tilde{M}^8) = 1881 - 3520m,$$

so that

$$\nu(\Sigma^7) \equiv \pm\left[\frac{5}{28} + \frac{1}{14}m(m+2)\right] \pmod 1.$$

Hence, for $m = 3$, $\nu(\Sigma^7) \equiv \pm 7/28 \pmod 1$.

With these results and Lemma 8, the proof of Theorem A is completed.

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